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THE EXPRESSIONS OF SOLUTIONS AND THE PERIODICITY OF SOME RATIONAL DIFFERENCE EQUATIONS SYSTEMS†

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Abstract. In this paper we deal with the expressions of solutions and the periodicity nature of some systems of nonlinear difference equations with order three with nonzero real numbers initial conditions.

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1. Introduction

In this paper, we investigate the periodic character and the form of the solutions of some rational difference equations systems of order three

$$x_{n+1}=\frac{x_{n-1}y_{n-2}}{y_n\left(-1\pm x_{n-1}y_{n-2}\right)},\ y_{n+1}=\frac{y_{n-1}x_{n-2}}{x_n\left(\pm 1\pm y_{n-1}x_{n-2}\right)},$$

with initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}\ and\ y_0$ are nonzero real numbers.

In recent years, rational difference equations have attracted the attention of many researchers for varied reasons. On the one hand, they provide examples of nonlinear equations which are, in some cases, treatable but whose dynamics present some new features with respect to the linear case. On the other hand, rational equations frequently appear in some biological models, and, hence, their study is of interest also due to their applications. The periodicity of the positive solutions of the rational difference equations systems

$$x_{n+1}=\tfrac{1}{y_n},\ \ y_{n+1}=\tfrac{y_n}{x_{n-1}y_{n-1}},$$

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has been obtained by Cinar [4].
Elabbasy et al. [7] has studied the solutions of particular cases of the following general system of difference equations

$$x_{n+1} = \frac{a_1+a_2y_n}{a_3z_n+a_4x_{n-1}z_n},\ y_{n+1} = \frac{b_1z_{n-1}+b_2z_n}{b_3x_ny_n+b_4x_ny_{n-1}},\ z_{n+1} = \frac{c_1z_{n-1}+c_2z_n}{c_3x_{n-1}y_{n-1}+c_4x_{n-1}y_n+c_5x_ny_n}.$$

The behavior of positive solutions of the following system

$$x_{n+1} = \frac{x_{n-1}}{1+x_{n-1}y_n},\ y_{n+1} = \frac{y_{n-1}}{1+y_{n-1}x_n}.$$

has been studied by Kurbanli et al. [24].

Kurbanli [25] investigated the behavior of the solution of the difference equation system

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_n-1},\ y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n-1},\quad z_{n+1} = \frac{1}{z_ny_n}.$$

Özban [26] has investigated the positive solution of the system of rational difference equations

$$x_{n+1} = \frac{a}{y_{n-3}},\ y_{n+1} = \frac{by_{n-3}}{x_{n-q}y_{n-q}}.$$

Also, Touafek et al. [28] studied the periodicity and gave the form of the solutions of the following systems

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm1\pm y_n)},\ y_{n+1} = \frac{x_n}{y_{n-1}(\pm1\pm x_n)}.$$

In [29] Yalçınkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$z_{n+1} = \frac{t_nz_{n-1}+a}{t_n+z_{n-1}},\ t_{n+1} = \frac{z_nt_{n-1}+a}{z_n+t_{n-1}},$$

In [35] Zhang et al. studied the boundedness, the persistence and global asymptotic stability of the positive solutions of the system of difference equations

$$x_{n+} = A + \frac{y_{n-m}}{x_n},\qquad y_{n+1} = A + \frac{x_{n-m}}{y_n}.$$

Similar to difference equations and nonlinear systems of rational difference equations were investigated, see [1]-[38].

Definition (Periodicity).

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2. The First System : $x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n(-1+x_{n-1}y_{n-2})},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n(1+y_{n-1}x_{n-2})}$

In this section, we get the form of the solutions of the system of the difference equations

$$x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n(-1+x_{n-1}y_{n-2})},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n(1+y_{n-1}x_{n-2})}, \tag{1}$$

where $n = 0, 1, 2, ...$ and the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are arbitrary nonzero real numbers with $x_0y_{-1} \neq 1,\ x_{-1}y_{-2} \neq 1$.

Theorem 1. *If $\{x_n, y_n\}$ are solutions of difference equation system (1). Then for $n = 0, 1, 2, \ldots,$*

$$x_{4n-2} = \frac{a^{2n}\prod_{i=0}^{n-1}(1+2ice)(1+(2i+1)ce)}{c^{2n-1}(-1+ae)^n}, \quad x_{4n-1} = \frac{bf^{2n}\prod_{i=0}^{n-1}(1+2ibd)(1+(2i+1)bd)}{d^{2n}(-1+bf)^n},$$

$$x_{4n} = \frac{a^{2n+1}\prod_{i=0}^{n-1}(1+(2i+1)ce)(1+(2i+2)ce)}{c^{2n}(-1+ae)^n}, \quad x_{4n+1} = \frac{bf^{2n+1}\prod_{i=0}^{n-1}(1+(2i+1)bd)(1+(2i+2)bd)}{d^{2n+1}(-1+bf)^{n+1}},$$

$$y_{4n-2} = \frac{d^{2n}(-1+bf)^n}{f^{2n-1}\prod_{i=0}^{n-1}(1+2ibd)(1+(2i+1)bd)}, \quad y_{4n-1} = \frac{ec^{2n}(-1+ae)^n}{a^{2n}\prod_{i=0}^{n-1}(1+(2i+1)ce)(1+(2i+2)ce)},$$

