STOCHASTIC INTEGRAL OF PROCESSES TAKING VALUES OF GENERALIZED OPERATORS†

BYOUNG JIN CHOI, JIN PIL CHOI AND UN CIG JI*

ABSTRACT. In this paper, we study the stochastic integral of processes taking values of generalized operators based on a triple $E \subset H \subset E^*$, where H is a Hilbert space, E is a countable Hilbert space and E^* is the strong dual space of E. For our purpose, we study E-valued Wiener processes and then introduce the stochastic integral of $\mathcal{L}(E, F^*)$-valued process with respect to an E-valued Wiener process, where F^* is the strong dual space of another countable Hilbert space F.

AMS Mathematics Subject Classification : 60H05.
Key words and phrases : countable Hilbert space, Q-Wiener process, generalized operator, stochastic integral.

1. Introduction

Since the stochastic integral for a standard Brownian motion initiated by K. Itô in [7], so called the Itô integral, the stochastic calculus has been successfully and extensively developed with wide applications to various fields with randomness. The stochastic integral is one of the main topics in the study of stochastic calculus. In Itô integral, the noise term to explain the diffusion part of a stochastic dynamics is corresponding to the white noise which is formally understood as the time derivative of a standard Brownian motion. Recently, to understand various random phenomena, the stochastic calculus associated with various noise processes has been studied by several authors.

In another direction to study of stochastic calculus, the stochastic integrals of operator-valued (stochastic) processes with respect to vector-valued noise processes have been studied by several authors [1, 2, 3, 5, 8, 10, 4], and the references cited therein. In particular, Applebaum [2], Curtain & Falb [3], Kunita [8] and

Da Prato & Zabczyk [4] studied the stochastic integral of operator-valued processes in a Hilbert space, and Albeverio & Rüdiger [1], Mamporia [10] and van Neerven & Weis [12] studied the stochastic integral of operator-valued processes with respect to a Banach space valued Wiener process. Also, Yadava [13] studied the stochastic integrals of operator-valued processes with respect to a Wiener process taking values in a locally convex space.

The main purpose of this paper is to develop the stochastic integral of generalized operator-valued stochastic process with respect to a Wiener process taking values in a countable Hilbert space. For our purpose, we focus on the following twofold:

(1) Wiener process values in a countable Hilbert space;
(2) the stochastic integral of processes taking values of generalized operators.

By taking $E = H$, from our Wiener process we construct a Wiener process taking values in a Hilbert space (see Remark 3.2). Therefore, our results are generalizations of the results for the case of Hilbert space. Also, our approach can be applied for the study of the Wiener process taking values in a locally convex space and the stochastic integral of processes taking values of generalized operators between locally convex spaces (see Remark 3.2).

This paper is organized as follows. In Section 2, we briefly review a standard construction of a countable Hilbert space E from a Hilbert space H and a triple: $E \subset H \subset E^*$, which is necessary for our study. In Section 3, we construct an E-valued Wiener process. In Section 4, we introduce a stochastic integral of processes taking values of generalized operators from a countable Hilbert space into the strong dual space of another countable Hilbert space.

2. Countable Hilbert Spaces

Let H be a real separable Hilbert space with the norm $\|\cdot\|_0$ induced by the inner product $\langle \cdot, \cdot \rangle$. Let A be a positive self-adjoint operator with the dense domain $\mathrm{Dom}\,(A) \subset H$ such that $\inf \mathrm{Spec}\,(A) > 0$. For each $p \geq 0$, the dense subspace $\mathrm{Dom}\,(A^p) \subset H$ equipped with the norm

$$\|\xi\|_p = \|A^p \xi\|_0, \quad \xi \in \mathrm{Dom}\,(A^p)$$

becomes a Hilbert space, which is denoted by E_p. Note that since $\inf \mathrm{Spec}\,(A) > 0$, A^{-1} is a bounded operator on H of which the operator norm is given by

$$\rho \equiv \|A^{-1}\|_{OP} = \{\inf \mathrm{Spec}\,(A)\}^{-1} < \infty.$$

For each $p \geq 0$, we define the norm $\|\cdot\|_{-p}$ on H by

$$\|\xi\|_{-p} = \|A^{-p} \xi\|_0, \quad \xi \in H.$$

Let E_{-p} be the completion of H with respect to the norm $\|\cdot\|_{-p}$. Then we have a chain of Hilbert spaces:

$$\cdots \subset E_p \subset \cdots \subset E_0 = H \subset \cdots \subset E_{-p} \subset \cdots, \tag{1}$$

where each inclusion is continuous and has a dense image, and

$$\|\xi\|_p \leq \rho^q \|\xi\|_{p+q}, \ \ \xi \in E, \ \ p \in \mathbf{R}, \ \ q \geq 0. \tag{2}$$

