J. Appl. Math. & Informatics Vol. 34(2016), No. 1 - 2, pp. 75 - 84
http://dx.doi.org/10.14317/jami.2016.075

GEOMETRIC ANALYSIS OF NET PRESENT VALUE AND INTERNAL RATE OF RETURN†

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Abstract. The objective of this work is to perform a geometric analysis of the net present value (NPV) and Internal Rate of Return (IRR), defining analytics and in verifying the relationship between geometric properties of such functions. For this simulation, was used the values of the cash flows for each period identical and equal to US$ 200.00 cash, the initial investment US$ 1,000.00 and investments of each identical and equal to US$ 50.00 period. In addition, the discount rate and time were considered a maximum of 2 years (24 months) at a rate between 0 and 100%. The geometric analysis of the characteristics obtained from the expressions of the Net Present Value and Internal Rate of Return possible to observe that besides the analytical dependence between these quantities , the geometric relationships are relevant when studied in relation to the zero NPV and expressed a great contribution the sense of a broad vision for the administrator in the analysis of analytical variables that influences the balance sheet of the company.

AMS Mathematics Subject Classification : 65H05, 65F10.
Key words and phrases : Analytics, Financial statements of the company, Appropriate interest rate.

1. Introduction

One of the most relevant concerns for any administrator is the selection of new investments for the firm. The Net Present Value (NPV), according to [10], is a highly employed method for the assessment of projects. NPV may be defined as the sum of values of the annual net assets accounted for during the period in which the project is being accomplished.

The best method for the assessment of investment, according [13], is the deterministic method, or rather, consolidated coefficients, also called Internal Return

Received February 23, 2015. Revised June 6, 2015. Accepted August 26, 2015. *Corresponding author. †This work was supported by the research grant of the University UNESP - Univ Estadual Paulista, Laboratório de Matemática Aplicada e Computacional, Campus de Tupã, São Paulo, Brasil.

Rate (IRR) and Net Present Value (NPV), will give financial and economic support to the investment. The preciseness of calculations require exact information and data, otherwise the project may be altered due to data oscillation [7].

In [14], is calculate current values of future cash flows by employing costs of capital or the return rate as discount rate. If the result is different than zero, the project is accepted since net present values are higher than the cost of the project's initial investment.

According to [16], NPV is the most adequate method to compare several economic proposals. However, many types of inefficiencies may be extant in the NPV method, with conflicts in the results during its use as a criterion for selection.

According to [1], the sum of all future cash flows is determined by present value, a.k.a. Net Present Value in which are included the influxes of both and of the exit, which are discounted at a rate. In other words, NPV of an investment is the sum of total cash flow of the present value in entries and exits discounted at a rate according to the investment risk.

According to [5], NPV carries, in an explicit way, the value of money over time. NPV, a highly developed technique for capital evaluation, is based on the relationship to calculate the present value of future payments discounted at an appropriate interest rate, minus the cost of the initial investment.

NPV may be obtained by the difference between the present value of the net assets foreseen for each period during the project's duration, and the present value of the investment [18], [6].

According to [9], NPV may be obtained by subtracting the initial investment of a project from the present value of cash entries, with a discount at a rate equal to the firm's cost of capital.

Calculation of NPV for an investment project may be obtained by the algebra sum of values discounting cash flow associated to it. In other words, it is the difference in value in assets minus cost value [15].

The calculation of NPV identifies the net value of the project at its end in which investment value, cash flow in the project's horizon and the risk associated with the project by the application of a discount rate should be taken into consideration to calculate net value.

According to [5] and [2], NPV is a tool for decision-taking for the acceptance or rejection, with the following criteria:

- If NPV is over $0, the project will be viable.
- If NPV is less than $0, the project will not be viable.

So that NPV may be over $0, the firm will obtain a return over the cost of its investment capital. The market value of the firm will increase and thus the owners' richness.

A positive NPV shows a gain higher than the acceptable minimum, whereas a negative NPV indicates a return lower than the minimum rate required for investments. Therefore, the value represents updated economical value of the investment project.

Internal Rate of Return (IRR) is an indicator that evaluates the project's return as a function of costs of capital. IRR is obtained when the cost of capital eliminates NPV.

According to [11], the Internal Return Rate (IRR) is a return rate applied to cash flow during one year so that all initial investment is reduced to zero during the period in which the project is being accomplished.

Analysis of IRR shows, according to [4], the financial return of investment and the discount rate by which the value of the project becomes zero. Investment becomes attractive when the rate is higher than the cost of the initial investment.

