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TRIPLE AND FIFTH PRODUCT OF DIVISOR FUNCTIONS AND TREE MODEL†

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ABSTRACT. It is known that certain convolution sums can be expressed as a combination of divisor functions and Bernoulli formula. In this article, we consider relationship between fifth-order combinatoric convolution sums of divisor functions and Bernoulli polynomials. As applications of these identities, we give a concrete interpretation in terms of the procedural modeling method.

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1. Introduction

Throughout this paper, the symbols $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and the ring of integers respectively. The classical Bernoulli polynomials $B_n(x)$ is usually defined by means of the following generating function:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \qquad (|t|<2\pi).$$

The corresponding Bernoulli number B_n is given by

$$B_n:=B_n(0)=(-1)^nB_n(1)=\left(2^{1-n}-1\right)^{-1}B_n\left(\frac{1}{2}\right) \quad (n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}).$$

The Bernoulli polynomial is expressed through the respective numbers and polynomials

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}, \tag{1.1}$$

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$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k}. \tag{1.2}$$

For $n\in\mathbb{N},\ k\in\mathbb{Z}$ and $l\in\{0,1\}$ we define some divisor functions

$$\sigma_k(n):=\sum_{d|n}d^k,\quad \sigma_k^*(n):=\sum_{\substack{d|n\\ \frac{n}{d}\text{ odd}}}d^k,\quad \sigma_{k,l}(n;2):=\sum_{\substack{d|n\\ d\equiv l(\text{mod}2)}}d^k. \tag{1.3}$$

The identity

$$\sum_{k=1}^{n-1}\sigma_1(k)\sigma_1(n-k)=\frac{5}{12}\sigma_3(n)+\left(\frac{1}{12}-\frac{1}{2}n\right)\sigma_1(n) \tag{1.4}$$

for the basic convolution sum first appeared in a letter from *Besge to Liouville* in 1862 (See [2]). Recently, the study of convolution formulas for divisor functions can be found in B.C. Berndt [1], J.W.L. Glaisher [3], H. Hahn [4], J.G. Huard et al. [5], D. Kim et al. [8], G. Melfi [12] and K.S. Williams [13]. We are motivated by Ramanujan's recursion formula for sums of the product of two Eisenstein series (See [1]) and its proof, and also the following identities (See [13]):

$$\begin{aligned}&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\ &=\frac{2k+3}{4k+2}\sigma_{2k+1}(n)+\left(\frac{k}{6}-n\right)\sigma_{2k-1}(n)+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k-2j+1}(n).\end{aligned} \tag{1.5}$$

In this paper we focus on the combinatorial convolution sums. For positive integers l and N, the combinatorial convolution sum

$$\sum_{\substack{a_1+a_2+\cdots+a_n=k\\ a_1,a_2,\ldots,a_{n-1}\text{ odd}}}\binom{k}{a_1,a_2,\ldots,a_n}\sum_{\substack{m_1+m_2+\cdots+m_s=N\\ m_i,\ldots,m_j\text{ odd}}}\sigma_{a_1}(m_1)\cdots\sigma_{a_s}(m_s) \tag{1.6}$$

can be evaluated explicitly in terms of divisor functions. The aim of this article is to study fourth and fifth-order combinatorial convolution sums of the analogous types of (1.5) and (1.6). More precisely, we prove the following results.

Theorem 1.1. *For $k,q,n\in\mathbb{N}$ and $k,n\geq 2$, we have*

(a)

$$\begin{aligned}&\sum_{\substack{a,b\ odd\\ 1\leq m\leq n-1}}\binom{k}{a,b,c}\sigma_a^*(m)\sigma_b^*(n-m)\sigma_c^*(q)\\ &=\frac{1}{4(k+1)}\sum_{\substack{x|q,\ \frac{q}{x}\ odd\\ y|n,\ \frac{n}{y}\ odd}}\left(y-\frac{n}{y}\right)\times\Big(B_{k+1}(x+y+1)-B_{k+1}(x+y)\\ &\qquad+B_{k+1}(x-y+1)-B_{k+1}(x-y)\Big)-\frac{1}{2}\sigma_k^*(q)\sigma_1(\frac{n}{2}).\end{aligned}$$