From a chain (1), we obtain the projective limit space:

$$E = \underset{p \to \infty}{\text{proj}\lim} E_p = \bigcap_{p \geq 0} E_p$$

equipped with the locally convex topology generated by the family $\{\|\cdot\|_p\}_{p \geq 0}$ of Hilbertian norms. Note that E is sequentially complete. Since the norms are linearly ordered as given in (2), we may choose a countable set of defining norms and so E becomes a countable Hilbert space. The strong dual space E^* of E is identified with the inductive limit space of $\{E_{-p}\}_{p \geq 0}$, i.e.,

$$E^* \cong \underset{p \to \infty}{\text{ind}\lim} E_{-p} = \bigcup_{p \geq 0} E_{-p}.$$

Thus, we have a triple:

$$E \subset H \subset E^*. \tag{3}$$

Note that if there exists $p > 0$ such that A^{-p} is a Hilbert-Schmidt operator, then E becomes a countable Hilbert nuclear space, and in this case, the triple given as in (3) is called a Gelfand triple. The canonical real bilinear form in $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$. For further study of the triple, we refer to [6, 11].

3. E-valued Wiener Processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A function $X : \Omega \to E$ is said to be Bochner measurable if there exists a sequence $\{X_n\}_{n=1}^{\infty}$ of simple functions from Ω to E such that $X_n(\omega)$ converges to $X(\omega)$ in E for almost all $\omega \in \Omega$. A function $X : \Omega \to E$ is said to be Bochner integrable if there exists a sequence $\{X_n\}_{n=1}^{\infty}$ of simple functions from Ω to E such that $X_n(\omega)$ converges to $X(\omega)$ in E for almost all $\omega \in \Omega$ and

$$\lim_{n \to \infty} \int_{\Omega} \|X_n - X\|_p \, dP = 0 \tag{4}$$

for any $p \geq 0$, and in this case, we define

$$\int_{\Omega} X dP = \lim_{n \to \infty} \int_{\Omega} X_n dP.$$

A Bochner integrable function $X : \Omega \to E$ is called a (E-valued) random variable and for which, we denote

$$\mathbf{E}[X] = \int_{\Omega} X dP,$$

which is called the expectation of X with respect to P.

For each $1 \leq \ell < \infty$, we denote $L^\ell(\Omega, \mathcal{F}, P; E)$ the space of all Bochner integrable functions $X : \Omega \to E$ such that $\int_{\Omega} \|X\|_p^\ell dP < \infty$ for all $p \geq 0$.

For two locally convex spaces E and F, we denote $\mathcal{L}(E, F)$ the space of all continuous linear operators from E into F. If E is a nuclear Fréchet space and F is a Fréchet space, then

$$\mathcal{L}(E, F^*) \cong E^* \otimes_\pi F^*$$

by the kernel theorem (see [11]), where $E^* \otimes_\pi F^*$ is the π-tensor product of E^* and F^*. Therefore, an operator in $\mathcal{L}(E, F^*)$ is sometimes called a generalized operator. It is well-known that for each $\Phi \in \mathcal{L}(E, F^*)$, there exist $p, q \geq 0$ such that $\Phi \in \mathcal{L}(E_q, F_{-p})$. In this case, the operator norm of Φ is denoted by $\|\Phi\|_{q,-p}$.

From now on, let $Q \in \mathcal{L}(E^*, E)$ be given by

$$Q = S^* S$$

for some $S \in \mathcal{L}(E^*, E)$, and let $\{\xi_n\}_{n=1}^\infty$ be a sequence of mutually independent real-valued Brownian motions.

