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A SPLIT LEAST-SQUARES CHARACTERISTIC MIXED FINITE ELEMENT METHOD FOR THE CONVECTION DOMINATED SOBOLEV EQUATIONS†

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Abstract. In this paper, we present a split least-squares characteristic mixed finite element method(MFEM) to get the approximate solutions of the convection dominated Sobolev equations. First, to manage both convection term and time derivative term efficiently, we apply a least-squares characteristic MFEM to get the system of equations in the primal unknown and the flux unknown. Then, we obtain a split least-squares characteristic MFEM to convert the coupled system in two unknowns derived from the least-squares characteristic MFEM into two uncoupled systems in the unknowns. We theoretically prove that the approximations constructed by the split least-squares characteristic MFEM converge with the optimal order in L^2 and H^1 normed spaces for the primal unknown and with the optimal order in L^2 normed space for the flux unknown. And we provide some numerical results to confirm the validity of our theoretical results.

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1. Introduction

In this paper we consider the following convection dominated Sobolev equation:

$$\begin{cases} c(\boldsymbol{x})u_t + \boldsymbol{d}(\boldsymbol{x}) \cdot \nabla u - \nabla \cdot (a(\boldsymbol{x})\nabla u_t + b(\boldsymbol{x})\nabla u) \\ \qquad\qquad\qquad = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T], \\ u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \partial\Omega \times (0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded convex domain in $\mathbb{R}^m$ with $1 \leq m \leq 3$ with boundary $\partial\Omega$, $c(\boldsymbol{x}), \boldsymbol{d}(\boldsymbol{x}), a(\boldsymbol{x}), b(\boldsymbol{x}), f(\boldsymbol{x}, t)$, and $u_0(\boldsymbol{x})$ are given functions. The Sobolev equation which represents the flow of fluids through fissured rock, the migration of the moisture in soil, the physical phenomena of thermodynamics and other applications as described in [2, 19, 20], is one of most principal partial differential equations. For the existence and uniqueness results of the solutions of the equation (1.1), refer to [8].

For the problems with no convection term, mixed finite element methods [11, 16, 18, 22], least-squares methods [12, 18, 21, 22], and discontinuous Galerkin methods [14, 15] were used for numerical treatments. In the case that a conventional (least-squares) MFEM is applied, we generally needs to solve the coupled system of equations in two unknowns, which brings to difficulties in some extent. So, in [18], a split least-squares mixed finite element method for reaction-diffusion problems was firstly introduced to solve the uncoupled systems of equations in the unknowns.

For the partial differential equations with a convection term, a characteristic (mixed) finite element method is one of the useful methods [1, 3, 4, 5, 6, 7, 10, 13] because it reflects well the physical character of a convection term and also it treats efficiently both convection term and time derivative term. Gao and Rui [9] introduced a split least-squares characteristic MFEM to approximate the primal unknown u and the flux unknown $-a\nabla u$ of the equation (1.1) and obtained the optimal convergence in $L^2(\Omega)$ norm for the primal unknown and in $H(div, \Omega)$ norm for the flux unknown. And Zhang and Guo [23] introduced a split least-squares characteristic mixed element method for nonlinear nonstationary convection-diffusion problem to approximate the primal unknown and the flux unknown and obtained the optimal convergence in $L^2(\Omega)$ norm for the primal unknown and in $H(div, \Omega)$ norm for the flux unknown.

In this paper, we apply a split least-squares characteristic characteristic mixed finite element method (MFEM) to achieve two uncoupled system of equations, one of which is for approximations to the primal unknown u and the other of which is for ones to the flux unknown $\boldsymbol{\sigma} = -(a(\boldsymbol{x})\nabla u_t + b(\boldsymbol{x})\nabla u)$ of the equation (1.1). And we analyze the optimal order of convergence in L^2 and H^1 normed spaces for the approximations. In section 2, we introduce necessary assumptions and notations, and in section 3, we construct finite element spaces on which we compose the approximations of two unknowns. In section 4, by adopting a split least-squares characteristic MFEM, we construct the approximations of the primal unknown and the unknown flux and establish the convergence of optimal order in L^2 and H^1 normed spaces for the primal unknown and the convergence of optimal order in L^2 normed space for the flux unknown. In section 5, we provide some numerical results to confirm the validity of the theoretical results obtained in section 4.

2. Assumption and notations

For an $s \geq 0$ and $1 \leq p \leq \infty$, we denoted by $W^{s,p}(\Omega)$ the Sobolev space endowed with the norm $\|\phi\|_{s,p}^p = \sum_{|\boldsymbol{k}|\leq s} \int_\Omega |D^{\boldsymbol{k}}\phi|^p dx$ where $\boldsymbol{k} = (k_1, k_2, \cdots, k_m)$, $|\boldsymbol{k}| = k_1 + k_2 + \cdots + k_m$, $D^{\boldsymbol{k}}\phi = \frac{\partial^{|\boldsymbol{k}|}\phi}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_m^{k_m}}$ and k_i is a nonnegative integer, for each i, $1 \leq i \leq m$. If $p = 2$, we simply denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $\|\phi\|_s = \|\phi\|_{s,2}$. And also in case that $s = 0$, we simply write $\|\phi\|$. We let $\boldsymbol{H}^s(\Omega) = \{\boldsymbol{u} = (u_1, u_2, \cdots, u_m) \mid u_i \in H^s(\Omega), 1 \leq i \leq m\}$ with the norm $\|\boldsymbol{u}\|_s^2 = \sum_{i=1}^m \|u_i\|_s^2$. Let $V = H_0^1(\Omega)$ and $\boldsymbol{W} = H(div, \Omega)$.

If $\phi(x,t)$ belongs to a Sobolev space equipped with a norm $\|\cdot\|_X$ for each t, then we let

$$\|\phi(x,t)\|_{L^p(0,t_0:X)}^p = \int_0^{t_0} \|\phi(x,t)\|_X^p dt, \text{ for } 1 \leq p < \infty,$$
$$\|\phi(x,t)\|_{L^\infty(0,t_0:X)} = \text{ess} \sup_{0\leq t\leq t_0} \|\phi(x,t)\|_X.$$

In case that $t_0 = T$, we denote $L^p(0, T : X)$ and $L^\infty(0, T : X)$ by $L^p(X)$ and $L^\infty(X)$, respectively. Let $H^{q,\infty}(X) = \{\phi(x,t) \mid \phi(x,t), \phi_t(x,t), \cdots, \phi_q(x,t) \in L^\infty(X)\}$ for a nonnegative integer q.

We consider the problem (1.1) with the coefficients satisfying the following assumption:

(A). There exist $c_*, c^*, d^*, a_*, a^*, b_*$, and b_* such that $0 < c_* < c(\boldsymbol{x}) \leq c^*, 0 < |d(\boldsymbol{x})| \leq d^*$, $0 < a_* < a(\boldsymbol{x}) \leq a^*$, and $0 < b_* < b(\boldsymbol{x}) \leq b^*$, for all $\boldsymbol{x} \in \Omega$, where $|d(\boldsymbol{x})| = \sum_{i=1}^m d_i^2(\boldsymbol{x})$.

3. Finite element spaces

Before preceding the numerical scheme, we let $\mathcal{E}_h = \{E_1, E_2, \cdots, E_{N_h}\}$ be a family of regular finite element subdivision of Ω. We let h denote the maximum of the diameters of the elements of $\mathcal{E}_h$. If $m = 2$, then E_i is a triangle or a quadrilateral, and if $m = 3$, then E_i is a 3-simplex or 3-rectangle. Boundary elements are allowed to have a curvilinear edge (or a curved surface).

We denote by $V_h \times \boldsymbol{W}_h$ the Raviart-Thomas-Nedlec space associated with $\mathcal{E}_h$. For each triangle (or 3-simplex) element $E \in \mathcal{E}_h$, we define $V_h(E) = P_k(E)$, and $\boldsymbol{W}_h(E) = P_k(E)^m \oplus (x_1, x_2, \cdots, x_m)^T P_k(E)$ where $P_k(E)$ is the set of polynomials of total defree $\leq k$ difined on E. Now we define the finite element spaces

$$V_h = \{v \in V \mid v|_E \in V_h(E), \forall E \in \mathcal{E}_h\},$$
$$\boldsymbol{W}_h = \{w \in \boldsymbol{W} \mid w|_E \in \boldsymbol{W}_h(E), \forall E \in \mathcal{E}_h\}.$$

And also in case that E is a rectangle (or a parallelogram), we adopt analogous modification to construct V_h and $\boldsymbol{W}_h$.