(b)

$$\sum_{\substack{a,b\ odd\\1\le m\le n-1}} 2^c\binom{k}{a,b,c}\sigma_{a,1}(m;2)\sigma_{b,1}(n-m;2)\sigma_{c,1}(q;2)$$

$$=\frac{1}{8(k+1)}\sum_{\substack{x|n,\ x\ even\\y|q,\ y\ odd}} x\times\Big\{B_{k+1}(x+2y+1)-B_{k+1}(x+2y)+(-1)^k\big(B_{k+1}(x-2y+1)$$

$$-B_{k+1}(x-2y)\big)\Big\}+\frac{2^{k-1}}{k+1}\sum_{\substack{x|n,\ x\ odd\\y|q,\ y\ odd}}\left\{B_{k+1}\left(\frac{x+1}{2}+y\right)+(-1)^kB_{k+1}\left(\frac{x+1}{2}-y\right)\right\}$$

$$-2^{k-1}\sigma_1^*(n)\sigma_{k,1}(q;2).$$

Theorem 1.2. *Let $n\ge 4$ be an even integer with $l,q\in\mathbb{N}-\{1\}$. Then*

$$\sum_{\substack{a+b+c+d+e=2l+1\\a,b,c,d,e\ odd}}\binom{2l}{a-1,b,c,d,e}$$

$$\times\sum_{\substack{m_1+m_2+m_3=n\\m_4+m_5=q\\m_3\ even}}(-1)^{m_1+1}\sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)\sigma_{d,1}(m_4;2)\sigma_{e,1}(m_5;2)$$

$$=\frac{n}{256(2l+1)}\sum_{\substack{x|q,\ x\ even\\y|n,\ \frac{n}{y}\ odd}}\Big(xy-\frac{2nx}{y}\Big)$$

$$\times\Big(B_{2l+1}(x+y+1)-B_{2l+1}(x+y)+B_{2l+1}(x-y+1)-B_{2l+1}(x-y)\Big)$$

$$+\frac{2^{2l}n}{64(2l+1)}\sum_{\substack{x|q,\ x\ odd\\y|n,\ \frac{n}{y}\ odd}}\Big(y-\frac{2n}{y}\Big)\Big(B_{2l+1}\left(\frac{x+y+1}{2}\right)+B_{2l+1}\left(\frac{x-y+1}{2}\right)\Big)$$

$$-\frac{n}{64}\Big\{\Big(\sigma_{2l+1}^*(n)-2n\sigma_{21-1}^*(n)\Big)\Big(\sigma_1\left(\frac{q}{2}\right)+\sigma_{1,1}(q;2)\Big)$$

$$+\Big(\sigma_1^*(n)-2n\sigma_{-1}^*(n)\Big)\Big(2^{2l}\sigma_{2l+1}\left(\frac{q}{2}\right)+\frac{2^{2l+1}}{2l+1}\sum_{x|q,\ x\ odd}B_{2l+1}\left(\frac{x+1}{2}\right)\Big)\Big\}.$$

Finally, we introduce a divisor tree model using Theorem 1.1 in Section 3.

2. Proof of Theorems

In this section, we will discuss some relationships between the Bernoulli polynomials and the combinatoric convolution sums of divisor functions.

Propositin 2.1 ([6, 7, 9])**.** *Let $n\ge 2$, $N\ge 4$, $k,l\in\mathbb{N}$ and N be an even integer. Then we have*