Proposition 3.1. *Suppose that for any $p \geq 0$,*

$$C(p) := \sum_{n=1}^\infty \|S^* e_n\|_p^2 < \infty \tag{5}$$

for a complete orthonormal basis $\{e_n\}_{n=1}^\infty \subset E$ for H. Then there exists an E-valued stochastic process $\{W_t\}_{t \geq 0} \subset L^2(\Omega, \mathcal{F}, P; E)$ such that

   (i) $\mathbf{E}[W_t] = 0$ *for all $t \geq 0$;*
   (ii) *for any $p \geq 0$, it holds that*

$$E\left[\|W_t - W_s\|_p^2\right] = C(p)|t - s| \tag{6}$$

   for any $s, t \geq 0$, where $C(p)$ is given as in (5);
   (iii) *for any $\phi, \psi \in E^*$, $\mathbf{E}\left[\langle \phi, W_s \rangle \langle \psi, W_t \rangle\right] = \min\{s, t\}\langle \phi, Q\psi \rangle$;*
   (iv) $\{W_t\}_{t \geq 0}$ *has independent increments.*

Proof. The proof is a modification of the proof of Proposition 4.2 in [4]. Let $\{e_n\}_{n=1}^\infty \subset E$ be a complete orthonormal basis for H such that (5) holds. For each $t \geq 0$, we can easily see that the series of E-valued random variables:

$$\sum_{n=1}^\infty S^*(e_n)\xi_n(t)$$

is summable in E almost surely on Ω. In fact, for each $p \geq 0$ and any $m, n \in \mathbb{N}$, we obtain that

$$\mathbf{E}\left[\left\|\sum_{k=m}^{m+n} S^*(e_k)\xi_k(t)\right\|_p^2\right] = t\left(\sum_{k=m}^{m+n} \|S^*(e_k)\|_p^2\right), \tag{7}$$

which implies that the sequence $\{\sum_{k=1}^n S^*(e_n)\xi_n(t)\}_{n=1}^\infty$ converges in $L^2(\Omega, \mathcal{F}, P; E_p)$ for any $p \geq 0$, and so converges in $L^2(\Omega, \mathcal{F}, P; E)$. Therefore, for each $t \geq 0$, we define W_t by

$$W_t = \sum_{n=1}^\infty S^*(e_n)\xi_n(t) \in E.$$

Then $\{W_t\}_{t \geq 0}$ is an E-valued stochastic process and it is obvious that $\mathbf{E}[W_t] = 0$ for all $t \geq 0$. On the other hand, by applying the dominated convergence theorem, we obtain that

$$\mathbf{E}\left[\left\|\sum_{n=1}^{\infty} S^*(e_n)\left(\xi_n(t) - \xi_n(s)\right)\right\|_p^2\right] = (t - s)\left(\sum_{n=1}^{\infty} \|S^*(e_n)\|_p^2\right), \qquad (8)$$

which proves (6). For any $\phi, \psi \in E^*$, we obtain that

$$\mathbf{E}\left[\langle \phi,\, W_s \rangle \langle \psi,\, W_t \rangle\right] = \sum_{m,n=1}^{\infty} \mathbf{E}\left[\xi_n(s)\xi_m(t)\right]\langle \phi,\, S^*(e_n)\rangle \langle \psi,\, S^*(e_m)\rangle$$

$$= \min\{s,t\}\left(\sum_{n=1}^{\infty} \langle S\phi,\, e_n\rangle \langle S\psi,\, e_n\rangle\right)$$

$$= \min\{s,t\}\langle \phi,\, S^*S\psi \rangle,$$

which gives the proof of (iii). Let $0 \leq s < t \leq u < v$ and $\phi, \psi \in E^*$. Then we obtain that

$$\mathbf{E}\left[\langle \phi,\, W_t - W_s \rangle \langle \psi,\, W_v - W_u \rangle\right]$$

$$= \sum_{m,n=1}^{\infty} \mathbf{E}\left[\left(\xi_n(t) - \xi_n(s)\right)\left(\xi_m(v) - \xi_m(u)\right)\right]\langle \phi,\, S^*(e_n)\rangle \langle \psi,\, S^*(e_m)\rangle$$

$$= \mathbf{E}\left[\left\langle \phi,\, \sum_{n=1}^{\infty} S^*(e_n)\left(\xi_n(t) - \xi_n(s)\right)\right\rangle\right]\mathbf{E}\left[\left\langle \psi,\, \sum_{m=1}^{\infty} S^*(e_m)\left(\xi_m(v) - \xi_m(u)\right)\right\rangle\right]$$

$$= \mathbf{E}\left[\langle \phi,\, W_t - W_s \rangle\right]\mathbf{E}\left[\langle \psi,\, W_v - W_u \rangle\right],$$

which means that $\{W_t\}_{t \geq 0}$ has independent increments. $\qquad \square$