Let $P_h \times \mathbf{\Pi}_h : V \times \boldsymbol{W} \to V_h \times \boldsymbol{W}_h$ denote the Raviart-Thomas [17] projection which satisfies

$$(\nabla \cdot \boldsymbol{w} - \nabla \cdot \mathbf{\Pi}_h \boldsymbol{w}, \chi) = 0, \qquad \forall \chi \in V_h, \tag{3.1}$$

$$(v - P_h v, \chi) = 0, \qquad \forall \chi \in V_h. \tag{3.2}$$

Then, obviously, $(\nabla \cdot \boldsymbol{w}, v - P_h v) = 0$ holds for each $v \in V$ and each $\boldsymbol{w} \in \boldsymbol{W}_h$ and $\mathrm{div}\mathbf{\Pi}_h = P_h \mathrm{div}$ is a function from $\boldsymbol{W}$ onto V_h. It is proved that the following approximation properties hold [17]:

$$\|v - P_h v\| + h\|v - P_h v\|_1 \le K h^r \|v\|_r, \ \forall v \in V \cap H^r(\Omega), \ 1 \le r \le k+1, \tag{3.3}$$

$$\|\boldsymbol{w} - \mathbf{\Pi}_h \boldsymbol{w}\| \le K h^r \|\boldsymbol{w}\|_r, \ \forall \boldsymbol{w} \in \boldsymbol{W} \cap \boldsymbol{H}^r(\Omega), \ 1 \le r \le k+1, \tag{3.4}$$

$$\|\nabla \cdot (\boldsymbol{w} - \mathbf{\Pi}_h \boldsymbol{w})\| \le K h^r \|\nabla \cdot \boldsymbol{w}\|_r, \ \forall \boldsymbol{w} \in \boldsymbol{W} \cap \boldsymbol{H}^r(\Omega), \ 0 \le r \le k+1. \tag{3.5}$$

4. Optimal L^2 error analysis

Let $\psi(\boldsymbol{x}) = \Big(c^2(\boldsymbol{x}) + |\boldsymbol{d}(\boldsymbol{x})|^2\Big)^{\frac{1}{2}}$ with $|\boldsymbol{d}(\boldsymbol{x})|^2 = \sum_{i=1}^{m} d_i^2(\boldsymbol{x})$ and $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x}, t)$ be the unit vector in the direction of $(\boldsymbol{d}(\boldsymbol{x}), c(\boldsymbol{x}))$. Then, we have

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{c(\boldsymbol{x})}{\psi(\boldsymbol{x})} \frac{\partial u}{\partial t} + \frac{\boldsymbol{d}(\boldsymbol{x})}{\psi(\boldsymbol{x})} \cdot \nabla u.$$

Hence the problem (1.1) can be written in the form

$$\begin{cases} \psi(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{\nu}} - \nabla \cdot (a(\boldsymbol{x}) \nabla u_t + b(\boldsymbol{x}) \nabla u) = f(\boldsymbol{x}, t), & \text{in } \Omega \times (0, T], \\ u(\boldsymbol{x}, t) = 0, & \text{on } \partial\Omega \times (0, T], \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \text{in } \Omega. \end{cases} \tag{4.1}$$

By introducing the flux term $\boldsymbol{\sigma} = -(a(\boldsymbol{x}) \nabla u_t + b(\boldsymbol{x}) \nabla u)$, the problem (4.1) can be rewritten as follows:

$$\begin{cases} \psi(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{\nu}} + \nabla \cdot \boldsymbol{\sigma} = f(\boldsymbol{x}, t), & \text{in } \Omega \times (0, T], \\ \boldsymbol{\sigma} + a(\boldsymbol{x}) \nabla u_t + b(\boldsymbol{x}) \nabla u = 0, & \text{in } \Omega \times (0, T]. \end{cases} \tag{4.2}$$

For a positive integer N, let $\Delta t = T/N$ and $t^n = n\Delta t, \ n = 0, 1, \cdots, N$. Choosing $t = t^n$ in (4.2) and discretizing it with respect to t by applying the backward Euler method along ν-characteristic tangent at $(\boldsymbol{x}, t^n)$, we get

$$\psi(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, t^n) \cong \psi(\boldsymbol{x}) \frac{u(\boldsymbol{x}, t^n) - u(\hat{\boldsymbol{x}}, t^{n-1})}{\sqrt{|\frac{\boldsymbol{d}(\boldsymbol{x})}{c(\boldsymbol{x})} \Delta t|^2 + (\Delta t)^2}} = c(\boldsymbol{x}) \frac{u(\boldsymbol{x}, t^n) - u(\hat{\boldsymbol{x}}, t^{n-1})}{\Delta t},$$

where $\hat{\boldsymbol{x}} = \boldsymbol{x} - \tilde{\boldsymbol{d}}(\boldsymbol{x}) \Delta t$ with $\tilde{\boldsymbol{d}}(\boldsymbol{x}) = \frac{\boldsymbol{d}(\boldsymbol{x})}{c(\boldsymbol{x})}$. Therefore we have

$$\begin{cases} c(\boldsymbol{x}) \frac{u^n - \hat{u}^{n-1}}{\Delta t} + \nabla \cdot \boldsymbol{\sigma}^n = f^n + E_1^n, \\ \boldsymbol{\sigma}^n + a(\boldsymbol{x}) \frac{\nabla u^n - \nabla u^{n-1}}{\Delta t} + b(\boldsymbol{x}) \nabla u^n = E_2^n, \end{cases} \tag{4.3}$$

where $u^n = u(\boldsymbol{x}, t^n), \ \hat{u}^{n-1} = u(\hat{\boldsymbol{x}}, t^{n-1}), \ E_1^n = c(\boldsymbol{x}) \frac{u^n - \hat{u}^{n-1}}{\Delta t} - \psi(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, t^n)$ and $E_2^n = a(\boldsymbol{x}) \Big(\frac{\nabla u^n - \nabla u^{n-1}}{\Delta t} - \nabla u_t^n \Big)$.

Now let $\tilde{a}(\boldsymbol{x}) = a(\boldsymbol{x}) + b(\boldsymbol{x})\Delta t$. By multiplying the first equation of (4.3) by $c^{-\frac{1}{2}}\Delta t$ and the second equation by $(\tilde{a})^{-\frac{1}{2}}\Delta t$, we have the equivalent system of equations

$$\begin{cases} c^{-\frac{1}{2}}[cu^n + \Delta t\ \nabla\cdot\boldsymbol{\sigma}^n - (c\hat{u}^{n-1} + \Delta t f^n + \Delta t E_1^n)] = 0, \\ (\tilde{a})^{-\frac{1}{2}}[\Delta t\ \boldsymbol{\sigma}^n + (\tilde{a})\nabla u^n - (a\nabla u^{n-1} + \Delta t E_2^n)] = 0. \end{cases} \tag{4.4}$$

For $(v,\boldsymbol{\tau}) \in V\times \boldsymbol{W}$, we define a least-squares functional $J(v,\boldsymbol{\tau})$ as follows

$$\begin{aligned} J(v,\boldsymbol{\tau}) =& \|c^{-\frac{1}{2}}[cu^n + \Delta t\ \nabla\cdot\boldsymbol{\sigma}^n - (c\hat{u}^{n-1} + \Delta t f^n + \Delta t E_1^n)]\|^2 \\ &+ \|(\tilde{a})^{-\frac{1}{2}}[\Delta t\ \boldsymbol{\sigma}^n + (\tilde{a})\nabla u^n - (a\nabla u^{n-1} + \Delta t E_2^n)]\|^2. \end{aligned}$$

Then the least-squares minimization problem is to find s solution $(u^n,\boldsymbol{\sigma}^n) \in V\times \boldsymbol{W}$ such that

$$J(u^n,\boldsymbol{\sigma}^n) = \inf_{(v,\boldsymbol{\tau})\in V\times \boldsymbol{W}} J(v,\boldsymbol{\tau}).$$

