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INTERVAL VALUED (α, β)-INTUITIONISTIC FUZZY BI-IDEALS OF SEMIGROUPS

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Abstract. The concept of quasi-coincidence of interval valued intuitionistic fuzzy point with an interval valued intuitionistic fuzzy set is considered. By using this idea, the notion of interval valued (α, β)-intuitionistic fuzzy bi-ideals, (1,2)ideals in a semigroup introduced and consequently, a generalization of interval valued intuitionistic fuzzy bi-ideals and intuitionistic fuzzy bi-ideals is defined. In this paper, we study the related properties of the interval valued (α, β)-intuitionistic fuzzy bi-ideals, (1,2) ideals and in particular, an interval valued $(\in, \in \vee q)$-fuzzy bi-ideals and (1,2) ideals in semigroups will be investigated.

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1. Introduction

The concept of a fuzzy set was first initiated by Zadeh [1]. Fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, real analysis, measure theory etc. After the introduction of the concept of fuzzy sets by Zadeh, several researches conducted the researches on the generalizations of the notion of fuzzy set with huge applications in computer, logics, automata and many branches of pure and applied mathematics. The notion of i-v fuzzy sets

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was first introduced by Zadeh [1] as an extension of fuzzy sets in which the values of the membership degrees are intervals of numbers instead of the numbers. Thus, i-v fuzzy sets provide a more adequate description of uncertainty than the traditional fuzzy sets. I-v fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Rosenfeld studied fuzzy subgroups of a group [2]. The study of fuzzy semigroups was studied by Kuroki in his classical paper [3] and Kuroki initiated fuzzy ideals, bi-ideals, semi-prime ideals, quasi-ideals of semigroups [4, 5, 6, 7, 8, 9, 10]. A systematic exposition of fuzzy semigroup was given by Mordeson et.al. [11], and they have find theoretical results on fuzzy semigroups and their use in fuzzy finite state machines, fuzzy languages and fuzzy coding. Mordeson and Malik studied monograph in [12] deals with the application of fuzzy approch to the concepts of formal languages and automata. In 2008, Shabir and Khan introduced the concept of i-v fuzzy ideals generated by i-v fuzzy subset in ordered semigroup [43]. Using the notions "*belong to*" relation ($\in$) introduced by Pu and Lia [19]. In [20], Morali proposed the concept of a fuzzy point belonging to a fuzzy subset under natural equivalence on fuzzy subset. Bhakat and Das introduced the concepts of (α, β)-fuzzy subgroups by using the "*belong to*" relation ($\in$) and "*quasi-coincident with*" relation (q) between a fuzzy point and a fuzzy subgroup, and defined an ($\in$, $\in \vee q$)-fuzzy subgroup of a group [21]. Kazanci and Yamak [22] studied generalized types fuzzy bi-ideals of semigroups and defined ($\overline{\in}$, $\overline{\in} \vee \overline{q}$)-fuzzy bi-ideals of semigroups. Jun and Song [23] studied generalized fuzzy interior ideals of semigroups. In [24], Shabir et. al. characterized regular semigroups by the properties of (α, β)-fuzzy ideals, bi-ideals and quasi-ideals. In [25], Shabir and Yasir characterized regular semigroups by the properties of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals, generalized bi-ideals and quasi-ideals of a semigroup. M. Shabir et. al. defined some types of $(\in, \in \vee q_k)$-fuzzy ideals of semigroups and characterized regular semigroups by these ideals [26]. In [27], M. Shabir and T. Mehmood studied $(\in, \in \vee q_k)$-fuzzy h-ideals of hemirings and characterized different classes of hemirings by the properties of $(\in, \in \vee q_k)$-fuzzy h-ideals. Recently, M. Aslam et al. [28] initiated the concept of (α,β)-fuzzy Γ-ideals of Γ-LA-semigroups and given some characterization of Γ-LA-semigroups by (α,β)-fuzzy Γ-ideals.