Remark 3.2. Since every absolutely summable sequence is square summable, the condition that

$$\sum_{n=1}^{\infty} \|S^*e_n\|_p < \infty \qquad (9)$$

is stronger than the condition given as in (5). For a special case of the operator S and a locally convex space E, a condition similar to the condition given as in (9) can be found in [13]. In the case of Hilbert space valued Wiener processes studied in [4], the condition given as in (9) implies that Q is a Hilbert-Schmidt operator. But for the study of Hilbert space valued Wiener processes, it is enough to assume that Q is a trace-class operator (see [4]). In this sense, the condition given as in (5) is natural.

Remark 3.3. In Proposition 3.1, the identity given as in (7) is essential. If E is a locally convex space, then as a counterpart of (7), by applying the Cauchy-Schwarz inequality, we obtain that

$$\mathbf{E}\left[\left\|\sum_{k=m}^{m+n} S^*(e_k)\xi_k(t)\right\|_p^2\right] \leq \mathbf{E}\left[\left(\sum_{k=m}^{m+n} \|S^*(e_k)\|_p |\xi_k(t)|\right)^2\right]$$

$$\leq \mathbf{E}\left[\left(\sum_{k=m}^{m+n} \|S^*(e_k)\|_p |\xi_k(t)|^2\right)\left(\sum_{k=m}^{m+n} \|S^*(e_k)\|_p\right)\right]$$

$$= t\left(\sum_{k=m}^{m+n} \|S^*(e_k)\|_p\right)^2,$$

for which we used only the triangle inequality and the Cauchy-Schwarz inequality, and so it is a sharp iequality. Therefore, for the study of Q-Wiener processes in a locally convex space, the condition given as in (9) is natural.

Definition 3.4. Let $Q = S^*S \in \mathcal{L}(E^*, E)$ be a nonnegative symmetric operator for an operator $S \in \mathcal{L}(E^*, E)$ satisfying (5). An E-valued stochastic process $\{W_t\}_{t \geq 0} \subset L^2(\Omega, \mathcal{F}, P; E)$ is called a Q-Wiener process if

    (i) $W_0 = 0$ a.s.;

    (ii) W has continuous sample paths;

    (iii) for every $0 \leq s < t$, $W_t - W_s$ is a Gaussian random variable with mean 0 and the covariance operator $(t-s)Q$ given by the following:

$$\mathbf{E}\left[\langle \phi, W_t - W_s \rangle \langle \psi, W_t - W_s \rangle\right] = (t-s)\langle \psi, Q\phi \rangle \qquad (10)$$

        for any $\phi, \psi \in E^*$;

    (iv) W has independent increments.

Theorem 3.5. *Let* $Q = S^*S \in \mathcal{L}(E^*, E)$ *be a nonnegative symmetric operator for an operator* $S \in \mathcal{L}(E^*, E)$ *satisfying* (5). *Then there exists an* E-valued Q-Wiener process $\{W_t\}_{t \geq 0}$.

Proof. The proof is immediate from Proposition 3.1. $\square$

Proposition 3.6. *Let* $\{W_t\}_{t \geq 0}$ *be an* E-*valued* Q-*Wiener process. Suppose that there exist a complete orthonormal basis* $\{e_n\}_{n=1}^{\infty}$ *for* H *and a sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *of nonnegative real numbers such that for any* $p \geq 0$,

$$\sum_{n=1}^{\infty} \lambda_n \|e_n\|_p^2 < \infty \quad and \quad Qe_n = \lambda_n e_n, \qquad n \in \mathbb{N}.$$

Then there exists a sequence $\{\xi_n\}_{n=1}^{\infty}$ *of mutually independent real-valued Wiener processes such that*

$$W_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \xi_n(t) e_n, \qquad (11)$$

where the series given as in (11) *converges in* $L^2(\Omega, \mathcal{F}, P; E)$.