If we define the bilinear form A on $(V\times \boldsymbol{W})^2$ by

$$\begin{aligned} A(u,\boldsymbol{w};v,\boldsymbol{\tau}) = & \Big(c^{-1}(cu + \Delta t\ \nabla\cdot\boldsymbol{w}), cv + \Delta t\ \nabla\cdot\boldsymbol{\tau}\Big) \\ & + \Big(\tilde{a}^{-1}(\Delta t\ \boldsymbol{w} + \tilde{a}\nabla u), \Delta t\ \boldsymbol{\tau} + \tilde{a}\nabla v\Big), \end{aligned} \tag{4.5}$$

then the weak formulation of the minimization problem becomes as follows: find $(u^n,\boldsymbol{\sigma}^n)\in V\times \boldsymbol{W}$ such that

$$\begin{aligned} A(u^n,\boldsymbol{\sigma}^n;v,\boldsymbol{\tau}) = & \Big(c^{-1}(c\hat{u}^{n-1} + \Delta t\ f^n + \Delta t\ E_1^n), cv + \Delta t\ \nabla\cdot\boldsymbol{\tau}\Big) \\ & + \Big(\tilde{a}^{-1}(a\nabla u^{n-1} + \Delta t\ E_2^n), \Delta t\ \boldsymbol{\tau} + \tilde{a}\nabla v\Big),\ \forall\ (v,\boldsymbol{\tau})\in V\times \boldsymbol{W}. \end{aligned} \tag{4.6}$$

Based on (4.6), we derive the following least-squares characteristic MFEM scheme: find $(u_h^n,\boldsymbol{\sigma}_h^n)\in V_h\times \boldsymbol{W}_h$ satisfying

$$\begin{aligned} A(u_h^n,\boldsymbol{\sigma}_h^n;v_h,\boldsymbol{\tau}_h) = & \Big(c^{-1}(c\hat{u}_h^{n-1} + \Delta t f^n), cv_h + \Delta t\ \nabla\cdot\boldsymbol{\tau}_h\Big) \\ & + \Big(\tilde{a}^{-1}(a\nabla u_h^{n-1}), \Delta t\ \boldsymbol{\tau}_h + \tilde{a}\nabla v_h\Big),\ \forall\ (v_h,\boldsymbol{\tau}_h)\in V_h\times \boldsymbol{W}_h. \end{aligned} \tag{4.7}$$

Lemma 4.1. *For $(v,\boldsymbol{\tau})\in V\times \boldsymbol{W}$, we have*

$$\begin{aligned} A(u^n,\boldsymbol{\sigma}^n;v,\boldsymbol{\tau}) =& (cu^n,v) + (c^{-1}\Delta t\ \nabla\cdot\boldsymbol{\sigma}^n, \Delta t\ \nabla\cdot\boldsymbol{\tau}) + (\tilde{a}^{-1}\Delta t\ \boldsymbol{\sigma}^n, \Delta t\ \boldsymbol{\tau}) \\ & + (\tilde{a}\nabla u^n, \nabla v). \end{aligned}$$

Proof. From the definition of the bilinear form (4.5), we have

$$\begin{aligned} & A(u^n,\boldsymbol{\sigma}^n;v,\boldsymbol{\tau}) \\ =& (cu^n,v) + (u^n,\Delta t\ \nabla\cdot\boldsymbol{\tau}) + (\Delta t\ \nabla\cdot\boldsymbol{\sigma}^n, v) + (c^{-1}\Delta t\ \nabla\cdot\boldsymbol{\sigma}^n, \Delta t\ \nabla\cdot\boldsymbol{\tau}) \\ & + (\tilde{a}^{-1}\Delta t\ \boldsymbol{\sigma}^n, \Delta t\ \boldsymbol{\tau}) + (\Delta t\ \boldsymbol{\sigma}^n, \nabla v) + (\nabla u^n, \Delta t\ \boldsymbol{\tau}) + (\tilde{a}\nabla u^n, \nabla v) \\ =& (cu^n,v) + (c^{-1}\Delta t\ \nabla\cdot\boldsymbol{\sigma}^n, \Delta t\ \nabla\cdot\boldsymbol{\tau}) + (\tilde{a}^{-1}\Delta t\ \boldsymbol{\sigma}^n, \Delta t\ \boldsymbol{\tau}) + (\tilde{a}\nabla u^n, \nabla v). \end{aligned}$$

□

Letting $v_h = 0$ in (4.7) and applying the definition of the bilinear form A, we have

$$\begin{aligned}&(c^{-1}\Delta t\ \nabla\cdot\boldsymbol{\sigma}_h^n, \Delta t\ \nabla\cdot\boldsymbol{\tau}_h) + (\tilde{a}^{-1}\Delta t\ \boldsymbol{\sigma}_h^n, \Delta t\ \boldsymbol{\tau}_h)\\ =&\Big(c^{-1}(c\hat{u}_h^{n-1} + \Delta t\ f^n), \Delta t\ \nabla\cdot\boldsymbol{\tau}_h\Big) + (\tilde{a}^{-1}a\nabla u_h^{n-1}, \Delta t\ \boldsymbol{\tau}_h),\end{aligned}$$

which implies that

$$\begin{aligned}&(\Delta t)^2\{(c^{-1}\nabla\cdot\boldsymbol{\sigma}_h^n, \nabla\cdot\boldsymbol{\tau}_h) + (\tilde{a}^{-1}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h)\}\\ =&\Delta t(\hat{u}_h^{n-1}, \nabla\cdot\boldsymbol{\tau}_h) + (\Delta t)^2(c^{-1}f^n, \nabla\cdot\boldsymbol{\tau}_h) + \Delta t(\tilde{a}^{-1}a\nabla u_h^{n-1}, \boldsymbol{\tau}_h).\end{aligned}$$

Since $-\frac{1}{\Delta t} + \frac{1}{\Delta t}\tilde{a}^{-1}a = \tilde{a}^{-1}(\frac{a}{\Delta t} - \tilde{a}\frac{1}{\Delta t}) = \tilde{a}^{-1}(-b)$, we have

$$\begin{aligned}&(c^{-1}\nabla\cdot\boldsymbol{\sigma}_h^n, \nabla\cdot\boldsymbol{\tau}_h) + (\tilde{a}^{-1}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h)\\ =&\frac{1}{\Delta t}(\hat{u}_h^{n-1}, \nabla\cdot\boldsymbol{\tau}_h) + (c^{-1}f^n, \nabla\cdot\boldsymbol{\tau}_h) + \frac{1}{\Delta t}(\tilde{a}^{-1}a\nabla u_h^{n-1}, \boldsymbol{\tau}_h)\\ =&\frac{1}{\Delta t}\Big(\nabla(u_h^{n-1} - \hat{u}_h^{n-1}), \boldsymbol{\tau}_h\Big) + (c^{-1}f^n, \nabla\cdot\boldsymbol{\tau}_h) - (\tilde{a}^{-1}b\nabla u_h^{n-1}, \boldsymbol{\tau}_h).\end{aligned}$$

Letting $\boldsymbol{\tau}_h = 0$ in (4.7) and applying the definition of the bilinear form A, we have

$$(cu_h^n, v_h) + (\tilde{a}\nabla u_h^n, \nabla v_h) = (c\hat{u}_h^{n-1}, v_h) + \Delta t(f^n, v_h) + (a\nabla u_h^{n-1}, \nabla v_h).$$

Finally, we derive a split least-squares characteristic MFEM: find $\{u_h^n, \boldsymbol{\sigma}_h^n\} \in V_h \times \boldsymbol{W}_h$ satisfying:

$$(cu_h^n, v_h) + (\tilde{a}\nabla u_h^n, \nabla v_h) = (c\hat{u}_h^{n-1}, v_h) + \Delta t(f^n, v_h) + (a\nabla u_h^{n-1}, \nabla v_h), \tag{4.8}$$

$$\begin{aligned}&(c^{-1}\nabla\cdot\boldsymbol{\sigma}_h^n, \nabla\cdot\boldsymbol{\tau}_h) + (\tilde{a}^{-1}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h)\\ &\quad= \frac{1}{\Delta t}\Big(\nabla(u_h^{n-1} - \hat{u}_h^{n-1}), \boldsymbol{\tau}_h\Big) + (c^{-1}f^n, \nabla\cdot\boldsymbol{\tau}_h) - (\tilde{a}^{-1}b\nabla u_h^{n-1}, \boldsymbol{\tau}_h).\end{aligned} \tag{4.9}$$

For the error analysis, we define an elliptic projection $\tilde{u}(x,t)$ of $u(x,t)$ onto V_h satisfying

$$\begin{cases}(a(x)\nabla(u-\tilde{u})_t, \nabla v_h) + (b(x)\nabla(u-\tilde{u}), \nabla v_h) = 0, \ \ \forall v_h \in V_h\\ (\tilde{u}(0), v) = (u_0, v), \ \ \forall v_h \in V_h.\end{cases} \tag{4.10}$$

Obviously, by the assumption (A), there exists a unique elliptic projection $\tilde{u}(x,t) \in V_h$. Now we let $\eta = u - \tilde{u}$ and $\xi = u_h - \tilde{u}$ so that $u - u_h = \eta - \xi$.