$$y_{4n} = \frac{d^{2n+1}(-1+bf)^n}{f^{2n}\prod_{i=0}^{n-1}(1+(2i+1)bd)(1+(2i+2)bd)}, \quad y_{4n+1} = \frac{ec^{2n+1}(-1+ae)^n}{a^{2n+1}(1+ce)\prod_{i=0}^{n-1}(1+(2i+2)ce)(1+(2i+3)ce)},$$

where $x_{-2} = c,\ x_{-1} = b,\ x_0 = a,\ y_{-2} = f,\ y_{-1} = e$ and $y_0 = d$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n-1$. that is,

$$x_{4n-6} = \frac{a^{2n-2}\prod_{i=0}^{n-2}(1+2ice)(1+(2i+1)ce)}{c^{2n-3}(-1+ae)^n}, \qquad x_{4n-5} = \frac{bf^{2n-2}\prod_{i=0}^{n-2}(1+2ibd)(1+(2i+1)bd)}{d^{2n-2}(-1+bf)^{n-1}},$$

$$x_{4n-4} = \frac{a^{2n-1}\prod_{i=0}^{n-2}(1+(2i+1)ce)(1+(2i+2)ce)}{c^{2n-2}(-1+ae)^{n-1}}, \qquad x_{4n-3} = \frac{bf^{2n-1}\prod_{i=0}^{n-2}(1+(2i+1)bd)(1+(2i+2)bd)}{d^{2n-1}(-1+bf)^n},$$

$$y_{4n-6} = \frac{d^{2n-2}(-1+bf)^{n-1}}{f^{2n-3}\prod_{i=0}^{n-2}(1+2ibd)(1+(2i+1)bd)}, \qquad y_{4n-5} = \frac{ec^{2n-2}(-1+ae)^{n-1}}{a^{2n-2}\prod_{i=0}^{n-2}(1+(2i+1)ce)(1+(2i+2)ce)},$$

$$y_{4n-4} = \frac{d^{2n-1}(-1+bf)^{n-1}}{f^{2n-2}\prod_{i=0}^{n-2}(1+(2i+1)bd)(1+(2i+2)bd)}, \qquad y_{4n-3} = \frac{ec^{2n-1}(-1+ae)^{n-1}}{a^{2n-1}(1+ce)\prod_{i=0}^{n-2}(1+(2i+2)ce)(1+(2i+3)ce)}.$$

Now we obtain from Eq.(1) that

$$\begin{aligned} x_{4n-2} &= \frac{x_{4n-4}y_{4n-5}}{y_{4n-3}(-1+x_{4n-4}y_{4n-5})} \\ &= \frac{ae}{\left(\frac{ec^{2n-1}(-1+ae)^{n-1}}{a^{2n-1}(1+ce)\prod_{i=0}^{n-2}(1+(2i+2)ce)(1+(2i+3)ce)}\right)(-1+ae)} \\ &= \frac{aea^{2n-1}(1+ce)\prod_{i=0}^{n-2}(1+(2i+2)ce)(1+(2i+3)ce)}{ec^{2n-1}(-1+ae)^{n-1}(-1+ae)} = \frac{a^{2n}\prod_{i=0}^{n-1}(1+(2i)ce)(1+(2i+1)ce)}{c^{2n-1}(-1+ae)^n}, \end{aligned}$$

$$y_{4n-2} = \frac{y_{4n-4}x_{4n-5}}{x_{4n-3}(1+y_{4n-4}x_{4n-5})} = \frac{\frac{bd}{\prod_{i=0}^{n-2}(1+(2i+2)bd)}\prod_{i=0}^{n-2}(1+(2i)bd)}{\frac{bf^{2n-1}\prod_{i=0}^{n-2}(1+(2i+1)bd)(1+(2i+2)bd)}{d^{2n-1}(-1+bf)^n}\left(1+\frac{bd\left(\prod_{i=0}^{n-2}(1+(2i)bd)\right)}{\prod_{i=0}^{n-2}(1+(2i+2)bd)}\right)}$$

$$= \frac{\left(\frac{bd}{(1+(2n-2)bd)}\right)}{\left(\frac{bf^{2n-1}\prod_{i=0}^{n-2}(1+(2i+1)bd)(1+(2i+2)bd)}{d^{2n-1}(-1+bf)^n}\right)\left(1+\frac{bd}{(1+(2n-2)bd)}\right)}$$

$$= \frac{d^{2n}(-1+bf)^n}{\left(f^{2n-1}\prod_{i=0}^{n-2}(1+(2i+1)bd)(1+(2i+2)bd)\right)(1+(2n-1)bd)}$$

$$= \frac{d^{2n}(-1+bf)^n}{f^{2n-1}\prod_{i=0}^{n-1}(1+(2i+1)bd)(1+(2i)bd)}.$$

Also, we see from Eq.(1) that

$$x_{4n-1} = \frac{x_{4n-3}y_{4n-4}}{y_{4n-2}(-1+x_{4n-3}y_{4n-4})} = \frac{\left(\frac{bf}{-1+bf}\right)}{\left(\frac{d^{2n}(-1+bf)^n}{f^{2n-1}\prod_{i=0}^{n-1}(1+(2i)bd)(1+(2i+1)bd)}\right)\left(-1+\frac{bf}{-1+bf}\right)}$$