Proof. The proof is a modification of the proof of Proposition 4.1 in [4]. For each $n \in \mathbb{N}$, define

$$\zeta_n(t) = \langle e_n, W_t \rangle, \ t \geq 0,$$

and put $\Lambda = \{n \in \mathbb{N} \,|\, \langle Qe_n, e_n \rangle > 0\}$. Then it is easy to see that $\{\zeta_n\}_{\in \Lambda}$ is a sequence of mutually independent Gaussian process such that for any $m, n \in \Lambda$,

$$\mathbf{E}\left[\zeta_n(s)\zeta_m(t)\right] = \min\{s, t\} \langle Q(e_n), e_m \rangle = \min\{s, t\}\lambda_n\delta_{mn}. \tag{12}$$

Therefore, we define

$$\xi_n(t) = \begin{cases} \frac{1}{\sqrt{\lambda_n}}\zeta_n(t), & n \in \Lambda, \\ \eta_n(t), & n \in \mathbb{N} \setminus \Lambda, \end{cases}$$

where $\{\eta_n\}_{n \in \mathbb{N} \setminus \Lambda}$ is any sequence of mutually independent real-valued Brownian motions. Then it holds that

$$\sum_{n=1}^{\infty} \langle e_n, W_t \rangle e_n = \sum_{n \in \Lambda} \langle e_n, W_t \rangle e_n + \sum_{n \in \mathbb{N} \setminus \Lambda} \langle e_n, W_t \rangle e_n$$

$$= \sum_{n=1}^{\infty} \sqrt{\lambda_n}\xi_n(t)e_n,$$

where the last identity is as in $L^2(\Omega, \mathcal{F}, P; E)$. In fact, it holds that

$$\mathbf{E}\left[\left\|\sum_{n \in \mathbb{N} \setminus \Lambda} \langle e_n, W_t \rangle e_n\right\|_p^2\right] = \sum_{n \in \mathbb{N} \setminus \Lambda} \langle Qe_n, e_n \rangle \|e_n\|_p^2 = 0.$$

Therefore, to complete the proof, it is enough to see the identity:

$$W_t = \sum_{n=1}^{\infty} \langle e_n, W_t \rangle e_n,$$

and then, since the identity holds as in $L^2(\Omega, \mathcal{F}, P; H)$, it is enough to see that the series converges in $L^2(\Omega, \mathcal{F}, P; E)$, which follows from that

$$\lim_{n \to \infty} \mathbf{E}\left[\left\|\sum_{k=n}^{\infty} \sqrt{\lambda_k}\xi_k(t)e_k\right\|_p^2\right] = \lim_{n \to \infty} \mathbf{E}\left[\left\|\sum_{k=n}^{\infty} \sqrt{\lambda_k}\xi_k(t)A^pe_k\right\|_0^2\right]$$

$$= t \lim_{n \to \infty} \sum_{k=n}^{\infty} \lambda_k\|e_k\|_p^2 = 0.$$

Hence, the proof is completed. $\qquad\square$

4. Stochastic Integration of $\mathcal{L}(E, F^*)$-valued Process

Let H and K be Hilbert spaces. Then we have chains such as (1): for $p, q \geq 0$,

$$E \subset \cdots \subset E_p \subset \cdots \subset E_0 = H \subset \cdots \subset E_{-p} \subset \cdots \subset E^*,$$
$$F \subset \cdots \subset F_q \subset \cdots \subset F_0 = K \subset \cdots \subset F_{-q} \subset \cdots \subset F^*.$$

Let $\{W_t\}_{t \geq 0}$ be an E-valued Q-Wiener process. For each $t \geq 0$, we denote $\mathcal{F}_t$ the σ-algebra generated by $\{W_s | 0 \leq s \leq t\}$. Then $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is a filtration, i.e., an increasing family of σ-algebras.

A $\mathcal{L}(E, F^*)$-valued stochastic process $\{\Phi_t\}_{t \geq 0}$ is said to be $\mathcal{F}$-adapted (or simply adapted) if for each $t \geq 0$, Φ_t is $\mathcal{F}_t$-measurable.