Hereafter a constant K denotes a generic positive constant depending on Ω and u, but independent of h and Δt, and also any two Ks in different places don't need to be the same. We state the error bounds of η below, the proofs of which can be found in [14, 15].

Theorem 4.2 ([14]). *If $u_t \in L^2(H^s(\Omega))$ and $u_0 \in H^s(\Omega)$, then there exists a constant K, independent of h, such that*
(i) $\|\eta\| + h\|\eta\|_1 \le Kh^\mu(\|u_t\|_{L^2(H^s)} + \|u_0\|_s)$,
(ii) $\|\eta_t\| + h\|\eta_t\|_1 \le Kh^\mu(\|u_t\|_{L^2(H^s)} + \|u_0\|_s)$,
where $\mu = \min(k+1, s)$.

Theorem 4.3 ([15]). *If $u_t \in L^2(H^s(\Omega))$, $u_{tt}(t) \in H^s(\Omega)$, and $u_0 \in H^s(\Omega)$, then there exists a constant K, independent of h, such that*

$$\|\eta_{tt}\|_1 \le Ch^{\mu-1}\{\|u_t\|_{L^2(H^s)} + \|u_{tt}\|_s + \|u_0\|_s\},$$

where $\mu = \min(k+1, s)$.

Lemma 4.4. *If $u \in H^{1,\infty}(H^2(\Omega))$ and $u_{tt}(t) \in L^2(\Omega)$, then*

$$\|E_1^n\| \le K\Delta t \text{ and } \|E_2^n\| \le K\Delta t.$$

Proof. By applying Taylor's expansion, we obviously have the estimations of E_1^n and E_2^n. □

Theorem 4.5. *In addition to the hypotheses of Theorem 4.2 and 4.3, if $u(t) \in H^s(\Omega)$, $u \in H^{1,\infty}(H^2(\Omega))$, and $\Delta t = O(h)$, then*

$$\|u^n - u_h^n\|_l \le K(h^{\mu-l} + \Delta t), \quad l = 0, 1,$$

where $\mu = \min(k+1, s)$.

Proof. Subtracting (4.1) at $t = t^n$ from (4.8), we get the equation

$$\begin{aligned}
&\Big(c(x)\frac{\xi^n - \xi^{n-1}}{\Delta t}, v_h\Big) + \Big(a(x)\frac{\nabla\xi^n - \nabla\xi^{n-1}}{\Delta t}, \nabla v_h\Big) + (b(x)\nabla\xi^n, \nabla v_h) \\
&= \Big(c(x)\frac{\hat{\xi}^{n-1} - \xi^{n-1}}{\Delta t}, v_h\Big) + \Big(c(x)\frac{\eta^n - \hat{\eta}^{n-1}}{\Delta t}, v_h\Big) \\
&\quad + \Big(\psi\frac{\partial u^n}{\partial \nu} - c(x)\frac{u^n - \hat{u}^{n-1}}{\Delta t}, v_h\Big) + \Big(b(x)\nabla\eta^n, \nabla v_h\Big) \\
&\quad + \Big(a(x)\nabla\frac{\eta^n - \eta^{n-1}}{\Delta t}, \nabla v_h\Big) + \Big(a(x)\Big(u_t^n - \frac{u^n - u^{n-1}}{\Delta t}\Big), v_h\Big) \\
&= \sum_{i=1}^{6} R_i.
\end{aligned} \tag{4.11}$$

Now we set $v_h = \partial_t\xi^n = \frac{\xi^n - \xi^{n-1}}{\Delta t}$ in (4.11). Then letting three terms of the left-hand side of (4.11) by L_1, L_2, and L_3, respectively, we get the following estimates for L_1, L_2, and L_3

$$\begin{aligned}
L_1 =& \Big(c(x)\partial_t\xi^n, \partial_t\xi^n\Big) \ge c_*\|\partial_t\xi^n\|^2, \\
L_2 =& \Big(a(x)\nabla\partial_t\xi^n, \nabla\partial_t\xi^n\Big) \ge a_*\|\nabla\partial_t\xi^n\|^2, \\
L_3 =& \Big(b(x)\nabla\xi^n, \frac{\nabla\xi^n - \nabla\xi^{n-1}}{\Delta t}\Big) \\
=& \frac{1}{\Delta t}\|\sqrt{b(x)}\nabla\xi^n\|^2 - \frac{1}{\Delta t}\Big(\sqrt{b(x)}\nabla\xi^n, \sqrt{b(x)}\nabla\xi^{n-1}\Big) \\
\ge& \frac{1}{\Delta t}\|\sqrt{b(x)}\nabla\xi^n\|^2 - \frac{1}{2\Delta t}\Big(\|\sqrt{b(x)}\nabla\xi^n\|^2 + \|\sqrt{b(x)}\nabla\xi^{n-1}\|^2\Big) \\
=& \frac{1}{2\Delta t}\Big(\|\sqrt{b(x)}\nabla\xi^n\|^2 - \|\sqrt{b(x)}\nabla\xi^{n-1}\|^2\Big).
\end{aligned}$$

Now let $\epsilon > 0$ be sufficiently small, but independent of h and Δt. Since

$$\begin{aligned}\hat{\xi}^{n-1} - \xi^{n-1} &= \xi^{n-1}(\boldsymbol{x} - \tilde{\boldsymbol{d}}\Delta t) - \xi^{n-1}(\boldsymbol{x}) \\ &= \xi^{n-1}(\boldsymbol{x}) - \tilde{\boldsymbol{d}}\Delta t \nabla \xi^{n-1}(\hat{\boldsymbol{x}}_*) - \xi^{n-1}(\boldsymbol{x})\end{aligned}$$

for some $\hat{\boldsymbol{x}}_* \in (\boldsymbol{x} - \tilde{\boldsymbol{d}}\Delta t, \boldsymbol{x})$, R_1 can be estimated as follows:

$$R_1 \le K\|\nabla \xi^{n-1}\|^2 + \epsilon\|\partial_t \xi^n\|^2.$$

By noting that

$$\begin{aligned}\eta^n - \hat{\eta}^{n-1} &= (\eta^n - \eta^{n-1}) + (\eta^{n-1} - \hat{\eta}^{n-1}) \\ &= \Delta t \eta_t(t_*^{n-1}) + \tilde{\boldsymbol{d}}\Delta t \nabla \eta(\hat{\boldsymbol{x}}_*, t^{n-1})\end{aligned}$$

for some $t_*^{n-1} \in (t^{n-1}, t^n)$ and $\hat{\boldsymbol{x}}_* \in (\hat{\boldsymbol{x}}, \boldsymbol{x})$, we can estimate R_2 as follows:

$$\begin{aligned}R_2 &= \Big(c(x)[\eta_t(x, t_*^{n-1}) + \tilde{\boldsymbol{d}}\nabla\eta(\hat{x}_*, t^{n-1})], \partial_t \xi^n\Big) \\ &= \Big(c(x)\eta_t(x, t_*^{n-1}), \partial_t \xi^n\Big) + \Big(\eta(\hat{x}_*, t^{n-1}), -\nabla\cdot(c(x)\tilde{\boldsymbol{d}}\partial_t\xi^n)\Big) \\ &\le K\Big(\|\eta_t\|^2_{L^\infty(L^2)} + \|\eta^{n-1}\|^2\Big) + \epsilon\|\partial_t\xi^n\|^2 + \epsilon\|\nabla\partial_t\xi^n\|^2 \\ &\le K(h^{2\mu}) + \epsilon\|\partial_t\xi^n\|^2 + \epsilon\|\nabla\partial_t\xi^n\|^2.\end{aligned}$$

By Lemma 4.4, we obviously get

$$R_3 \le K\|E_1^n\|^2 + \epsilon\|\partial_t\xi^n\|^2 \le K(\Delta t)^2 + \epsilon\|\partial_t\xi^n\|^2.$$