$$= \frac{bf^{2n}\prod_{i=0}^{n-1}(1+(2i)bd)(1+(2i+1)bd)}{d^{2n}(-1+bf)^n},$$

and

$$y_{4n-1} = \frac{y_{4n-3}x_{4n-4}}{x_{4n-2}\left(1+y_{4n-3}x_{4n-4}\right)}$$

$$= \frac{\frac{ec}{(1+ce)\prod_{i=0}^{n-2}(1+(2i+3)ce)}\prod_{i=0}^{n-2}(1+(2i+1)ce)}{\frac{a^{2n}\prod_{i=0}^{n-1}(1+(2i)ce)(1+(2i+1)ce)}{c^{2n-1}(-1+ae)^n}\left(1+\frac{ec\left(\prod_{i=0}^{n-2}(1+(2i+1)ce)\right)}{(1+ce)\prod_{i=0}^{n-2}(1+(2i+3)ce)}\right)}$$

$$= \frac{c^{2n-1}(-1+ae)^n ec}{\left(a^{2n}\prod_{i=0}^{n-1}(1+(2i)ce)(1+(2i+1)ce)\right)(1+(2n-1)ce+ce)}$$

$$= \frac{ec^{2n}(-1+ae)^n}{\left(a^{2n}\prod_{i=0}^{n-1}(1+(2i)ce)(1+(2i+1)ce)\right)(1+(2n)ce)} = \frac{ec^{2n}(-1+ae)^n}{a^{2n}\prod_{i=0}^{n-1}(1+(2i+1)ce)(1+(2i+2)ce)}.$$

Also, we can prove the other relations. This completes the proof. □

3. The Second System : $x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n(-1+x_{n-1}y_{n-2})}$, $y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n(-1+y_{n-1}x_{n-2})}$

In this section, we get the solutions of the system of the difference equations

$$x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n(-1+x_{n-1}y_{n-2})},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n(-1+y_{n-1}x_{n-2})}, \tag{2}$$

where $n = 0, 1, 2, ...$ and the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are arbitrary nonzero real numbers with $x_{-2}y_{-1},\ x_0y_{-1},\ x_{-1}y_0,\ x_{-1}y_{-2} \neq 1$.

Theorem 2. *If* $\{x_n, y_n\}$ *are solutions of difference equation system (2). Then for* $n = 0, 1, 2, ...,$

$$x_{4n-2} = \frac{a^{2n}(-1+ce)^n}{c^{2n-1}(-1+ae)^n}, \qquad x_{4n-1} = \frac{bf^{2n}(-1+bd)^n}{d^{2n}(-1+bf)^n},$$
$$x_{4n} = \frac{a^{2n+1}(-1+ce)^n}{c^{2n}(-1+ae)^n}, \qquad x_{4n+1} = \frac{bf^{2n+1}(-1+bd)^n}{d^{2n+1}(-1+bf)^{n+1}},$$
$$y_{4n-2} = \frac{d^{2n}(-1+bf)^n}{f^{2n-1}(-1+bd)^n}, \qquad y_{4n-1} = \frac{ec^{2n}(-1+ae)^n}{a^{2n}(-1+ce)^n},$$
$$y_{4n} = \frac{d^{2n+1}(-1+bf)^n}{f^{2n}(-1+bd)^n}, \qquad y_{4n+1} = \frac{ec^{2n+1}(-1+ae)^n}{a^{2n+1}(-1+ce)^{n+1}}.$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. that is,

$$x_{4n-6} = \frac{a^{2n-2}(-1+ce)^{n-1}}{c^{2n-3}(-1+ae)^{n-1}}, \qquad x_{4n-5} = \frac{bf^{2n-2}(-1+bd)^{n-1}}{d^{2n-2}(-1+bf)^{n-1}},$$
$$x_{4n-4} = \frac{a^{2n-1}(-1+ce)^{n-1}}{c^{2n-2}(-1+ae)^{n-1}}, \qquad x_{4n-3} = \frac{bf^{2n-1}(-1+bd)^{n-1}}{d^{2n-1}(-1+bf)^{n}},$$
$$y_{4n-6} = \frac{d^{2n-2}(-1+bf)^{n-1}}{f^{2n-3}(-1+bd)^{n-1}}, \qquad y_{4n-5} = \frac{ec^{2n-2}(-1+ae)^{n-1}}{a^{2n-2}(-1+ce)^{n-1}},$$
$$y_{4n-4} = \frac{d^{2n-1}(-1+bf)^{n-1}}{f^{2n-2}(-1+bd)^{n-1}}, \qquad y_{4n-3} = \frac{ec^{2n-1}(-1+ae)^{n-1}}{a^{2n-1}(-1+ce)^{n}}.$$

Now it follows from Eq.(2) that

$$\begin{aligned} x_{4n-2} &= \frac{x_{4n-4}y_{4n-5}}{y_{4n-3}(-1+x_{4n-4}y_{4n-5})} \\ &= \frac{\left(\frac{a^{2n-1}(-1+ce)^{n-1}}{c^{2n-2}(-1+ae)^{n-1}}\right)\left(\frac{ec^{2n-2}(-1+ae)^{n-1}}{a^{2n-2}(-1+ce)^{n-1}}\right)}{\left(\frac{ec^{2n-1}(-1+ae)^{n-1}}{a^{2n-1}(-1+ce)^{n}}\right)\left(-1+\left(\frac{a^{2n-1}(-1+ce)^{n-1}}{c^{2n-2}(-1+ae)^{n-1}}\right)\left(\frac{ec^{2n-2}(-1+ae)^{n-1}}{a^{2n-2}(-1+ce)^{n-1}}\right)\right)} \\ &= \frac{aea^{2n-1}(-1+ce)^n}{(ec^{2n-1}(-1+ae)^{n-1})(-1+ae)} = \frac{a^{2n}(-1+ce)^n}{c^{2n-1}(-1+ae)^n}, \end{aligned}$$