When the value is zero, the tax associated to investment is called Internal Rate of Return (IRR) which represents an important help in deciding on the best investment. According to [2], the rate represents the relative gain of an investment project expressed in terms of an equivalent periodic interest rate

As the discount rate increases, NPV decreases. IRR is the point where NPV is positive and the project is accepted. The opposite occurs when capital costs are higher than IRR, or rather, NPV is less than zero and thus the project is refused. When capital costs are equal to IRR, the indicator is neutral to judge the project and is open to complementary indicators.

In the meantime, no well-established methods are extant to analyze the investment by using as tools the estimate of coupling of yield between the two deterministic methodologies, the Internal Return Rate (IRR) and the Net Present Rate (NPR). An imprecise analysis of data would ensue according to [12].

In the case of NPR rates close to zero, internal rates of return, also close to IRR, may be detected. Relationships between the two amounts may be observed geometrically to produce other forms of basis for decision taking by the administrator.

Current research aims at a geometrical analysis of NPV and IRR by defining analytic functions in $\mathbb{R}^3$ and verifying the relationship of the geometrical properties between these functions.

2. Materials and Methods

Definitions involving the analytic expressions of NPV and IRR are required for the development of current assay. The mathematical expression of NPV is given by:

$$NPV = \left[\sum_{t=1}^{n} \frac{FC_t}{(1+k)^t}\right] - \left[I_0 + \sum_{t=1}^{n} \frac{I_t}{(1+k)^t}\right] \tag{1}$$

Where:

- FC_t is the cash flow (benefit) for each period;
- k is the project's discount rate;
- I_0 is the initial investment;

- I_t is the value of the investment foreseen in each subsequent period.

IRR is implicitly obtained by

$$I_0 + \sum_{t=1}^{n} \frac{I_t}{(1+IRR)^t} = \sum_{t=1}^{n} \frac{FC_t}{(1+IRR)^t} \tag{2}$$

A system of Cartesian coordinates in $\mathbb{R}^3$ with functions type $f(u,v) = (f_1(u,v)$ $f_2(u,v), f_3(u,v))$ of two variables n and k that vary within a subset of $\mathbb{R}^3$, in which R is the set of true numbers was taken into account for the establishment of a function associated with NPV. For each, determines a point in $\mathbb{R}^3$.

A contour map representing sets for several values of f was designed to determine the contour levels of the function represented by all n and k values.

Software Mathematica 5.2 (Wolfram Research Inc) was used to prepare surfaces and analytic calculations, following [3].

3. Results and Discussion

When the Net Present Value (NPV) is dependent only on the discount rate k and on project duration time n, the function of two variables is defined, $NPV = V(n,k)$. For (3) was have:

$$V(n,k) = NPV = \left[\sum_{t=1}^{n} \frac{FC_t}{(1+k)^t}\right] - \left[I_0 + \sum_{t=1}^{n} \frac{I_t}{(1+k)^t}\right] \tag{3}$$

$$V(n,k) = NPV = \sum_{t=1}^{n} \frac{FC_t - I_t}{(1+k)^t} - I_0 \tag{4}$$

In which, in the case of simulations, I_0 is constant and also FC_t and I_t are constant for all t, with $1 \leqslant t \leqslant n$. The function may be analyzed when there is a fixed cash flow and investments and variations in time and discount rate of the project are simulated.

In current simulation, rates of cash flow of each identical period and equal to FC_t = US\$ 200.00, initial investment I_0 =US\$ 1,000.00 and investments for each identical period and equal to I_t = US\$ 50.00 are employed. Further, time and discount rate (n,k) go through the set $[0,24] \times [0,1] \subset \mathbb{R}^2$, representing maximum time of 2 years (24 months) at a rate between 0 and 100%. (Figure 1).

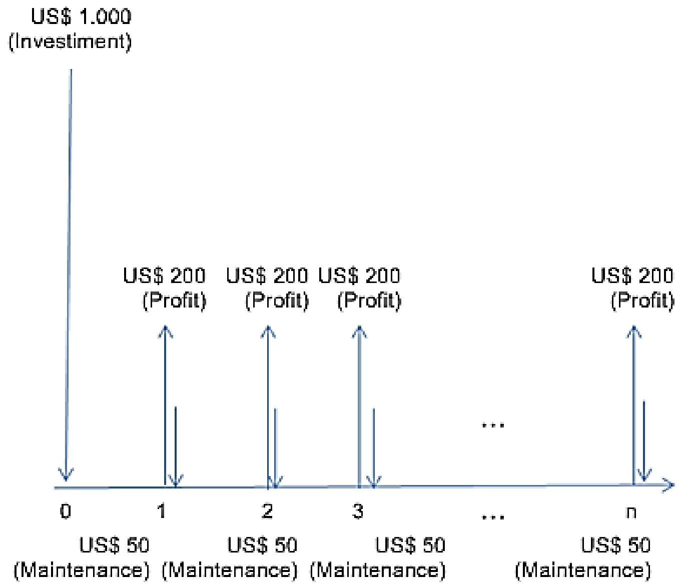

FIGURE 1. Diagram of cash flow.