By (4.10), Theorem 4.3, and the Taylor expansion, we have

$$\begin{aligned}R_4 + R_5 &= \Big(b(x)\nabla\eta^n, \nabla\partial_t\xi^n\Big) + \Big(a(x)\frac{\nabla\eta^n - \nabla\eta^{n-1}}{\Delta t}, \nabla\partial_t\xi^n\Big) \\ &= \Big(a(x)\Big(\frac{\nabla\eta^n - \nabla\eta^{n-1}}{\Delta t} - \eta_t^n\Big), \nabla\partial_t\xi^n\Big) \\ &= \Big(a(x)\Delta t\nabla\eta_{tt}(t_\theta^{n-1}), \nabla\partial_t\xi^n\Big) \\ &\le K(\Delta t)^2 h^{2(\mu-1)} + \epsilon\|\nabla\partial_t\xi^n\|^2 \le Kh^{2\mu} + \epsilon\|\nabla\partial_t\xi^n\|^2,\end{aligned}$$

where $t_\theta^{n-1} \in (t^{n-1}, t^n)$. By the Taylor expansion, we get

$$\begin{aligned}R_6 &= \Big(a(x)\Big(u_t^n - \frac{u^n - u^{n-1}}{\Delta t}\Big), \partial_t\xi^n\Big) \\ &= \Big(a(x)\Delta t u_{tt}(t_*^{n-1}), \partial_t\xi^n\Big) \le K(\Delta t)^2 + \epsilon\|\partial_t\xi^n\|^2\end{aligned}$$

for some $t_*^{n-1} \in (t^{n-1}, t^n)$. Now by applying the bounds of $L_1 \sim L_3$ and $R_1 \sim R_6$ to (4.11), we obtain

$$\begin{aligned}&c_*\|\partial_t\xi^n\|^2 + a_*\|\nabla\partial_t\xi^n\|^2 + \frac{1}{2\Delta t}\Big(\|\sqrt{b(x)}\nabla\xi^n\|^2 - \|\sqrt{b(x)}\nabla\xi^{n-1}\|^2\Big) \\ &\le K\|\nabla\xi^{n-1}\|^2 + 4\epsilon\|\partial_t\xi^n\|^2 + 2\epsilon\|\nabla\partial_t\xi^n\|^2 + K(h^{2\mu} + (\Delta t)^2),\end{aligned}$$

which yields that for sufficiently small $\epsilon > 0$

$$c_* \Delta t \|\partial_t \xi^n\|^2 + a_* \Delta t \|\nabla \partial_t \xi^n\|^2 + \Big(\|\sqrt{b(x)} \nabla \xi^n\|^2 - \|\sqrt{b(x)} \nabla \xi^{n-1}\|^2 \Big)$$
$$\leq K \Delta t \|\nabla \xi^{n-1}\|^2 + K(h^{2\mu} + (\Delta t)^2). \tag{4.12}$$

Now we sum up both sides of (4.12) from $n = 1$ to $n = N$ to get

$$\|\sqrt{b(x)} \nabla \xi^N\|^2 + \Delta t \Big[c_* \sum_{n=1}^{N} \|\partial_t \xi^n\|^2 + a_* \sum_{n=1}^{N} \|\nabla \partial_t \xi^n\|^2 \Big]$$
$$\leq K \Delta t \Big(\sum_{n=1}^{N} \|\nabla \xi^{n-1}\|^2 \Big) + K \Big(h^{2\mu} + (\Delta t)^2 \Big).$$

By the discrete-type Gronwall inequality, we get

$$\|\nabla \xi^N\|^2 \leq K(h^{2\mu} + (\Delta t)^2),$$

from which we get by Poincare's inequality

$$\|\xi^N\|^2 \leq K(h^{2\mu} + (\Delta t)^2).$$

Therefore, by using Theorem 4.2 and the triangular inequality, we obtain

$$\|u^n - u_h^n\|_l \leq K(h^{\mu - l} + \Delta t), \ l = 0, 1.$$

□

By applying Lemma 4.1 to (4.6), we get

$$(cu^n, v) + (c^{-1} \Delta t \ \nabla \cdot \boldsymbol{\sigma}^n, \Delta t \ \nabla \cdot \boldsymbol{\tau}) + (\tilde{a}^{-1} \Delta t \ \boldsymbol{\sigma}^n, \Delta t \ \boldsymbol{\tau}) + (\tilde{a} \nabla u^n, \nabla v)$$
$$= (c^{-1}(c\hat{u}^{n-1} + \Delta t \ f^n + \Delta t \ E_1^n), cv + \Delta t \ \nabla \cdot \boldsymbol{\tau})$$
$$+ (\tilde{a}^{-1}(a \nabla u^{n-1} + \Delta t \ E_2^n), \Delta t \ \boldsymbol{\tau} + \tilde{a} \nabla v)$$

and hence, letting $v = 0$, we obtain

$$(c^{-1} \nabla \cdot \boldsymbol{\sigma}^n, \nabla \cdot \boldsymbol{\tau}) + (\tilde{a}^{-1} \boldsymbol{\sigma}^n, \boldsymbol{\tau}) = \frac{1}{\Delta t} (\hat{u}^{n-1}, \nabla \cdot \boldsymbol{\tau}) + (c^{-1} f^n, \nabla \cdot \boldsymbol{\tau})$$
$$+ (c^{-1} E_1^n, \nabla \cdot \boldsymbol{\tau}) + \frac{1}{\Delta t} (\tilde{a}^{-1} a \nabla u^{n-1}, \boldsymbol{\tau}) + (\tilde{a}^{-1} E_2^n, \boldsymbol{\tau}). \tag{4.13}$$

And letting $v_h = 0$ in (4.7) and applying the definition of the bilinear form A, we get

$$\Big(c^{-1}(cu_h^n + \Delta t \ \nabla \cdot \boldsymbol{\sigma}_h^n), \Delta t \ \nabla \cdot \boldsymbol{\tau}_h \Big) + \Big(\tilde{a}^{-1}(\Delta t \ \boldsymbol{\sigma}_h^n + \tilde{a} \nabla u_h^n), \Delta t \ \boldsymbol{\tau}_h \Big)$$
$$= \Big(c^{-1}(c\hat{u}_h^{n-1} + \Delta t \ f^n), \Delta t \ \nabla \cdot \boldsymbol{\tau}_h \Big) + \Big(\tilde{a}^{-1}(a \nabla u_h^{n-1}), \Delta t \ \boldsymbol{\tau}_h \Big),$$

which implies that

$$\Delta t (u_h^n, \nabla \cdot \boldsymbol{\tau}_h) + \Delta t^2 (c^{-1} \nabla \cdot \boldsymbol{\sigma}_h^n, \nabla \cdot \boldsymbol{\tau}_h) + \Delta t^2 (\tilde{a}^{-1} \boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{n}}, \boldsymbol{\tau}_h) + \Delta t (\nabla u_h^n, \boldsymbol{\tau}_h)$$
$$= \Delta t (\hat{u}_h^{n-1}, \nabla \cdot \boldsymbol{\tau}_h) + \Delta t^2 (c^{-1} f^n, \nabla \cdot \boldsymbol{\tau}_h) + \Delta t (\tilde{a}^{-1} a \nabla u_h^{n-1}, \boldsymbol{\tau}_h).$$

Therefore we have

$$\begin{aligned}&(c^{-1}\nabla\cdot\boldsymbol{\sigma}_h^n,\nabla\cdot\boldsymbol{\tau}_h)+(\tilde a^{-1}\boldsymbol{\sigma}_h^n,\boldsymbol{\tau}_h)\\&=\frac{1}{\Delta t}(\hat u_h^{n-1},\nabla\cdot\boldsymbol{\tau}_h)+(c^{-1}f^n,\nabla\cdot\boldsymbol{\tau}_h)+\frac{1}{\Delta t}(\tilde a^{-1}a\nabla u_h^{n-1},\boldsymbol{\tau}_h).\end{aligned}\tag{4.14}$$

For $\boldsymbol{\sigma}\in\boldsymbol{W}$, we define an elliptic projection $\tilde{\boldsymbol{\sigma}}\in\boldsymbol{W}_h$ of $\boldsymbol{\sigma}$ satisfying

$$\Big(c^{-1}\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}),\nabla\cdot\boldsymbol{\tau}_h\Big)+\lambda(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}},\boldsymbol{\tau}_h)=0,\ \forall\boldsymbol{\tau}_h\in\boldsymbol{W}_h,\tag{4.15}$$

where λ is a positive real number. By applying the Lax-Milgram lemma, the existence of $\tilde{\boldsymbol{\sigma}}$ can be obtained.