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(n-m;2)\\
=&\frac{1}{4}\sigma_{2k+1,0}(n;2)+\frac{2^{2k}}{2k+1}\sum_{\substack{d|n\\ d\ odd}}B_{2k+1}\left(\frac{d+1}{2}\right),\\
&\sum_{s=o}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma^*_{2s+1}(n-m)\\
=&\frac{1}{2}\Big(\sigma^*_{2k+1}(n)-n\sigma^*_{2k-1}(n)\Big)
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
&\sum_{\substack{a+b+c=2l+1\\ a,b,c\ odd}} a\binom{2l+1}{a,b,c}\sum_{\substack{m_1+m_2+m_3=N\\ m_3\ even}}(-1)^{m_1+1}\sigma^*_a(m_1)\sigma^*_b(m_2)\sigma^*_c(m_3)\\
=&\frac{(2l+1)N}{32}\{\sigma^*_{2l+1}(N)-2N\sigma^*_{2l-1}(N)\}.
\end{aligned} \tag{2.2}$$

Remark 2.2. The different expression of (2.1) is in [8, Theorem 3].

To prove of Theorem 1.1, we need the following lemma.

Lemma 2.3. *Let k be a positive integer. Then*

$$\begin{aligned}
(a)\quad &\sum_{s=0}^{[\frac{k-1}{2}]-1}\binom{k}{2s+1}B_{k-2s-1}(x)y^{2s+1}\\
=&\frac{1}{2}\left[B_k(x+y)-B_k(x-y)-\{k(2x-1)\}^{\frac{1+(-1)^k}{2}}y^{2[\frac{k-1}{2}]+1}\right].\\
(b)\quad &\sum_{s=0}^{[\frac{k}{2}]-1}\binom{k}{2s}B_{k-2s}(x)y^{2s}\\
=&\frac{1}{2}\left[B_k(x+y)+B_k(x-y)-\{k(2x-1)\}^{\frac{1-(-1)^k}{2}}y^{2[\frac{k}{2}]}\right].
\end{aligned} \tag{2.3}$$

Proof. By (1.2) we directly get this lemma. □

Proof of Theorem1.1 For $k, n, q \in \mathbb{N}$ and $k, n \geq 2$.
(a) If k be an odd positive integer and let $k = 2l + 1$, then by Proposition 2.1, we have

$$
\begin{aligned}
&T(2l+1,n,q)\\
:=&\sum_{\substack{a,b \text{ odd}\\ 1\le m\le n-1}} \binom{2l+1}{a,b,c}\sigma_a^*(m)\sigma_b^*(n-m)\sigma_c^*(q)\\
=&\sum_{\substack{c \text{ odd}\\ 1\le c\le 2l-1}} \binom{2l+1}{c}\Big\{\sum_{j=0}^{\frac{2l-c+1}{2}-1}\binom{2l-c+1}{2j+1}\Big(\sum_{m=1}^{n-1}\sigma_{2l-c-2j}^*(m)\sigma_{2j+1}^*(n-m)\Big)\Big\}\sigma_c^*(q)\\
=&\sum_{\substack{c \text{ odd}\\ 1\le c\le 2l-1}} \binom{2l+1}{c}\frac{1}{2}\Big(\sigma_{2l-c+2}^*(n)-n\sigma_{2l-c}^*(n)\Big)\sum_{\substack{x|q\\ \frac{q}{x} \text{ odd}}} x^c\\
=&\frac{1}{2}\sum_{s=0}^{l-1}\binom{2l+1}{2s+1}\sum_{\substack{x|q\\ \frac{q}{x} \text{ odd}}} x^{2s+1}\Big(\sum_{\substack{y|n\\ \frac{n}{y} \text{ odd}}} y^{2l-2s+1}-n\sum_{\substack{y|n\\ \frac{n}{y} \text{ odd}}} y^{2l-2s-1}\Big)\\
=&\frac{1}{2}\sum_{\substack{x|q,\ \frac{q}{x} \text{ odd}\\ y|n,\ \frac{n}{y} \text{ odd}}}\Big(y-\frac{n}{y}\Big)\sum_{s=0}^{l-1}\binom{2l+1}{2s+1}y^{2l-2s}x^{2s+1},
\end{aligned}
$$

where $b=2j+1,\ c=2s+1$.