$$\begin{aligned} y_{4n-2} &= \frac{y_{4n-4}x_{4n-5}}{x_{4n-3}(-1+y_{4n-4}x_{4n-5})} \\ &= \frac{\left(\frac{d^{2n-1}(-1+bf)^{n-1}}{f^{2n-2}(-1+bd)^{n-1}}\right)\left(\frac{bf^{2n-2}(-1+bd)^{n-1}}{d^{2n-2}(-1+bf)^{n-1}}\right)}{\left(\frac{bf^{2n-1}(-1+bd)^{n-1}}{d^{2n-1}(-1+bf)^{n}}\right)\left(-1+\left(\frac{d^{2n-1}(-1+bf)^{n-1}}{f^{2n-2}(-1+bd)^{n-1}}\right)\left(\frac{bf^{2n-2}(-1+bd)^{n-1}}{d^{2n-2}(-1+bf)^{n-1}}\right)\right)} \\ &= \frac{bd}{\left(\frac{bf^{2n-1}(-1+bd)^{n-1}}{d^{2n-1}(-1+bf)^{n}}\right)(-1+bd)} = \frac{d^{2n}(-1+bf)^n}{f^{2n-1}(-1+bd)^n}. \end{aligned}$$

Also, we see from Eq.(2) that

$$\begin{aligned} x_{4n-1} &= \frac{x_{4n-3}y_{4n-4}}{y_{4n-2}(-1+x_{4n-3}y_{4n-4})} \\ &= \frac{\left(\frac{bf^{2n-1}(-1+bd)^{n-1}}{d^{2n-1}(-1+bf)^{n}}\right)\left(\frac{d^{2n-1}(-1+bf)^{n-1}}{f^{2n-2}(-1+bd)^{n-1}}\right)}{\left(\frac{d^{2n}(-1+bf)^{n}}{f^{2n-1}(-1+bd)^{n}}\right)\left(-1+\left(\frac{bf^{2n-1}(-1+bd)^{n-1}}{d^{2n-1}(-1+bf)^{n}}\right)\left(\frac{d^{2n-1}(-1+bf)^{n-1}}{f^{2n-2}(-1+bd)^{n-1}}\right)\right)} \end{aligned}$$

$$= \frac{\left(\frac{bf}{-1+bf}\right)}{\left(\frac{d^{2n}(-1+bf)^{n}}{f^{2n-1}(-1+bd)^{n}}\right)\left(-1+\frac{bf}{-1+bf}\right)} = \frac{bf^{2n}(-1+bd)^{n}}{d^{2n}(-1+bf)^{n}},$$

$$y_{4n-1} = \frac{y_{4n-3}x_{4n-4}}{x_{4n-2}(-1+y_{4n-3}x_{4n-4})}$$

$$= \frac{\left(\frac{ec^{2n-1}(-1+ae)^{n-1}}{a^{2n-1}(-1+ce)^{n}}\right)\left(\frac{a^{2n-1}(-1+ce)^{n-1}}{c^{2n-2}(-1+ae)^{n-1}}\right)}{\left(\frac{a^{2n}(-1+ce)^{n}}{c^{2n-1}(-1+ae)^{n}}\right)\left(-1+\left(\frac{ec^{2n-1}(-1+ae)^{n-1}}{a^{2n-1}(-1+ce)^{n}}\right)\left(\frac{a^{2n-1}(-1+ce)^{n-1}}{c^{2n-2}(-1+ae)^{n-1}}\right)\right)}$$

$$= \frac{\left(\frac{ec}{-1+ce}\right)}{\left(\frac{a^{2n}(-1+ce)^{n}}{c^{2n-1}(-1+ae)^{n}}\right)\left(-1+\left(\frac{ec}{-1+ce}\right)\right)} = \frac{c^{2n-1}(-1+ae)^{n}(ec)}{a^{2n}(-1+ce)^{n}(1-ce+ce)} = \frac{ec^{2n}(-1+ae)^{n}}{a^{2n}(-1+ce)^{n}}.$$

By the same way we can prove the other relations. The proof is complete. □

Lemma 1. *The solution of system (2) is unbounded except in the following case.*

Theorem 3. *System (2) has a periodic solution of period four iff* $d = f,\ a = c$ *and it will be taken the following form* $\{x_n\} = \left\{c, b, a, \frac{b}{-1+bf}, c, ...\right\}$, $\{y_n\} = \left\{f, e, d, \frac{e}{-1+ce}, f, e, ...\right\}$.

Proof. First suppose that there exists a prime period four solution

$$\{x_n\} = \left\{c, b, a, \frac{b}{-1+bf}, c, b, a, ...\right\},\ \{y_n\} = \left\{f, e, d, \frac{e}{-1+ce}, f, e, d, ...\right\},$$

of system (2), we see from the form of the solution of system (2) that

$$\begin{aligned}
x_{4n-2} &= c = \frac{a^{2n}(-1+ce)^{n}}{c^{2n-1}(-1+ae)^{n}}, & x_{4n-1} &= b = \frac{bf^{2n}(-1+bd)^{n}}{d^{2n}(-1+bf)^{n}},\\
x_{4n} &= a = \frac{a^{2n+1}(-1+ce)^{n}}{c^{2n}(-1+ae)^{n}}, & x_{4n+1} &= \frac{b}{(-1+bf)} = \frac{bf^{2n+1}(-1+bd)^{n}}{d^{2n+1}(-1+bf)^{n+1}},\\
y_{4n-2} &= f = \frac{d^{2n}(-1+bf)^{n}}{f^{2n-1}(-1+bd)^{n}}, & y_{4n-1} &= e = \frac{ec^{2n}(-1+ae)^{n}}{a^{2n}(-1+ce)^{n}},\\
y_{4n} &= d = \frac{d^{2n+1}(-1+bf)^{n}}{f^{2n}(-1+bd)^{n}}, & y_{4n+1} &= \frac{e}{-1+ce} = \frac{ec^{2n+1}(-1+ae)^{n}}{a^{2n+1}(-1+ce)^{n+1}}.
\end{aligned}$$