For a $\mathcal{L}(E, F^*)$-valued (adapted) simple process $\Phi = \{\Phi_t\}_{t \geq 0}$ of the form

$$\Phi_t = \sum_{i=1}^{N-1} \Phi_{t_i} \mathbf{1}_{(t_i, t_{i+1}]}(t), \tag{13}$$

where $0 = t_1 < t_2 < \cdots < t_N = T$ and Φ_{t_i} $(i = 1, \cdots, N-1)$ is $\mathcal{F}_{t_i}$-measurable, the stochastic integral of Φ with respect to the E-valued Q-Wiener process $\{W_t\}_{t \geq 0}$ is a F^*-valued stochastic process defined by

$$\int_0^T \Phi_t dW_t = \sum_{i=1}^{N-1} \Phi_{t_i}(W_{t_{i+1}} - W_{t_i}).$$

Proposition 4.1. *For a $\mathcal{L}(E, F^*)$-valued simple process $\Phi = \{\Phi_t\}_{t \geq 0}$ of the form given as in (13), we have*

(i) $\mathbf{E}\left[\int_0^T \Phi_t dW_t\right] = 0$;

(ii) *for some $p \geq 0$,*

$$\mathbf{E}\left[\left\|\int_0^T \Phi_t dW_t\right\|_{-p}^2\right] = \int_0^T \mathbf{E}\left[\sum_{n=1}^\infty \langle \Phi_t S^*(e_n), \Phi_t S^*(e_n)\rangle_{-p}\right] dt;$$

(iii) $\mathbf{E}\left[\left\|\int_0^T \Phi_t dW_t\right\|_{-p}^2\right] \leq C(q) \int_0^T \mathbf{E}\left[\|\Phi_t\|_{q,-p}^2\right] dt$ *for some $p, q \geq 0$, where* $C(q)$ *is given as in (5).*

Proof. (i) For any $\zeta \in F$, we obtain that

$$\left\langle \int_0^T \Phi_t dW_t, \zeta \right\rangle_{F^* \times F} = \sum_{i=1}^{N-1} \left\langle \Phi_{t_i}(W_{t_{i+1}} - W_{t_i}), \zeta \right\rangle_{F^* \times F}$$
$$= \sum_{i=1}^{N-1} \left\langle \Phi_{t_i}^*(\zeta), W_{t_{i+1}} - W_{t_i} \right\rangle_{E^* \times E}.$$

Therefore, the proof of (i) is immediate since $\{W_t\}_{t\geq 0}$ is an E-valued Q-Wiener process and

$$\mathbf{E}\left[\langle \phi, W_t \rangle\right] = 0$$

for any $\phi \in E^*$.

(ii) For some $p, q \geq 0$, $\{\Phi_t\}_{t\geq 0} \subset \mathcal{L}(E_q, F_{-p})$ and so we obtain that

$$\mathbf{E}\left[\left\|\int_0^T \Phi_t dW_t\right\|_{-p}^2\right]$$

$$= \mathbf{E}\left[\sum_{m,n=1}^\infty \sum_{i,j=1}^{N-1} \langle \Phi_{t_i}(W_{t_{i+1}} - W_{t_i}), f_n \rangle \langle \Phi_{t_j}(W_{t_{j+1}} - W_{t_j}), f_m \rangle \langle f_n, f_m \rangle_{-p}\right],$$

where $\langle \cdot, \cdot \rangle$ is the bilinear form on $F^* \times F$ and $\{f_n\}_{n=1}^\infty \subset F$ is a complete orthonormal basis for the given Hilbert space K. Therefore, we have

$$\mathbf{E}\left[\left\|\int_0^T \Phi_t dW_t\right\|_{-p}^2\right]$$

$$= \sum_{i=1}^{N-1} (t_{i+1} - t_i) \mathbf{E}\left[\sum_{m,n=1}^\infty \langle S(\Phi_{t_i}^*(f_m)), S(\Phi_{t_i}^*(f_n)) \rangle \langle f_n, f_m \rangle_{-p}\right],$$

where $\{f_n\}_{n=1}^\infty \subset F$ is a complete orthonormal basis for the given Hilbert space K. On the other hand, we obtain that

$$\sum_{m,n=1}^\infty \langle S(\Phi_{t_i}^*(f_m)), S(\Phi_{t_i}^*(f_n)) \rangle \langle f_n, f_m \rangle_{-p}$$