From the binomial theorem it follows that

$$
\begin{aligned}
&T(2l+1,n,q)\\
=&\frac{1}{2}\sum_{\substack{x|q,\ \frac{q}{x} \text{ odd}\\ y|n,\ \frac{n}{y} \text{ odd}}}\Big(y-\frac{n}{y}\Big)\frac{1}{2}\Big\{(y+x)^{2l+1}-(y-x)^{2l+1}-2x^{2l+1}\Big\}\\
=&\frac{1}{4}\sum_{\substack{x|q,\ \frac{q}{x} \text{ odd}\\ y|n,\ \frac{n}{y} \text{ odd}}}\Big(y-\frac{n}{y}\Big)\Big\{(x+y)^{2l+1}+(x-y)^{2l+1}\Big\}-\frac{1}{2}\sigma_{2l+1}^*(q)\Big(\sigma_1^*(n)-\sigma_{1,1}(n;2)\Big).
\end{aligned}
$$

By the property of Bernoulli polynomial

$$
B_n(x+1)-B_n(x)=nx^{n-1}, \tag{2.4}
$$

we get

$$
\begin{aligned}
&T(2l+1,n,q)\\
=&\frac{1}{4(2l+2)}\sum_{\substack{x|q,\ \frac{q}{x} \text{ odd}\\ y|n,\ \frac{n}{y} \text{ odd}}}\Big(y-\frac{n}{y}\Big)\times\Big(B_{2l+2}(x+y+1)-B_{2l+2}(x+y)\\
&\qquad +B_{2l+2}(x-y+1)-B_{2l+2}(x-y)\Big)-\frac{1}{2}\sigma_{2l+1}^*(q)\sigma_1\left(\frac{n}{2}\right).
\end{aligned}
$$

Similarly, we obtain

$$
T(2l,n,q)
$$

$$
\begin{aligned}
&= \sum_{\substack{a,b \text{ odd} \\ 1\le m\le n-1}} \binom{2l}{a,b,c} \sigma_a^*(m)\sigma_b^*(n-m)\sigma_c^*(q) \\
&= \frac{1}{4(2l+1)} \sum_{\substack{x|q,\ y|n \\ \frac{q}{x},\frac{n}{y} \text{ odd}}} \Big(y-\frac{n}{y}\Big) \times \Big(B_{2l+1}(x+y+1) - B_{2l+1}(x+y) \\
&\qquad + B_{2l+1}(x-y+1) - B_{2l+1}(x-y)\Big) - \frac{1}{2}\sigma_{2l}^*(q)\sigma_1\left(\frac{n}{2}\right).
\end{aligned}
$$

This completes the proof of (a).
(b) If k be an odd positive integer and let $k=2l+1$, then by Proposition 2.1

$$
\begin{aligned}
&Y(2l+1,n,q) \\
:=& \sum_{\substack{a,b \text{ odd} \\ 1\le m\le n-1}} 2^c \binom{2l+1}{a,b,c} \sigma_{a,1}(m;2)\sigma_{b,1}(n-m;2)\sigma_{c,1}(q;2) \\
=& \sum_{\substack{c \text{ odd} \\ 1\le c\le 2l-1}} 2^c \binom{2l+1}{c} \\
&\times \Big\{ \sum_{j=0}^{\frac{2l-c+1}{2}-1} \binom{2l-c+1}{2j+1} \Big(\sum_{m=1}^{n-1} \sigma_{2l-c-2k,1}(m;2)\sigma_{2k+1,1}(n-m;2)\Big)\Big\} \sigma_{c,1}(q;2) \\
=& \sum_{\substack{c \text{ odd} \\ 1\le c\le 2l-1}} 2^c \binom{2l+1}{c} \Big(\frac{1}{4}\sigma_{2l-c+2,0}(n;2) + \frac{2^{2l-c+1}}{2l-c+2} \sum_{\substack{x|n \\ x \text{ even}}} B_{2l-c+2}\left(\frac{x+1}{2}\right)\Big) \sum_{\substack{y|q \\ y \text{ odd}}} y^c \\
=& \frac{1}{4} \sum_{s=0}^{l-1} \binom{2l+1}{2s+1} \sum_{\substack{x|n,\ x \text{ even} \\ y|q,\ y \text{ odd}}} x^{2l-2s+1}(2y)^{2s+1} \\
&+ \frac{2^{2l+1}}{2l+2} \sum_{\substack{x|n,\ x \text{ odd} \\ y|q,\ y \text{ odd}}} \sum_{s=0}^{l-1} \binom{2l+2}{2s+1} B_{2l-2s+1}\left(\frac{x+1}{2}\right) y^{2s+1},
\end{aligned}
$$