Then we get $d = f,\ a = c$. Second assume that $d = f,\ a = c$. Then we see from the form of the solution of system (1) that

$$\begin{aligned}
x_{4n-2} &= \frac{a^{2n}(-1+ae)^{n}}{c^{2n-1}(-1+ae)^{n}} = c, & x_{4n-1} &= \frac{bf^{2n}(-1+bd)^{n}}{d^{2n}(-1+bd)^{n}} = b,\\
x_{4n} &= \frac{a^{2n+1}(-1+ae)^{n}}{c^{2n}(-1+ae)^{n}} = a, & x_{4n+1} &= \frac{bf^{2n+1}(-1+bd)^{n}}{d^{2n+1}(-1+bd)^{n+1}} = \frac{b}{(-1+bd)},\\
y_{4n-2} &= \frac{d^{2n}(-1+bf)^{n}}{f^{2n-1}(-1+bd)^{n}} = f, & y_{4n-1} &= \frac{ec^{2n}(-1+ae)^{n}}{a^{2n}(-1+ce)^{n}} = e,\\
y_{4n} &= \frac{d^{2n+1}(-1+bf)^{n}}{f^{2n}(-1+bd)^{n}} = d, & y_{4n+1} &= \frac{ec^{2n+1}(-1+ae)^{n}}{a^{2n+1}(-1+ce)^{n+1}} = \frac{e}{(-1+ce)}.
\end{aligned}$$

Thus we have a periodic solution of period four and the proof is complete. □

Lemma 2. *System (2) has a periodic solution of period two iff* $d = f,\ a = c,\ bd = ce = 2$ *and it will be taken the following form* $\{x_n\} = \{c, b, c, b, ...\}$, $\{y_n\} = \{f, e, f, e, ...\}$.

Proof. The proof is consequently from the previous theorem and so, will be omitted. □

4. Other Systems

Here we study some systems of difference equations and the proof of all the Theorems are similar to above systems and so, will be omitted and in all cases we suppose that $x_{-2} = c,\ x_{-1} = b,\ x_0 = a,\ y_{-2} = f,\ y_{-1} = e$ *and* $y_0 = d$.

$$x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n(-1+x_{n-1}y_{n-2})},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n(1-y_{n-1}x_{n-2})}. \tag{3}$$

$$x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n(-1-x_{n-1}y_{n-2})},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n(1+y_{n-1}x_{n-2})}. \tag{4}$$

$$x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n(-1-x_{n-1}y_{n-2})},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n(1-y_{n-1}x_{n-2})}. \tag{5}$$

$$x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n(-1+x_{n-1}y_{n-2})},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n(-1-y_{n-1}x_{n-2})}. \tag{6}$$

$$x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n(-1-x_{n-1}y_{n-2})},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n(-1+y_{n-1}x_{n-2})}. \tag{7}$$

$$x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n(-1-x_{n-1}y_{n-2})},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n(-1-y_{n-1}x_{n-2})}. \tag{8}$$

Theorem 4. *The solutions of the following system (3) with* $x_0y_{-1} \neq 1,\ x_{-1}y_{-2} \neq 1$ *are given by the following formula for* $n = 0, 1, 2, ...,$

$$x_{4n-2} = \frac{a^{2n}\prod_{i=0}^{n-1}(1-(2i)ce)(1-(2i+1)ce)}{c^{2n-1}(-1+ae)^n},\ x_{4n-1} = \frac{bf^{2n}\prod_{i=0}^{n-1}(1-(2i)bd)(1-(2i+1)bd)}{d^{2n}(-1+bf)^n},$$

$$x_{4n} = \frac{a^{2n+1}\prod_{i=0}^{n-1}(1-(2i+1)ce)(1-(2i+2)ce)}{c^{2n}(-1+ae)^n},\ x_{4n+1} = \frac{bf^{2n+1}\prod_{i=0}^{n-1}(1-(2i+1)bd)(1-(2i+2)bd)}{d^{2n+1}(-1+bf)^{n+1}},$$

$$y_{4n-2} = \frac{d^{2n}(-1+bf)^n}{f^{2n-1}\prod_{i=0}^{n-1}(1-(2i)bd)(1-(2i+1)bd)},\ y_{4n-1} = \frac{ec^{2n}(-1+ae)^n}{a^{2n}\prod_{i=0}^{n-1}(1-(2i+1)ce)(1-(2i+2)ce)},$$

$$y_{4n} = \frac{d^{2n+1}(-1+bf)^n}{f^{2n}\prod_{i=0}^{n-1}(1-(2i+1)bd)(1-(2i+2)bd)},\ y_{4n+1} = \frac{ec^{2n+1}(-1+ae)^n}{a^{2n+1}(1-ce)\prod_{i=0}^{n-1}(1-(2i+2)ce)(1-(2i+3)ce)}.$$