$$= \sum_{l=1}^\infty \sum_{m,n=1}^\infty \langle S(\Phi_{t_i}^*(f_m)), e_l \rangle \langle e_l, S(\Phi_{t_i}^*(f_n)) \rangle \langle f_n, f_m \rangle_{-p}$$

$$= \sum_{l=1}^\infty \left\langle \sum_{n=1}^\infty \langle \Phi_{t_i} S^*(e_l), f_n \rangle f_n, \sum_{m=1}^\infty \langle \Phi_{t_i} S^*(e_l), f_m \rangle f_m \right\rangle_{-p}$$

$$= \sum_{l=1}^\infty \langle \Phi_{t_i} S^*(e_l), \Phi_{t_i} S^*(e_l) \rangle_{-p}.$$

Hence we have

$$\mathbf{E}\left[\left\|\int_0^T \Phi_t dW_t\right\|_{-p}^2\right] = \int_0^T \sum_{l=1}^\infty \mathbf{E}\left[\langle \Phi_t S^*(e_l), \Phi_t S^*(e_l) \rangle_{-p}\right] dt,$$

which gives the proof.

(iii) From (ii), we obtain that

$$\mathbf{E}\left[\left\|\int_0^T \Phi_t dW_t\right\|_{-p}^2\right] \leq \int_0^T \sum_{l=1}^\infty \mathbf{E}\left[\|\Phi_t\|_{q,-p}^2\right]\|S^*(e_l)\|_q^2 dt$$

$$= C(q)\int_0^T \mathbf{E}\left[\|\Phi_t\|_{q,-p}^2\right]dt,$$

which gives the proof.   □

Let $L^2(E, F^*)$ be the space of all (adapted) $\mathcal{L}(E, F^*)$-valued process such that

$$\int_0^T \mathbf{E}\left[\|\Phi_t\|_{q,-p}^2\right]dt < \infty$$

for some $p, q \geq 0$, and let $L_0^2(E, F^*)$ the space of all (adapted) simple process $\Phi \in L^2(E, F^*)$ of the form given as in (13). We define a map $I : L_0^2(E, F^*) \rightarrow L^2(\Omega, \mathcal{F}, P; F^*)$ by

$$I(\Phi) = \int_0^T \Phi_t dW_t.$$

From (iii) in Proposition 4.1, we see that I is a continuous linear map from $L_0^2(E, F^*)$ into $L^2(\Omega, \mathcal{F}, P; F^*)$. Since $L^2(\Omega, \mathcal{F}, P; F^*)$ is complete and $L_0^2(E, F^*)$ is dense in $L^2(E, F^*)$, we extend I as a continuous linear map from $L^2(E, F^*)$ into $L^2(\Omega, \mathcal{F}, P; F^*)$. Therefore, for $\Phi \in L^2(E, F^*)$, we define

$$I(\Phi) = \int_0^T \Phi_t \, dW_t,$$

which is called the stochastic integral of Φ with respect to Q-Wiener process $\{W_t\}_{t\geq 0}$.

Theorem 4.2. *For any $\mathcal{L}(E, F^*)$-valued process Φ in $L^2(E, F^*)$, we have*

(i) $\mathbf{E}\left[\int_0^T \Phi_t dW_t\right] = 0,$

(ii) $\mathbf{E}\left[\left\|\int_0^T \Phi_t dW_t\right\|_{-p}^2\right] \leq C(q)\int_0^T \mathbf{E}\left[\|\Phi_t\|_{q,-p}^2\right]dt$ *for some $p, q \geq 0$.*

Proof. (i) It is obvious from (i) in Proposition 4.1.

(ii) Let $\{\Phi_n\}_{n=1}^\infty$ be a sequence of simple processes in $L_0^2(E, F^*)$ such that Φ_n converges to Φ in $L^2(E, F^*)$ and $I(\Phi_n)$ converges to $I(\Phi)$. Then for some $p, q \geq 0$ and any arbitrary $\epsilon > 0$, by applying the Cauchy-Schwarz inequality, we obtain that

$$\mathbf{E}\left[\left\|\int_0^T \Phi_t dW_t\right\|_{-p}^2\right] \leq \left(1 + \frac{1}{\epsilon}\right)\mathbf{E}\left[\left\|\int_0^T (\Phi - \Phi_n)_t \, dW_t\right\|_{-p}^2\right]$$