Lemma 4.6. *If $\boldsymbol{\sigma}\in\boldsymbol{W}\cap\boldsymbol{H}^s(\Omega)$, then there exists a constant $K>0$ such that*

$$\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|_l\le Kh^{\mu-l}\|\boldsymbol{\sigma}\|_s,\ l=0,1,$$

where $\mu=\min(k+1,s)$.

Proof. By the difinition of $\tilde{\boldsymbol{\sigma}}$ and (3.5), we get

$$\begin{aligned}\|c^{-\frac12}\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\|^2&=\Big(c^{-1}\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}),\ \nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma})\Big)-\lambda(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}},\boldsymbol{\Pi}_h\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\\&\le\|c^{-\frac12}\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\|\|c^{-\frac12}\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma})\|+\lambda\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|\|\boldsymbol{\Pi}_h\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|\end{aligned}$$

and so

$$\begin{aligned}&\|c^{-\frac12}\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\|^2\\&\le(\|c^{-\frac12}\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma})\|^2)+\lambda(\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|^2+\|\boldsymbol{\Pi}_h\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|^2)\\&\le Kh^{2(\mu-1)}\|\boldsymbol{\sigma}\|_s^2+\lambda(\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|^2+2\|\boldsymbol{\Pi}_h\boldsymbol{\sigma}-\boldsymbol{\sigma}\|^2+2\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|^2).\end{aligned}$$

Therefore, by (3.4), we have

$$\|c^{-\frac12}\nabla(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\|\le Kh^{\mu-1}\|\boldsymbol{\sigma}\|_s+K\lambda\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|\tag{4.16}$$

for sufficiently small $\lambda>0$. We let $\boldsymbol{\varphi}\in\boldsymbol{H}^2(\Omega)$ be the solution of an elliptic problem

$$\begin{cases}-\nabla(c^{-1}\nabla\cdot\boldsymbol{\varphi})+\lambda\boldsymbol{\varphi}=\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}},\ \text{in }\Omega,\\(c^{-1}\nabla\cdot\boldsymbol{\varphi})\boldsymbol{n}=\boldsymbol{0},\ \text{on }\partial\Omega,\end{cases}\tag{4.17}$$

where $\boldsymbol{n}$ denotes the outward normal unit vector to $\partial\Omega$. By the regularity property of the elliptic problem, we have $\|\boldsymbol{\varphi}\|_2\le K\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|$. Using (3.4), (3.5), (4.15), (4.16), and (4.17), we obtain the following estimation

$$\begin{aligned}\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|^2&=\Big(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}},-\nabla(c^{-1}\nabla\cdot\boldsymbol{\varphi})+\lambda\boldsymbol{\varphi}\Big)\\&=\Big(\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}),c^{-1}\nabla\cdot\boldsymbol{\varphi}\Big)+\lambda(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}},\boldsymbol{\varphi})\\&=\Big(c^{-1}\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}),\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{\Pi}_h\boldsymbol{\varphi})\Big)+\lambda(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}},\boldsymbol{\varphi}-\boldsymbol{\Pi}_h\boldsymbol{\varphi})\\&\le K\|c^{-\frac12}\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\|\|c^{-\frac12}\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{\Pi}_h\boldsymbol{\varphi})\|+K\lambda h^2\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|\|\boldsymbol{\varphi}\|_2\end{aligned}$$

$$
\begin{aligned}
&\leq Kh\|c^{-\frac{1}{2}}\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\|\|\boldsymbol{\varphi}\|_2+K\lambda h^2\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|\|\boldsymbol{\varphi}\|_2\\
&\leq Kh\|c^{-\frac{1}{2}}\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\|\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|+K\lambda h^2\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|^2\\
&\leq Kh(h^{\mu-1}\|\boldsymbol{\sigma}\|_s+\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|)\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|+K\lambda h^2\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|^2.
\end{aligned}
$$

Now if we choose h sufficiently small, then we get $\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}\|\leq Kh^{\mu}\|\boldsymbol{\sigma}\|_s$. □

Theorem 4.7. *In addition to the hypotheses of Theorem 4.5, if* $\boldsymbol{\sigma}\in\boldsymbol{W}\cap\boldsymbol{H}^s(\Omega)$, *then*

$$\|\boldsymbol{\sigma}^n-\boldsymbol{\sigma}_h^n\|\leq K(h^{\mu}+\Delta t),$$

where $\mu=\min(k+1,s)$.

Proof. By subtracting (4.14) from (4.13), we have

$$
\begin{aligned}
&\Big(c^{-1}\nabla\cdot(\boldsymbol{\sigma}^n-\boldsymbol{\sigma}_h^n),\nabla\cdot\boldsymbol{\tau}_h\Big)+(\tilde{a}^{-1}(\boldsymbol{\sigma}^n-\boldsymbol{\sigma}_h^n),\boldsymbol{\tau}_h)\\
=&\frac{1}{\Delta t}(\hat{u}^{n-1}-\hat{u}_h^{n-1},\nabla\cdot\boldsymbol{\tau}_h)+\frac{1}{\Delta t}(\tilde{a}^{-1}a\nabla(u^{n-1}-u_h^{n-1}),\boldsymbol{\tau}_h)\\
&+(c^{-1}E_1^n,\nabla\cdot\boldsymbol{\tau}_h)+(\tilde{a}^{-1}E_2^n,\boldsymbol{\tau}_h)\\
=&\frac{1}{\Delta t}\Big((\hat{u}^{n-1}-\hat{u}_h^{n-1})-(u^{n-1}-u_h^{n-1}),\nabla\cdot\boldsymbol{\tau}_h\Big)\\
&-\frac{1}{\Delta t}\Big(\nabla(u^{n-1}-u_h^{n-1}),\boldsymbol{\tau}_h\Big)+\frac{1}{\Delta t}\Big(\tilde{a}^{-1}a\nabla(u^{n-1}-u_h^{n-1}),\boldsymbol{\tau}_h\Big)\\
&+(c^{-1}E_1^n,\nabla\cdot\boldsymbol{\tau}_h)+(\tilde{a}^{-1}E_2^n,\boldsymbol{\tau}_h)\\
=&\frac{1}{\Delta t}\Big((\hat{u}^{n-1}-\hat{u}_h^{n-1})-(u^{n-1}-u_h^{n-1}),\nabla\cdot\boldsymbol{\tau}_h\Big)\\
&-\Big(\tilde{a}^{-1}b\nabla(u^{n-1}-u_h^{n-1}),\boldsymbol{\tau}_h\Big)+(c^{-1}E_1^n,\nabla\cdot\boldsymbol{\tau}_h)+(\tilde{a}^{-1}E_2^n,\boldsymbol{\tau}_h).
\end{aligned}
\tag{4.18}
$$

Now we let $\boldsymbol{\pi}=\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}},\ \boldsymbol{\rho}=\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h$. From (4.18), we get

$$
\begin{aligned}
&\Big(c^{-1}\nabla\cdot(\boldsymbol{\pi}^n+\boldsymbol{\rho}^n),\nabla\cdot\boldsymbol{\tau}_h\Big)+\Big(\tilde{a}^{-1}(\boldsymbol{\pi}^n+\boldsymbol{\rho}^n),\boldsymbol{\tau}_h\Big)\\
=&\frac{1}{\Delta t}(\hat{\eta}^{n-1}-\hat{\xi}^{n-1}-(\eta^{n-1}-\xi^{n-1}),\nabla\cdot\boldsymbol{\tau}_h)\\
&-\Big(\tilde{a}^{-1}b\nabla(\eta^{n-1}-\xi^{n-1}),\boldsymbol{\tau}_h\Big)+(c^{-1}E_1^n,\nabla\cdot\boldsymbol{\tau}_h)+(\tilde{a}^{-1}E_2^n,\boldsymbol{\tau}_h).
\end{aligned}
\tag{4.19}
$$