Theorem 5. *If* $\{x_n, y_n\}$ *are solutions of the difference equation system (4) where the initial conditions* $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 *are arbitrary nonzero real numbers with* $x_0y_{-1} \neq -1,\ x_{-1}y_{-2} \neq -1$. *Then for* $n = 0, 1, 2, ...,$

$$x_{4n-2} = \frac{a^{2n}\prod_{i=0}^{n-1}(1+(2i)ce)(1+(2i+1)ce)}{c^{2n-1}(-1-ae)^n},\ x_{4n-1} = \frac{bf^{2n}\prod_{i=0}^{n-1}(1+(2i)bd)(1+(2i+1)bd)}{d^{2n}(-1-bf)^n},$$

$$x_{4n} = \frac{a^{2n+1}\prod_{i=0}^{n-1}(1+(2i+1)ce)(1+(2i+2)ce)}{c^{2n}(-1-ae)^n},\ x_{4n+1} = \frac{bf^{2n+1}\prod_{i=0}^{n-1}(1+(2i+1)bd)(1+(2i+2)bd)}{d^{2n+1}(-1-bf)^{n+1}},$$

$$y_{4n-2} = \frac{d^{2n}(-1-bf)^n}{f^{2n-1}\prod_{i=0}^{n-1}(1+(2i)bd)(1+(2i+1)bd)},\ y_{4n-1} = \frac{ec^{2n}(-1-ae)^n}{a^{2n}\prod_{i=0}^{n-1}(1+(2i+1)ce)(1+(2i+2)ce)},$$

$$y_{4n}=\frac{d^{2n+1}(-1-bf)^{n}}{f^{2n}\prod_{i=0}^{n-1}(1+(2i+1)bd)(1+(2i+2)bd)},\ y_{4n+1}=\frac{ec^{2n+1}(-1-ae)^{n}}{a^{2n+1}(1+ce)\prod_{i=0}^{n-1}(1+(2i+2)ce)(1+(2i+3)ce)}.$$

Theorem 6. *If $\{x_n, y_n\}$ are solutions of the difference equations system (5) where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ and y_0 are arbitrary nonzero real numbers with $x_0y_{-1}\neq -1,\ x_{-1}y_{-2}\neq -1$. Then for $n=0,1,2,...,$*

$$x_{4n-2}=\frac{a^{2n}\prod_{i=0}^{n-1}(1-(2i)ce)(1-(2i+1)ce)}{c^{2n-1}(-1-ae)^{n}},\ x_{4n-1}=\frac{bf^{2n}\prod_{i=0}^{n-1}(1-(2i)bd)(1-(2i+1)bd)}{d^{2n}(-1-bf)^{n}},$$

$$x_{4n}=\frac{a^{2n+1}\prod_{i=0}^{n-1}(1-(2i+1)ce)(1-(2i+2)ce)}{c^{2n}(-1-ae)^{n}},\ x_{4n+1}=\frac{bf^{2n+1}\prod_{i=0}^{n-1}(1-(2i+1)bd)(1-(2i+2)bd)}{d^{2n+1}(-1-bf)^{n+1}},$$

$$y_{4n-2}=\frac{d^{2n}(-1-bf)^{n}}{f^{2n-1}\prod_{i=0}^{n-1}(1-(2i)bd)(1-(2i+1)bd)},\ y_{4n-1}=\frac{ec^{2n}(-1-ae)^{n}}{a^{2n}\prod_{i=0}^{n-1}(1-(2i+1)ce)(1-(2i+2)ce)},$$

$$y_{4n}=\frac{d^{2n+1}(-1-bf)^{n}}{f^{2n}\prod_{i=0}^{n-1}(1-(2i+1)bd)(1-(2i+2)bd)},\ y_{4n+1}=\frac{ec^{2n+1}(-1-ae)^{n}}{a^{2n+1}(1-ce)\prod_{i=0}^{n-1}(1-(2i+2)ce)(1-(2i+3)ce)}.$$

Theorem 7. *Assume that $\{x_n, y_n\}$ are solutions of the system (6) with the initial conditions are arbitrary nonzero real numbers with $x_{-2}y_{-1},\ x_{-1}y_0\neq -1$, $x_0y_{-1},\ x_{-1}y_{-2}\neq 1$. Then for $n=0,1,2,...,$*

$$x_{4n-2}=\frac{a^{2n}(-1-ce)^{n}}{c^{2n-1}(-1+ae)^{n}},\ x_{4n-1}=\frac{bf^{2n}(-1-bd)^{n}}{d^{2n}(-1+bf)^{n}},$$

$$x_{4n}=\frac{a^{2n+1}(-1-ce)^{n}}{c^{2n}(-1+ae)^{n}},\ x_{4n+1}=\frac{bf^{2n+1}(-1-bd)^{n}}{d^{2n+1}(-1+bf)^{n+1}},$$

$$y_{4n-2}=\frac{d^{2n}(-1+bf)^{n}}{f^{2n-1}(-1-bd)^{n}},\ y_{4n-1}=\frac{ec^{2n}(-1+ae)^{n}}{a^{2n}(-1-ce)^{n}},$$

$$y_{4n}=\frac{d^{2n+1}(-1+bf)^{n}}{f^{2n}(-1-bd)^{n}},\ y_{4n+1}=\frac{ec^{2n+1}(-1+ae)^{n}}{a^{2n+1}(-1-ce)^{n+1}}.$$

Lemma 3. *The solution of equation system (6) is unbounded except in the following case.*

Theorem 8. *System (6) has a periodic solution of period four iff $d=-f$, $a=-c$ and it will be taken the following form $\{x_n\}=\left\{c,b,a,\frac{b}{1-bf},c,b,a,...\right\}$, $\{y_n\}=\left\{f,e,d,\frac{e}{1+ce},f,e,d,...\right\}$.*