The situation is commonly seen when an entrepreneur buys an equipment at I_0 = US$ 1,000.00 and invests I_F = US$ 50.00 for its maintenance and guarantee of continuous running. Investment will give a monthly profit return of US$ 200.00.

The entrepreneur tries all possible financial means to pay off the equipment with the above profits in a 24-month period. As previously informed, the rate is linked to the profit of the equipment by its use. If profit is low, the rate is low.

As previously explained, the rate is linked to the profit the equipment will return during its use. If profit is very low, the rate will be low too.

It is thus possible to give the graph of $V(n,k)$ comprising all the possible Net Present Values, with variations in the discount rate (k) and time (n), as below:

$$V(n,k) = \left[\sum_{t=1}^{n} \frac{200}{(1+k)^t}\right] - \left[1000 + \sum_{t=1}^{n} \frac{50}{(1+k)^t}\right] \tag{5}$$

$$V(n,k) = \sum_{t=1}^{n} \frac{200-50}{(1+k)^t} - 1000 \Rightarrow \left[V(n,k) \sum_{t=1}^{n} \frac{150}{(1+k)^t} - 1000\right] \tag{6}$$

The sum of $V(n,k)$ may be rewritten as a geometric progression, as bellow:

$$S_n = a_1 \frac{q^n - 1}{q - 1} = (1+k)^{-1} \frac{(1+k)^{-n} - 1}{(1+k)^{-1} - 1} \tag{7}$$

The sum is thus represented by the following expression:

$$V(n,k) = -1000 + \sum_{t=1}^{n} \frac{150}{(1+k)^t} = -1000 + 150\left(\frac{1-(1+k)^{-n}}{k}\right) \qquad (8)$$

The expression is thus expressed by the software Mathematica:

Plot3D[-1000+150(1+k)^(-1)((1+k)^(-n)-1)/((1+k)^(-1)-1),{n,0.00001,24},{k,0,1}, AxesLabel->{"n","k","NPV"}]

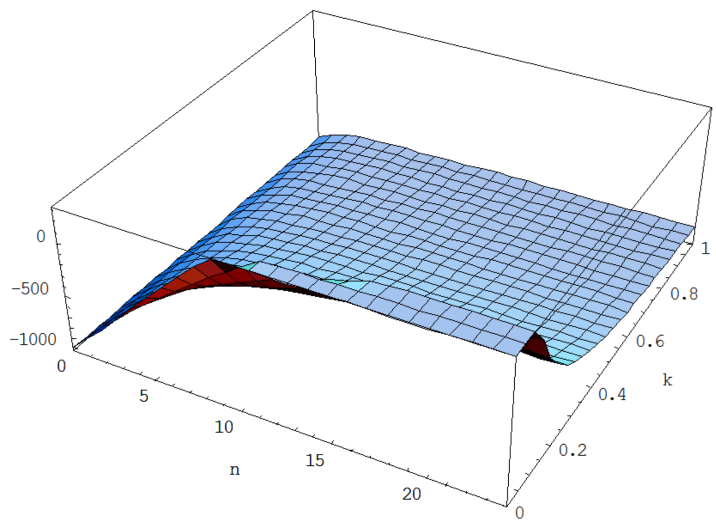

FIGURE 2. Simulation of NPV for $0 \leq n \leq 24$ and $0 \leq k \leq 1$.

Figure 2 illustrates all combinations of all financial situations up to 24 months at discount rate of [0,100%]. There are regions in which NPV is positive ($NPV > 0$).

Graph contour map of the function $V(n,k)$, illustrated in Figure 3, reveals the behavior of NPV according to n and k. Thus, a bi-dimensional curve may be observed (which defines a curve in $\mathbb{R}^2$) for each fixed rate of NPV, in which k varies due to n.

Graph contour curves of the function $V(n,k)$ forms the contour map 2 of the same shape by curves so that each point in the curve has the same NPV obtained by software Mathematica:

ContourPlot[-1000+150(1+k)^(-1)((1+k)^(-n)-1)/((1+k)^(-1)-1),{n,0.00001,24},{k,0,1}] {GrayLevel[1-#]&,10,"600","-1000", LegendPosition -> {1.1, -.4}}]

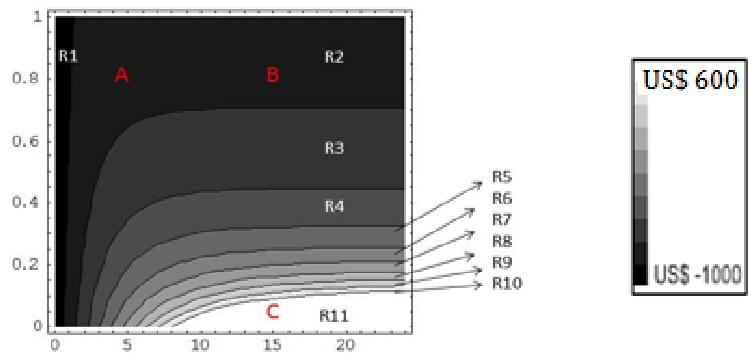

FIGURE 3. Contour map of net present value for $0 \leq n \leq 24$ and $0 \leq k \leq 1$.