Choosing $\boldsymbol{\tau}_h=\boldsymbol{\rho}^n$ in (4.19) and applying the integration by parts, we obtain

$$
\begin{aligned}
&(c^{-1}\nabla\cdot\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\rho}^n)+(\tilde{a}^{-1}\boldsymbol{\rho}^n,\boldsymbol{\rho}^n)\\
=&-(c^{-1}\nabla\cdot\boldsymbol{\pi}^n,\nabla\cdot\boldsymbol{\rho}^n)-(\tilde{a}^{-1}\boldsymbol{\pi}^n,\boldsymbol{\rho}^n)+\frac{1}{\Delta t}(\hat{\eta}^{n-1}-\eta^{n-1},\nabla\cdot\boldsymbol{\rho}^n)\\
&-\frac{1}{\Delta t}(\hat{\xi}^{n-1}-\xi^{n-1},\nabla\cdot\boldsymbol{\rho}^n)+\Big(\nabla(\tilde{a}^{-1}b)(\eta^{n-1}-\xi^{n-1}),\boldsymbol{\rho}^n\Big)\\
&+\Big(\tilde{a}^{-1}b(\eta^{n-1}-\xi^{n-1}),\nabla\cdot\boldsymbol{\rho}^n\Big)+(c^{-1}E_1^n,\nabla\cdot\boldsymbol{\rho}^n)+(\tilde{a}^{-1}E_2^n,\boldsymbol{\rho}^n).
\end{aligned}
\tag{4.20}
$$

Note that

$$\frac{1}{\Delta t}(\hat{\eta}^{n-1}-\eta^{n-1},\nabla\cdot\boldsymbol{\rho}^n)\le K\|\nabla\eta^{n-1}\|\|\nabla\cdot\boldsymbol{\rho}^n\|$$

and

$$\frac{1}{\Delta t}(\hat{\xi}^{n-1}-\xi^{n-1},\nabla\cdot\boldsymbol{\rho}^n)\le K\|\nabla\xi^{n-1}\|\|\nabla\cdot\boldsymbol{\rho}^n\|.$$

By applying (4.15) to (4.20), we get

$$\begin{aligned}
&\|c^{-\frac{1}{2}}\nabla\cdot\boldsymbol{\rho}^n\|^2+\|\tilde{a}^{-\frac{1}{2}}\boldsymbol{\rho}^n\|^2\\
\le& K\Big[\lambda\|\boldsymbol{\pi}^n\|\|\boldsymbol{\rho}^n\|+\|\boldsymbol{\pi}^n\|\|\tilde{a}^{-\frac{1}{2}}\boldsymbol{\rho}^n\|+\|\nabla\eta^{n-1}\|\|c^{-\frac{1}{2}}\nabla\cdot\boldsymbol{\rho}^n\|\\
&+\|\nabla\xi^{n-1}\|\|c^{-\frac{1}{2}}\nabla\cdot\boldsymbol{\rho}^n\|+(\|\eta^{n-1}\|+\|\xi^{n-1}\|)(\|\tilde{a}^{-\frac{1}{2}}\boldsymbol{\rho}^n\|+\|c^{-\frac{1}{2}}\nabla\cdot\boldsymbol{\rho}^n\|)\\
&+\|E_1^n\|\|c^{-\frac{1}{2}}\nabla\cdot\boldsymbol{\rho}^n\|+\|E_2^n\|\|\tilde{a}^{-\frac{1}{2}}\boldsymbol{\rho}^n\|\Big].
\end{aligned}$$

By using Lemma 4.4, Lemma 4.6, and Theorem 4.5, we get

$$\begin{aligned}
&\|c^{-\frac{1}{2}}\nabla\cdot\boldsymbol{\rho}^n\|^2+\|\tilde{a}^{-\frac{1}{2}}\boldsymbol{\rho}^n\|^2\\
\le& K\Big(\|\boldsymbol{\pi}^n\|^2+\|\nabla\eta^{n-1}\|^2+\|\nabla\xi^{n-1}\|^2+\|\eta^{n-1}\|^2+\|\xi^{n-1}\|^2+(\Delta t)^2\Big)\\
\le& K\Big(h^{2\mu}\|\boldsymbol{\sigma}^n\|_s^2+h^{2(\mu-1)}\|u^{n-1}\|_s^2+(\Delta t)^2\Big)\le K(h^{2(\mu-1)}+(\Delta t)^2). \qquad (4.21)
\end{aligned}$$

Let $\boldsymbol{\psi}^n\in\boldsymbol{H}^2(\Omega)$ be the solution of an elliptic problem

$$\begin{cases}\tilde{a}^{-1}\boldsymbol{\psi}^n-\nabla(c^{-1}\nabla\cdot\boldsymbol{\psi}^n)=\boldsymbol{\rho}^n, & \text{in } \Omega,\\ (c^{-1}\nabla\cdot\boldsymbol{\psi}^n)\boldsymbol{n}=\boldsymbol{0}, & \text{on } \partial\Omega,\end{cases} \qquad (4.22)$$

where $\boldsymbol{n}$ denotes the outward normal unit vector to $\partial\Omega$. By the regularity property of the elliptic problem, we have $\|\boldsymbol{\psi}^n\|_2\le K\|\boldsymbol{\rho}^n\|$. We let $\tilde{\boldsymbol{\psi}}^n$ be the elliptic projection of $\boldsymbol{\psi}^n$ onto $\boldsymbol{W}_h$ defined by exactly the same way as (4.15). Then using (4.19) and (4.22) with $\boldsymbol{\tau}_h=\tilde{\boldsymbol{\psi}}^n$, we get

$$\begin{aligned}
\|\boldsymbol{\rho}^n\|^2&=(\boldsymbol{\rho}^n,\tilde{a}^{-1}\boldsymbol{\psi}^n)-(\boldsymbol{\rho}^n,\nabla(c^{-1}\nabla\cdot\boldsymbol{\psi}^n))\\
&=(\tilde{a}^{-1}\boldsymbol{\rho}^n,\boldsymbol{\psi}^n)+(c^{-1}\nabla\cdot\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\psi}^n)\\
&=(\tilde{a}^{-1}\boldsymbol{\rho}^n,\boldsymbol{\psi}^n-\tilde{\boldsymbol{\psi}}^n)+(\tilde{a}^{-1}\boldsymbol{\rho}^n,\tilde{\boldsymbol{\psi}}^n)\\
&\quad+(c^{-1}\nabla\cdot\boldsymbol{\rho}^n,\nabla\cdot(\boldsymbol{\psi}^n-\tilde{\boldsymbol{\psi}}^n))+(c^{-1}\nabla\cdot\boldsymbol{\rho}^n,\nabla\cdot\tilde{\boldsymbol{\psi}}^n)\\
&=(\tilde{a}^{-1}\boldsymbol{\rho}^n,\boldsymbol{\psi}^n-\tilde{\boldsymbol{\psi}}^n)+(c^{-1}\nabla\cdot\boldsymbol{\rho}^n,\nabla\cdot(\boldsymbol{\psi}^n-\tilde{\boldsymbol{\psi}}^n))-(\tilde{a}^{-1}\boldsymbol{\pi}^n,\tilde{\boldsymbol{\psi}}^n)\\
&\quad-(c^{-1}\nabla\cdot\boldsymbol{\pi}^n,\nabla\cdot\tilde{\boldsymbol{\psi}}^n)+\frac{1}{\Delta t}(\hat{\eta}^{n-1}-\eta^{n-1},\nabla\cdot\tilde{\boldsymbol{\psi}}^n)\\
&\quad-\frac{1}{\Delta t}(\hat{\xi}^{n-1}-\xi^{n-1},\nabla\cdot\tilde{\boldsymbol{\psi}}^n)-(\tilde{a}^{-1}b\nabla(\eta^{n-1}-\xi^{n-1}),\tilde{\boldsymbol{\psi}}^n)\\
&\quad+(c^{-1}E_1^n,\nabla\cdot\tilde{\boldsymbol{\psi}}^n)+(\tilde{a}^{-1}E_2^n,\tilde{\boldsymbol{\psi}}^n)=\sum_{i=1}^{9}I_i. \qquad (4.23)
\end{aligned}$$