Theorem 9. *For $n=0,1,2,...,$ the solutions of system (7) with $x_{-2}y_{-1},\ x_{-1}y_0\neq 1,\ x_0y_{-1},\ x_{-1}y_{-2}\neq -1$ are given by the following relations*

$$x_{4n-2}=\frac{a^{2n}(-1+ce)^{n}}{c^{2n-1}(-1-ae)^{n}},\ x_{4n-1}=\frac{bf^{2n}(-1+bd)^{n}}{d^{2n}(-1-bf)^{n}},$$

$$x_{4n}=\frac{a^{2n+1}(-1+ce)^{n}}{c^{2n}(-1-ae)^{n}},\ x_{4n+1}=\frac{bf^{2n+1}(-1+bd)^{n}}{d^{2n+1}(-1-bf)^{n+1}},$$

$$y_{4n-2}=\frac{d^{2n}(-1-bf)^{n}}{f^{2n-1}(-1+bd)^{n}},\ y_{4n-1}=\frac{ec^{2n}(-1-ae)^{n}}{a^{2n}(-1+ce)^{n}},$$

$$y_{4n}=\frac{d^{2n+1}(-1-bf)^{n}}{f^{2n}(-1+bd)^{n}},\ y_{4n+1}=\frac{ec^{2n+1}(-1-ae)^{n}}{a^{2n+1}(-1+ce)^{n+1}}.$$

Lemma 4. *The solution of equation system (7) is unbounded except in the following case.*

Theorem 10. *System (7) has a periodic solution of period four iff* $d=-f$, $a=-c$ *and it will be taken the following form* $\{x_n\}=\left\{c,b,a,\frac{b}{1+bf},c,b,a,...\right\}$, $\{y_n\}=\left\{f,e,d,\frac{e}{1-ce},f,e,d,...\right\}$.

Theorem 11. *Suppose that the sequences* $\{x_n\}_{n=-2}^{\infty}$ *and* $\{y_n\}_{n=-2}^{\infty}$ *are solutions of system equations (8) with* $x_{-2}y_{-1},\ x_{-1}y_0,\ x_0y_{-1},\ x_{-1}y_{-2}\neq-1$, *then we obtain the following expressions of the solutions for* $n=0,1,2,...$,

$$\begin{aligned}
x_{4n-2} &= \frac{a^{2n}(-1-ce)^n}{c^{2n-1}(-1-ae)^n},\ x_{4n-1}=\frac{bf^{2n}(-1-bd)^n}{d^{2n}(-1-bf)^n},\\
x_{4n} &= \frac{a^{2n+1}(-1-ce)^n}{c^{2n}(-1-ae)^n},\ x_{4n+1}=\frac{bf^{2n+1}(-1-bd)^n}{d^{2n+1}(-1-bf)^{n+1}},\\
y_{4n-2} &= \frac{d^{2n}(-1-bf)^n}{f^{2n-1}(-1-bd)^n},\ y_{4n-1}=\frac{ec^{2n}(-1-ae)^n}{a^{2n}(-1-ce)^n},\\
y_{4n} &= \frac{d^{2n+1}(-1-bf)^n}{f^{2n}(-1-bd)^n},\ y_{4n+1}=\frac{ec^{2n+1}(-1-ae)^n}{a^{2n+1}(-1-ce)^{n+1}}.
\end{aligned}$$

Lemma 5. *The solution of equation system (8) is unbounded except in the following case.*

Theorem 12. *System (8) has a periodic solution of period four iff* $d=f$, $a=c$ *and it will be taken the following form* $\{x_n\}=\left\{c,b,a,\frac{b}{-1-bf},c,b,a,...\right\}$, $\{y_n\}=\left\{f,e,d,\frac{e}{-1-ce},f,e,d,...\right\}$.

Lemma 6. *System (8) has a periodic solution of period two iff* $d=f$, $a=c$, $bd=ce=-2$ *and it will be taken the following form* $\{x_n\}=\{c,b,c,b,...\}$, $\{y_n\}=\{f,e,f,e,...\}$.

5. Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to nonlinear difference equations.

Example 1. We consider numerical example for the difference system (1) with the initial conditions $x_{-2}=3.07,\ x_{-1}=0.13,\ x_0=0.4,\ y_{-2}=0.02,\ y_{-1}=0.7$ *and* $y_0=0.03$. (See Figure 1).

Example 2. We consider interesting example for the difference system (1) with the initial conditions $x_{-2}=0.07,\ x_{-1}=0.4,\ x_0=-0.04,\ y_{-2}=0.02,\ y_{-1}=-0.07$ *and* $y_0=0.03$. (See Figure 2).

Example 3. We consider numerical example for the difference system (2) with the initial conditions $x_{-2}=0.8,\ x_{-1}=0.4,\ x_0=0.9,\ y_{-2}=0.2,\ y_{-1}=0.7$ *and* $y_0=0.3$. (See Figure 3).