According to the contour map 3 from the graph $V(n, k)$, the behavior of NPV characteristics marked in eleven region may be perceived. Values range between US\$ -1,000 and US\$ 600, directly affected by return time for investment and the attractiveness rate of the project. The darker the region in which NPV lies, the lower will be the yield of the investment project; on the other hand, the brighter the region, the higher will be the return.

It should be underscored that the project becomes attractive when the time required for its return is over 7 months and the rate lower than 14%, shown in the bright section of the graph.

Let us take as examples the points A, B and C within the regions R2 and R11, in which A and B lie in the same region (R2) with similar rates but at different periods; point C has a predicted investment return equal to B (15 months), but different rates.

Note that the point A, was obtained from the function $V(n, k) = V(5, 0.8)$ that has value $NPV =$ US\$ -822.42, the value B of the function $V(15, 0.8)$ has a value of NPV in US\$ -812.53 and the point value C contained in the R11 region has its NPV value determined $V(15, 0.05) =$ US\$ 556.95.

In the wake of the analysis of all combinations between the investment rate and return time, the administrator will make the best decision on the return rate of the highest profit according to the time determined by him for the project.

For level $V(k, n) = 0$, or rather, the level in which NPV is nil, then

$$NPV = 0 \Rightarrow V(n, k) = 0 \tag{9}$$

$$\left[\sum_{t=1}^{n} \frac{FC_t}{(1+k)^t}\right] - \left[I_0 + \sum_{t=1}^{n} \frac{I_t}{(1+k)^t}\right] = 0 \tag{10}$$

$$\sum_{t=1}^{n} \frac{FC_t}{(1+k)^t} = I_0 + \sum_{t=1}^{n} \frac{I_t}{(1+k)^t} \tag{11}$$

The expression characterizes the curve $NPV = 0$ which returns several values of k due to n. Values of k define IRR and thus, in this case, $k = IRR$.

Since in this case, k depends on n, implicitly the curve $IRR = T(k)$ by

$$I_0 + \sum_{t=1}^{n} \frac{I_t}{(1+T)^t} = + \sum_{t=1}^{n} \frac{FC_t}{(1+T)^t} \quad or \quad \sum_{t=1}^{n} \frac{FC_t - I_t}{(1+T)^t} = I_0 \qquad (12)$$

where $T = T(k)$.

Expression (12) may be written:

$\frac{FC_1-I_1}{1+T} + \frac{FC_2-I_2}{(1+T)^2} + \frac{FC_3-I_3}{(1+T)^3} + ... + \frac{FC_n-I_n}{(1+T)^n} = I_0 = (FC_1 - I_1).(1+T)^{n-1} + (FC_2 - I_2).(1+T)^{n-2} + ... + (FC_n - I_n).(1+T)^n = I_0.(1+T)^n = I.(1+T)^n + (I_1 - FC_1).(1+T)^{n-1} + (I_2 - FC_2).(1+T)^{n-2} + ... + I_0.(1+T)^n = 0$

The expression characterizes a polynomial equation grade n, representing in the context of current assay an implicit function with independent variable n and dependent variable T. The function may be represented by $T = [1,n] \rightarrow R, T = T(n)$. Finally, for simplification and making $T = [1,n] \rightarrow R, T = T(n)$ and $C_0 = I_0$ and $C_0 = I_0$ and writing $\bar{T} = 1 + T$, was have (9) the function $T(n)$ or $T(n) = 1 + T_n$ also given by the polynomial equation of grade n : $c_0.\bar{T}^n + c_1.\bar{T}^{n-1} + c_2.\bar{T}^{n-2} + ... + c_n$

4. Conclusions

The geometrical analysis of the characteristics from NPV and IRR showed that, besides the analytic dependence between these quantities, geometric relations are relevant in studies related to zero level NPV.

Geometric dependence between the contour level and level zero of NPV surface highly contributes for a wider idea in light of the administrator in the analysis of analytic variables that affect the firm's financial equilibrium.

Contour map established in current assay makes possible the analysis of events close to IRR exact value which only occur when $NPV = 0$.

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