where $b=2j+1,\ c=2s+1$.
From the binomial theorem, Lemma 2.3 and (2.4) we obtain

$$
\begin{aligned}
&Y(2l+1,n,q) \\
=& \frac{1}{8(2l+2)} \sum_{\substack{x|n,\ x \text{ even} \\ y|q,\ y \text{ odd}}} x \times \Big(B_{2l+2}(x+2y+1) - B_{2l+2}(x+2y) - B_{2l+2}(x-2y+1) \\
&+ B_{2l+2}(x-2y)\Big) + \frac{2^{2l}}{2l+2} \sum_{\substack{x|n,\ x \text{ odd} \\ y|q,\ y \text{ odd}}} \Big(B_{2l+2}\left(\frac{x+1}{2}+y\right) - B_{2l+2}\left(\frac{x+1}{2}-y\right)\Big) \\
&- 2^{2l}\sigma_1^*(n)\sigma_{2l+1,1}(q;2)
\end{aligned}
$$

and

$$
\begin{aligned}
&Y(2l,n,q)\\
=&\frac{1}{8(2l+1)}\sum_{\substack{x|n,\ x \text{ even}\\ y|q,\ y \text{ odd}}} x\times\Big(B_{2l+1}(x+2y+1)-B_{2l+1}(x+2y)+B_{2l+1}(x-2y+1)\\
&-B_{2l+1}(x-2y)\Big)+\frac{2^{2l-1}}{2l+1}\sum_{\substack{x|n,\ x \text{ odd}\\ y|q,\ y \text{ odd}}}\Big(B_{2l+1}\Big(\frac{x+1}{2}+y\Big)+B_{2l+1}\Big(\frac{x+1}{2}-y\Big)\Big)\\
&-2^{2l-1}\sigma_1^*(n)\sigma_{2l,1}(q;2).
\end{aligned}
$$

□

Example 2.4. We can find some values of $T(k,n,q)$ and $Y(k,n,q)$ in Theorem 1.1.

Table 1. Values of $T(k,n,q)$.

k	n	q	$T(k,n,q)$	k	n	q	$T(k,n,q)$
2	2	3, 5	4	3	2	2	12
3	2	5	36	3	3	1	128
3	3	4	848	4	3	3	640

Table 2. Values of $Y(k,n,q)$.

k	n	q	$Y(k,n,q)$	k	n	q	$Y(k,n,q)$
2	2	3, 5	4	3	2	2	12
3	2	5	72	3	3	4	24
4	3	3	992	4	4	3	4848

Proof of Theorem 1.2 Let $n \geq 4$ be an even integer and $l,q \in \mathbb{N}-\{1\}$. By Proposition 2.1, we have