By using (4.21), Lemma 4.6, and the fact that $\|\boldsymbol{\psi}^n - \tilde{\boldsymbol{\psi}}^n\| \le ch^2\|\boldsymbol{\psi}^n\|_2$, we get the estimations of $I_1 \sim I_3$ as follows:

$$\begin{aligned}
I_1 =&(\tilde{a}^{-1}\boldsymbol{\rho}^n, \boldsymbol{\psi}^n - \tilde{\boldsymbol{\psi}}^n) \le K\|\boldsymbol{\rho}^n\|\|\boldsymbol{\psi}^n - \tilde{\boldsymbol{\psi}}^n\| \le Kh^2\|\boldsymbol{\rho}^n\|\|\boldsymbol{\psi}^n\|_2 \le Kh^2\|\boldsymbol{\rho}^n\|^2,\\
I_2 =&\Big(c^{-1}\nabla\cdot\boldsymbol{\rho}^n, \nabla\cdot(\boldsymbol{\psi}^n - \tilde{\boldsymbol{\psi}}^n)\Big) \le Kh\|\nabla\cdot\boldsymbol{\rho}^n\|\|\boldsymbol{\psi}^n\|_2 \le Kh\|\nabla\cdot\boldsymbol{\rho}^n\|\|\boldsymbol{\rho}^n\|\\
&\le Kh(h^{\mu-1}+\Delta t)\|\boldsymbol{\rho}^n\| \le K(h^\mu + \Delta t)\|\boldsymbol{\rho}^n\|,\\
I_3 =&-(\tilde{a}^{-1}\boldsymbol{\pi}^n, \tilde{\boldsymbol{\psi}}^n) \le K\|\boldsymbol{\pi}^n\|(\|\boldsymbol{\psi}^n - \tilde{\boldsymbol{\psi}}^n\| + \|\boldsymbol{\psi}^n\|)\\
&\le K\|\boldsymbol{\pi}^n\|(h^2\|\boldsymbol{\psi}^n\|_2 + \|\boldsymbol{\psi}^n\|)\\
&\le Kh^\mu\|\boldsymbol{\sigma}^n\|_s(h^2\|\boldsymbol{\rho}^n\| + \|\boldsymbol{\rho}^n\|) \le Kh^\mu\|\boldsymbol{\rho}^n\|.
\end{aligned}$$

By the definitions of $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\psi}}^n$ and Theorem 4.5, we have the estimations of $I_4 \sim I_6$ as follows:

$$\begin{aligned}
I_4 =&-(c^{-1}\nabla\cdot\boldsymbol{\pi}^n, \nabla\cdot\tilde{\boldsymbol{\psi}}^n) = \lambda(\boldsymbol{\pi}^n, \tilde{\boldsymbol{\psi}}^n) \le K\|\boldsymbol{\pi}^n\|\|\tilde{\boldsymbol{\psi}}^n\| \le Kh^\mu\|\boldsymbol{\rho}^n\|,\\
I_5 =&\frac{1}{\Delta t}(\hat{\eta}^{n-1} - \eta^{n-1}, \nabla\cdot\tilde{\boldsymbol{\psi}}^n) \le K\|\eta^{n-1}\|\|\tilde{\boldsymbol{\psi}}^n\|_2\\
\le&K\|\eta^{n-1}\|(\|\boldsymbol{\psi}^n - \tilde{\boldsymbol{\psi}}^n\|_2 + \|\boldsymbol{\psi}^n\|_2) \le K\|\eta^{n-1}\|\|\boldsymbol{\psi}^n\|_2 \le K(h^\mu)\|\boldsymbol{\rho}^n\|,\\
I_6 =&\frac{1}{\Delta t}(\hat{\xi}^{n-1} - \xi^{n-1}, \nabla\cdot\tilde{\boldsymbol{\psi}}^n) \le K\|\xi^{n-1}\|\|\boldsymbol{\psi}^n\|_2 \le K(h^\mu + \Delta t)\|\boldsymbol{\rho}^n\|.
\end{aligned}$$

Using the definition $\boldsymbol{\psi}_n$, Theorem 4.2, and Lemma 4.4, we estimate $I_7 \sim I_9$ as follows:

$$\begin{aligned}
I_7 =&-\Big(\tilde{a}^{-1}b\nabla(\eta^{n-1} - \xi^{n-1}), \tilde{\boldsymbol{\psi}}^n\Big)\\
=&\Big(\eta^{n-1} - \xi^{n-1}, (\nabla(\tilde{a}^{-1}b))\cdot\tilde{\boldsymbol{\psi}}^n\Big) + (\eta^{n-1} + \xi^{n-1}, (\tilde{a}^{-1}b)\nabla\cdot\tilde{\boldsymbol{\psi}}^n)\\
\le&K(\|\eta^{n-1}\| + \|\xi^{n-1}\|)\|\tilde{\boldsymbol{\psi}}^n\|_1\\
\le&K(\|\eta^{n-1}\| + \|\xi^{n-1}\|)(\|\boldsymbol{\psi}^n - \tilde{\boldsymbol{\psi}}^n\|_1 + \|\boldsymbol{\psi}^n\|_1) \le K(h^\mu + \Delta t)\|\boldsymbol{\rho}^n\|,\\
I_8 =&(c^{-1}E_1^n, \nabla\cdot\tilde{\boldsymbol{\psi}}^n) \le K\|E_1^n\|(\|\nabla\cdot(\boldsymbol{\psi}^n - \tilde{\boldsymbol{\psi}}^n)\| + \|\nabla\cdot\boldsymbol{\psi}^n\|) \le K\Delta t\|\boldsymbol{\rho}^n\|,\\
I_9 =&(\tilde{a}^{-1}E_2^n, \tilde{\boldsymbol{\psi}}^n) \le K\Delta t\|\boldsymbol{\rho}^n\|.
\end{aligned}$$

By applying the estimations of $I_1 \sim I_9$ to (4.23), we obtain

$$\|\boldsymbol{\rho}^n\|^2 \le Kh^2\|\boldsymbol{\rho}^n\|^2 + K(h^\mu + \Delta t)\|\boldsymbol{\rho}^n\|.$$

Therefore $\|\boldsymbol{\rho}^n\| \le K(h^\mu + \Delta t)$ holds for sufficiently small $h > 0$. Thus by the triangular inequality and Lemma 4.6, we obtain the result of this theorem. □

5. Numerical example

In this section, we will present some numerical results to verify the convergence order of the split least-squares CMFEM proposed in (4.8) and (4.9). For the sake of convenience, we consider the one dimensional convection dominated Sobolev equation (1.1) with $c(x) = d(x) = 1$, $a(x) = b(x) = 0.001$ and $\Omega = [0, 1]$.

We construct the approximation of $u(x,t)$ on the finite element space consisting of the piecewise linear polynomials defined on the uniform grids and the approximation of $\sigma(x,t)$ on the finite element space consisting of the piecewise quadratic polynomials defined on the uniform grids. Choose the exact solution $u(x,t)$ as follows:

$$u(x,t) = \begin{cases} (25(x-t-0.2)(0.6+t-x))^4, & 0.2 \le x-t \le 0.6, \\ 0, & \text{otherwise}, \end{cases} \tag{5.1}$$

and compute $f(x,t) = u_t + u_x - 10^{-3}u_{xx} - 10^{-3}u_{txx}$ by substituting $u(x,t)$ defined in (5.1). Notice that $u(x,t) \in H^4(\Omega)$ and $\sigma(x,t) \in H^2(\Omega)$

The numerical results for $u_h(x)$ at $T = 0.4$ are given in Table 1 in terms of the space mesh size h and the time mesh size $\triangle t$. We know from Table 1 that the convergence orders in L^2 and H^1 norms for u_h at $T = 0.4$ are consistent with the results in Theorem 4.5.

Table 1. The estimates for u_h

$(h, \triangle t)$	$\|u-u_h\|$	order(order/2)	$\|u-u_h\|_1$	order
$(1/20, 1/400)$	$0.236490e-1$		$1.037926e+0$	
$(1/40, 1/1600)$	$0.574480e-2$	2.04(1.02)	$0.509088e+0$	1.03
$(1/80, 1/6400)$	$0.141995e-2$	2.02(1.01)	$0.253263e+0$	1.01
$(1/160, 1/25600)$	$0.354024e-3$	2.00(1.00)	$0.126478e+0$	1.00
$(1/320, 1/102400)$	$0.886114e-4$	2.00(1.00)	$0.632202e-1$	1.00
$(1/640, 1/409600)$	$0.223377e-4$	1.99(1.00)	$0.316078e-1$	1.00

Table 2. The estimates for σ_h

$(h, \triangle t)$	$\|\sigma-\sigma_h\|$	order(order/2)
$(1/20, 1/400)$	$0.936948e-2$	
$(1/40, 1/1600)$	$0.236267e-2$	1.99(1.00)
$(1/80, 1/6400)$	$0.595235e-3$	1.99(1.00)
$(1/160, 1/25600)$	$0.150335e-3$	1.99(1.00)
$(1/320, 1/102400)$	$0.378825e-4$	1.99(1.00)
$(1/640, 1/409600)$	$0.952258e-5$	1.99(1.00)

The corresponding numerical results for σ_h at $T = 0.4$ are given in Table 2 in terms of the space mesh size h and the time mesh size $\triangle t$. We know from Table 2 that the convergence order in L^2 norm for σ_h at $T = 0.4$ is consistent with the result in Theorem 4.7.

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