$$+ (1 + \epsilon)\mathbf{E}\left[\left\|\int_0^T (\Phi_n)_t dW_t\right\|_{-p}^2\right]$$

$$\leq \left(1 + \frac{1}{\epsilon}\right) C(q) \int_0^T \mathbf{E}\left[\|(\Phi_n - \Phi)_t\|_{q,-p}^2 \, dt\right]$$

$$+ (1 + \epsilon)C(q) \int_0^T \mathbf{E}\left[\|(\Phi_n)_t\|_{q,-p}^2 \, dt\right],$$

from which, by taking limit as $n \to \infty$, we obtain that

$$\mathbf{E}\left[\left\|\int_0^T \Phi_t dW_t\right\|_{-p}^2\right] \leq (1 + \epsilon)C(q) \lim_{n\to\infty} \int_0^T \mathbf{E}\left[\|(\Phi_n)_t\|_{q,-p}^2 \, dt\right]$$

$$= (1 + \epsilon)C(q) \int_0^T \mathbf{E}\left[\|\Phi_t\|_{q,-p}^2 \, dt\right].$$

Since $\epsilon > 0$ is arbitrary, we prove the desired result. $\qquad\square$

The study of stochastic differential equations and stochastic control systems (see [9]) is in progress.

References

1. S. Albeverio and B. Rüdiger, *Stochastic integrals and the Lévy-Itô decomposition theorem on separable Banach space*, Stoch. Anal. and Appl. **23** (2005), 217–253.
2. D. Applebaum, *Matingale-valued measure, Ornstein-Uhlenbeck process with jump and operator self-decomposabliity in Hilbert space*, Lecture Notes in Math. **1874** (2006), 173–198.
3. R.F. Curtain and P.L. Falb, *Ito's lemma in infinite dimensions*, J. Math. Anal. Appl. **31** (1970), 434–448.
4. G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.
5. E. Dettweiler, *Banach space valued processes with independent increments and stochastic integration*, in "Probability in Banach spaces IV", Lecture Notes in Math. **990** (1982), 54–83.
6. I.M. Gelfand and N.Y. Vilenkin, *Generalized Functions, Vol. 4*, Academic Press, 1964.
7. K. Itô, *Stochastic integral*, Proc. Imp. Acad. Tokyo **20** (1944), 519–524.
8. H. Kunita, *Stochastic integrals based on martingales taking values in Hilbert space*, Nagoya Math. J. **38** (1970), 41–52.
9. P. Lu and C. Gao, *Existence-and-uniqueness and mean-square boundedness of the solution to stochastic control systems,* J. Appl. Math. & Informatics **31** (2013), 513–522.
10. B.I. Mamporia, *Wiener processes and stochastic integrals in a Banach space*, Probab. Math. Statist. **7** (1986), 59–75.
11. N. Obata, *White Noise Calculus and Fock Space*, Lecture Notes in Math. **1577** (1994).
12. J.M.A.M. van Neerven and L. Weis, *Stochastic integration of functions with values in a Banach space*, Studia Math. **166** (2005), 131–170.
13. S.L. Yadava, *Stochastic evolution equations in locally convex space*, Proc. Indian Acad. Sci. Math. Sci. **95** (1986), 79–96.

Un Cig Ji received M.Sc. in 1991 and Ph.D in 1996 from Sogang University. He is currently a professor at Chungbuk National University since 2001. His research interests are

Functional Analysis, Infinite Dimensional Analysis, White Noise Theory, Quantum Probability and Stochastic Analysis.

Department of Mathematics, Research institute of Mathematical Finance, Chungbuk National University, Cheongju 28644, Korea
uncigji@chungbuk.ac.kr

Byoung Jin Choi received M.Sc. and Ph.D. from Chunbuk National University. His research interests are Functional Analysis, Quantum Probability and Stochastic Analysis.

Department of Mathematics, Sungkyunkwan University, Suwon 16419, Korea
choibj@skku.edu

Jin Pil Choi received M.Sc. from Chunbuk National University. His research interests are Stochastic Analysis and Applications.

Department of Mathematics, Chungbuk National University, Cheongju 28644, Korea
jinpilchoi@chungbuk.ac.kr