$$
\begin{aligned}
F(l,n,q) :=& \sum_{\substack{a+b+c+d+e=2l+1\\ a,b,c,d,e \text{ odd}}}\binom{2l}{a-1,b,c,d,e}\\
&\times\sum_{\substack{m_1+m_2+m_3=n\\ m_4+m_5=q\\ m_3 \text{ even}}}(-1)^{m_1+1}\sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)\sigma_{d,1}(m_4;2)\sigma_{e,1}(m_5;2)\\
=&\sum_{\substack{a+b+c+d+e=2l+1\\ a,b,c,d,e \text{ odd}}}\binom{2l}{d+e}\binom{d+e}{e}\times\frac{a}{a+b+c}\binom{a+b+c}{a,b,c}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\substack{m_1+m_2+m_3=n\\ m_4+m_5=q\\ m_3 \text{ even}}} (-1)^{m_1+1}\sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)\sigma_{d,1}(m_4;2)\sigma_{e,1}(m_5;2)\\
=&\sum_{k=1}^{l-1}\binom{2l}{2k}\left\{\frac{n}{32}\Big(\sigma_{2l-2k+1}^*(n)-2n\sigma_{2l-2k-1}^*(n)\Big)\right\}\\
&\times\Big(\frac{1}{4}\sigma_{2k+1,0}(q;2)+\frac{2^{2k}}{2k+1}\sum_{x|q,\ x\text{ odd}} B_{2k+1}\left(\frac{x+1}{2}\right)\Big)\\
=&\frac{n}{128}\sum_{k=1}^{l-1}\binom{2l}{2k}\sigma_{2k+1,0}(q;2)\sigma_{2l-2k+1}^*(n)-\frac{n^2}{64}\sum_{k=1}^{l-1}\binom{2l}{2k}\sigma_{2k+1,0}(q;2)\sigma_{2l-2k-1}^*(n)\\
&+\frac{n}{32}\sum_{k=1}^{l-1}\frac{2^{2k}}{2k+1}\binom{2l}{2k}\sum_{x|q,\ x\text{ odd}} B_{2k+1}\left(\frac{x+1}{2}\right)\sigma_{2l-2k+1}^*(n)\\
&-\frac{n^2}{16}\sum_{k=1}^{l-1}\frac{2^{2k}}{2k+1}\binom{2l}{2k}\sum_{x|q,\ x\text{ odd}} B_{2k+1}\left(\frac{x+1}{2}\right)\sigma_{2l-2k-1}^*(n).
\end{aligned}
$$

By the same method in Theorem 1.1, we derive the following 4 terms below;

$$
\begin{aligned}
F_1 :=&\frac{n}{128}\sum_{k=1}^{l-1}\binom{2l}{2k}\sigma_{2k+1,0}(q;2)\sigma_{2l-2k+1}^*(n)\\
=&\frac{n}{256(2l+1)}\sum_{\substack{x|q,\ x\text{ even}\\ y|n,\ \frac{n}{y}\text{ odd}}} xy\times\Big(B_{2l+1}(x+y+1)-B_{2l+1}(x+y)+B_{2l+1}(x-y+1)\\
&-B_{2l+1}(x-y)\Big)-\frac{n}{128}\Big(2^{2l+1}\sigma_{2l+1}\left(\frac{q}{2}\right)\sigma_1^*(n)+2\sigma_1\left(\frac{q}{2}\right)\sigma_{2l+1}^*(n)\Big),\\
F_2 :=&\frac{n^2}{64}\sum_{k=1}^{l-1}\binom{2l}{2k}\sigma_{2k+1,0}(q;2)\sigma_{2l-2k-1}^*(n)\\
=&\frac{n^2}{128(2l+1)}\sum_{\substack{x|q,\ x\text{ even}\\ y|n,\ \frac{n}{y}\text{ odd}}} \frac{x}{y}\times\Big(B_{2l+1}(x+y+1)-B_{2l+1}(x+y)+B_{2l+1}(x-y+1)\\
&-B_{2l+1}(x-y)\Big)-\frac{n^2}{64}\Big(2^{2l+1}\sigma_{2l+1}\left(\frac{q}{2}\right)\sigma_{-1}^*(n)+2\sigma_1\left(\frac{q}{2}\right)\sigma_{2l-1}^*(n)\Big),\\
F_3 :=&\frac{n}{32}\sum_{k=1}^{l-1}\frac{2^{2k}}{2k+1}\binom{2l}{2k}\sum_{x|q,\ x\text{ odd}} B_{2k+1}\left(\frac{x+1}{2}\right)\sigma_{2l-2k+1}^*(n)\\
=&\frac{2^{2l}n}{64(2l+1)}\sum_{\substack{x|q,\ x\text{ odd}\\ y|n,\ \frac{n}{y}\text{ odd}}} y\Big(B_{2l+1}\left(\frac{x+y+1}{2}\right)+B_{2l+1}\left(\frac{x-y+1}{2}\right)\Big)\\
&-\frac{n}{64}\Big(\sigma_{1,1}(q;2)\sigma_{2l+1}^*(n)+\frac{2^{2l+1}}{2l+1}\sigma_1^*(n)\sum_{x|q,\ x\text{ odd}} B_{2l+1}\left(\frac{x+1}{2}\right)\Big),
\end{aligned}
$$