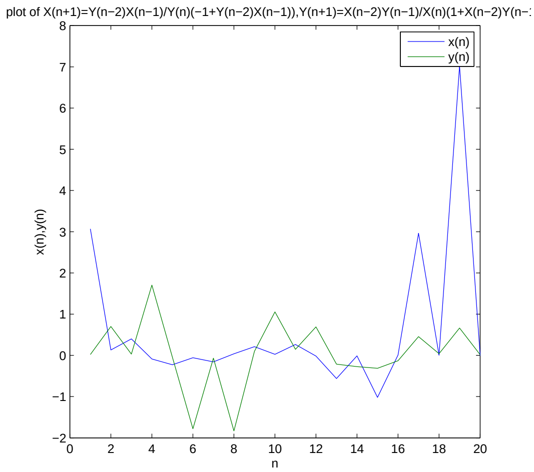

FIGURE 1

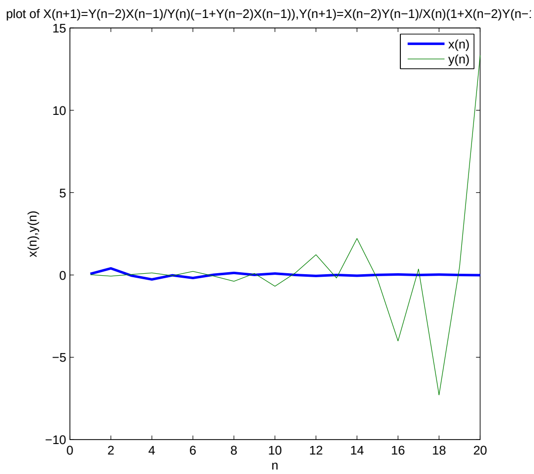

FIGURE 2

Example 4. See Figure (4) when we take system (2) with the initial conditions $x_{-2} = 9,\ x_{-1} = 0.5,\ x_0 = 7,\ y_{-2} = 8,\ y_{-1} = 2\ and\ y_0 = 4.$

Example 5. We assume the difference equations system (2)when we put the initial conditions $x_{-2} = 9,\ x_{-1} = 7,\ x_0 = 9,\ y_{-2} = 5,\ y_{-1} = 2\ and\ y_0 = 5$. See figure 5.

Example 6. Figure (6) shows the periodicity of the solution of the difference system (2) with the initial conditions $x_{-2} = -3,\ x_{-1} = 5,\ x_0 = -3,\ y_{-2} = 0.4,\ y_{-1} = -2/3\ and\ y_0 = 0.4.$

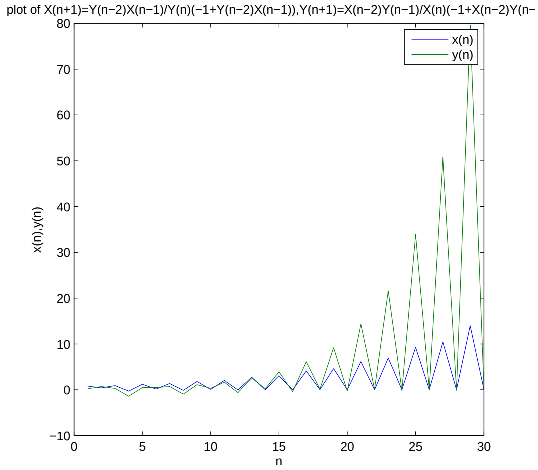

FIGURE 3

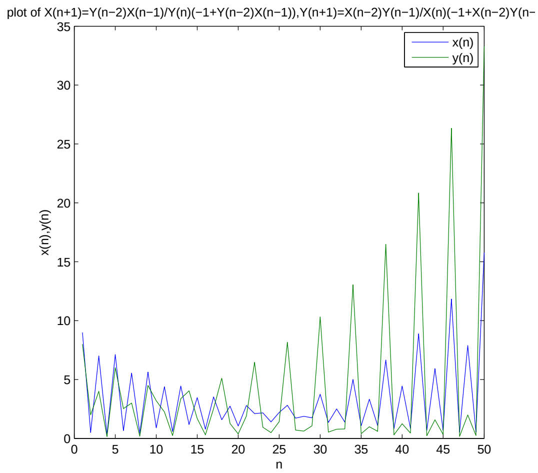

FIGURE 4

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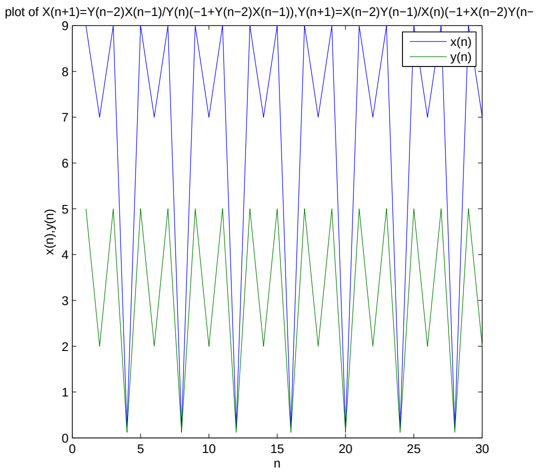

FIGURE 5

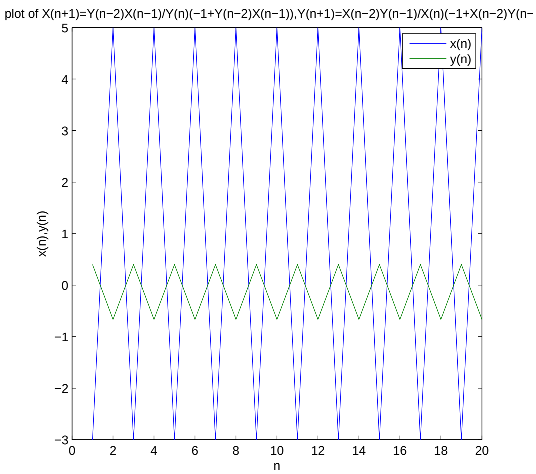

FIGURE 6

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