$$F_4 := \frac{n^2}{16}\sum_{k=1}^{l-1}\frac{2^{2k}}{2k+1}\binom{2l}{2k}\sum_{x|q,\ x\text{ odd}} B_{2k+1}\left(\frac{x+1}{2}\right)\sigma^*_{2l-2k-1}(n)$$
$$= \frac{2^{2l}n^2}{32(2l+1)}\sum_{\substack{x|q,\ x\text{ odd}\\ y|n,\ \frac{n}{y}\text{ odd}}}\frac{1}{y}\Big(B_{2l+1}\left(\frac{x+y+1}{2}\right)+B_{2l+1}\left(\frac{x-y+1}{2}\right)\Big)$$
$$-\frac{n^2}{32}\Big(\sigma_{1,1}(q;2)\sigma^*_{2l-1}(n)+\frac{2^{2l+1}}{2l+1}\sigma^*_{-1}(n)\sum_{x|q,\ x\text{ odd}}B_{2l+1}\left(\frac{x+1}{2}\right)\Big).$$

Thus we have

$$F(l,n,q) = F_1 - F_2 + F_3 - F_4 = \frac{n}{256(2l+1)}\sum_{\substack{x|q,\ x\text{ even}\\ y|n,\ \frac{n}{y}\text{ odd}}}\left(xy-\frac{2nx}{y}\right)$$
$$\times\Big(B_{2l+1}(x+y+1)-B_{2l+1}(x+y)+B_{2l+1}(x-y+1)-B_{2l+1}(x-y)\Big)$$
$$+\frac{2^{2l}n}{64(2l+1)}\sum_{\substack{x|q,\ x\text{ odd}\\ y|n,\ \frac{n}{y}\text{ odd}}}\left(y-\frac{2n}{y}\right)\left(B_{2l+1}\left(\frac{x+y+1}{2}\right)+B_{2l+1}\left(\frac{x-y+1}{2}\right)\right)$$
$$-\frac{n}{64}\left\{\Big(\sigma^*_{2l+1}(n)-2n\sigma^*_{21-1}(n)\Big)\Big(\sigma_1\left(\frac{q}{2}\right)+\sigma_{1,1}(q;2)\Big)\right.$$
$$\left.+\Big(\sigma^*_1(n)-2n\sigma^*_{-1}(n)\Big)\Big(2^{2l}\sigma_{2l+1}\left(\frac{q}{2}\right)+\frac{2^{2l+1}}{2l+1}\sum_{x|q,\ x\text{ odd}}B_{2l+1}\left(\frac{x+1}{2}\right)\Big)\right\}.$$

□

Example 2.5. We can find some values of $F(l,n,q)$ in Theorem 1.2.

Table 3. Values of $F(l,n,q)$.

l	n	q	$F(l,n,q)$	l	n	q	$F(l,n,q)$
2	4	2	48	2	4	3	96
2	6	2	288	2	6	3	576
3	4	2	2400	3	4	3	4800

3. The tree-modeling method

Th procedural modeling method using convolution sums of divisor functions (MCD) was suggested for a variety of natural trees in a virtual ecosystem [10], [11]. Th basic structure of MCD is that it defines the growth grammar including the branch propagation, a growth pattern of branches and leaves, and a process of growth deformation for various generations of tree. Theorems 1.1 gives us a basic background for efficient and diverse generations and expressions of trees composing virtual ecosystem or real-time animation processing.

In order to apply MCD to the growth structure of a tree model, (1.3) is modified and expressed in

$$\sum_{\substack{a,b \text{ odd} \\ 1\le m\le n-1}} \binom{k}{a,b,c} D^i(B^i(x,y)a)D^i(T^i(x,y)b)D^i(L^i(x,y)c)$$

where D represents various divisor functions, $i$is the current growth step, and $n-1$ is the final iteration number of the ith growth step. Here, $D^i(B^i(x,y)a)$ is a divisor function that determines the pattern of the number of branches, $D^i(T^i(x,y)a)$ is a divisor function that determines the number of twigs, and $D^i(L^i(x,y)a)$ is a divisor function that determines the number of leaves with l different types of trees and grasses in the virtual system. We put $D^i = \sigma^*$. Using Theorem 1.1, we obtain approximate total numbers for MCD. We suggest an example for

$$\sum_{m=1}^{N-1} \sigma_1^*(m)\sigma_1^*(N-m)\sigma_1^*(q)$$

The basic models of tree consist of main column, bough, twig, and leaf. $\sigma_1^*(m)$ means the bough grown in the main column, $\sigma_1^*(N-m)$ means the twig grown in the bough, $\sigma_1^*(q)$ means leaf grown in the twig. If $N=2, q=2$, then

$$\sum_{m=1}^{1} \sigma_1^*(m)\sigma_1^*(2-m)\sigma_1^*(2) = \sigma_1^*(1)\sigma_1^*(1)\sigma_1^*(2).$$

One bough ($\sigma_1^*(1)=1$) are grown in the main column, and one twig ($\sigma_1^*(1)=1$) are grown in the bough, and two leaves ($\sigma_1^*(2)=2$) are grown in the twig.

Fig. 1. $N=2, q=2$.

If $N=3, q=2$, then

$$\sum_{m=1}^{2} \sigma_1^*(m)\sigma_1^*(3-m)\sigma_1^*(2) = \sigma_1^*(1)\sigma_1^*(2)\sigma_1^*(2) + \sigma_1^*(2)\sigma_1^*(1)\sigma_1^*(2). \tag{3.1}$$

Consider the first sum of right hand side of (3.1) $\sigma_1^*(1)\sigma_1^*(2)\sigma_1^*(2) = 1\times 2\times 2$. This number represents the following : New twigs are grown in the first bough, and two leaves are grown in the new twigs(See Figure 2).

Similarly, we consider the second sum of right hand side of (3.1) $\sigma_1^*(2)\sigma_1^*(1)\sigma_1^*(2) = 2 \times 1 \times 2$. This number represents the following : Two new boughs are grown in the main column, and one twig are grown each in the bough, and two leaves are grown each in the twig (see Figure 2).

Fig. 2. $N = 3, q = 2$.

Similarly, we obtain Figure 3.

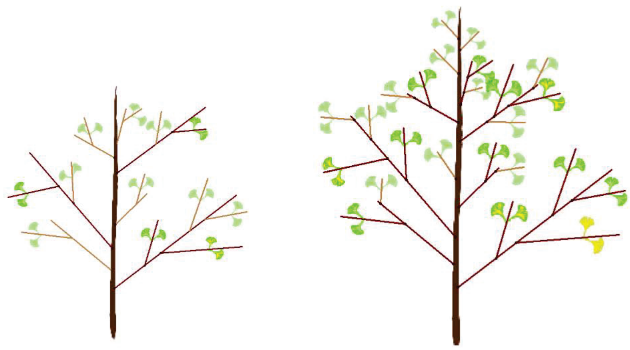

Fig. 3. $N = 4, q = 2$. $N = 5, q = 2$.

Using this divisor model, we can find the total number of leaves (see Table 4).

Table 4. Total leaves of tree $\sum_{m=1}^{N-1} \sigma_1^*(m)\sigma_1^*(N-m)\sigma_1^*(q)$.

N	2	3	4	5	6	7	8	9
$q = 2$	2	8	26	48	88	144	224	320
$q = 3$	4	16	52	96	176	288	448	640
$q = 5$	6	24	78	144	264	432	672	960

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