In fuzzy sets theory, there is no means to incorporate the hesitation or uncertainty in the membership degrees. In 1986, Atanassov [29] premised the concept of an intuitionistic fuzzy set (IFS) and more operations defined in [30]. An Atanassov intuitionistic fuzzy set is deliberated as a generalization of fuzzy set [1] and has been found to be useful to deal with vagueness. In the sense of an IFS is characterized by a pair of functions valued in $[0, 1]$: the membership function and the non-membership function. The evaluation degrees of membership and non-membership are independent. Thus, an Atanassov intuitionistic fuzzy set is more material and concise to describe the essence of fuzziness, and Atanassov intuitionistic fuzzy set theory may be more suitable than fuzzy set theory for dealing with imperfect knowledge in many problems. The concept has been applied to various algebraic structures. Atanassov and Gargov [31] initiated the notion of i-v intuitionistic fuzzy sets which is a generalization of both intuitionistic fuzzy sets and interval valued fuzzy sets. In [**?**], Akram and Dudek have defined interval valued intuitionistic fuzzy Lie ideals of Lie algebras and some interesting results are abtained. Biswas [32] introduced the notion of intuitionistic fuzzy subgroup of a group by using the notion of intuitionistic fuzzy sets. In [33], Kim and Jun defined intuitionistic fuzzy ideals of semigroups. Kim and Lee [34] studied intuitionistic fuzzy bi-ideals of semigroups. Kim and Jun initiated the concept intuitionistic fuzzy interior ideals of semigroups [35]. Coker and Demirci introduced the notion of intuitionistic fuzzy point [36] of 1995. Jun [37] introduced the notion of (Φ, Ψ)-intuitionistic fuzzy subgroups, where Φ, Ψ are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\Phi \neq \in \wedge q$, and related properties are investigated. Aslam and Abdullah [38] introduced the concept of (Φ, Ψ)-intuitionistic fuzzy ideals of semigroups and obtained some properties of (Φ, Ψ)-intuitionistic fuzzy ideals. Recently, Abdullah et.al., initiated the concept of (α, β)-intuitionistic fuzzy ideals of hemirings by using the "*belong to*" relation $(\in)$ and "*quasi-coincident with*" relation (q) between an intuitionistic fuzzy point and an intuitionistics fuzzy set, and they defined prime (semi-prime) (α, β)-intuitionistic fuzzy ideals of hemirings [39].

In this paper, we introduce the concept of interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of semigroup and (α, β)-intuitionistic fuzzy bi-ideal of semigroup where Φ, Ψ are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, by using *belong* to relation $(\in)$ and *quasi-coincidence with* relation (q) between intuitionistic fuzzy point and intuitionistic fuzzy set, and investigated related properties. We also prove that in regular semigroup, every $(\in, \in \vee q)$-intuitionistic fuzzy $(1, 2)$ ideal of semigroup S is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of semigroup S.

2. Preliminaries

Definition 1 ([1, 2])**.** Let X be a non-empty fixed set. An interval valued intuitionistic fuzzy set (briefly, IVIFS) A is an object having the form

$$A = \{\langle x, \widehat{\mu}_A(x), \widehat{\lambda}_A(x)\rangle : x \in X\}$$

where the functions $\widehat{\mu}_A : X \longrightarrow [0, 1]$ and $\widehat{\lambda}_A : X \longrightarrow [0, 1]$ denote the degree of membership (namely $\widehat{\mu}_A(x)$) and the degree of non-membership (namely $\widehat{\lambda}_A(x)$) of each element $x \in X$ to the set A, respectively, and $\widehat{\mu}_A(x) + \widehat{\lambda}_A(x) \leq \widehat{1}$ for all $x \in S$ for the sake of simplicity, we use the symbol $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A\rangle$ for the IVIFS $A = \{(x, \widehat{\mu}_A(x), \widehat{\lambda}_A(x) : x \in X\}$.

Definition 2 ([4])**.** Let c be a point in a non-empty set X. If $\widetilde{t} \in D(0, 1]$ and $\widetilde{s} \in D[0, 1)$ are two interval numbers such that $\widetilde{0} \leq \widetilde{t} + \widetilde{s} \leq \widetilde{1}$, and at same time both values $\widetilde{t}$ and $\widetilde{s}$ does not less than $\widetilde{0.5}$. Then, the IFS

$$c(\widetilde{t}, \widetilde{s}) = \langle x, c_{\widetilde{t}}, \widetilde{1} - c_{\widetilde{1}-\widetilde{s}}\rangle$$

is called an interval valued intuitionistic fuzzy point (IVIFP for short) in X, where $\widetilde{t}$ (resp, $\widetilde{s}$) is the degree of membership (resp, non-membership) of $c(\widetilde{t},\widetilde{s})$ and $c \in X$ is the support of $c(\widetilde{t},\widetilde{s})$. Let $c(\widetilde{t},\widetilde{s})$ be an IVIFP in X. and let $A = \langle x, \widetilde{\mu}_A, \widetilde{\lambda}_A\rangle$ be an interval valued IFS in X. Then, $c(\widetilde{t},\widetilde{s})$ is said to *belong* to A, written $c(\widetilde{t},\widetilde{s}) \in A$, if $\widetilde{\mu}_A(c) \geq \widetilde{t}$ and $\widetilde{\lambda}_A(c) \leq \widetilde{s}$. We say that $c(\widetilde{t},\widetilde{s})$ is quasi-coincident with A, written $c(\widetilde{t},\widetilde{s})qA$, if $\widetilde{\mu}_A(c) + \widetilde{t} > \widetilde{1}$ and $\widetilde{\lambda}_A(c) + \widetilde{s} < \widetilde{1}$. To say that $c(\widetilde{t},\widetilde{s}) \in \vee qA$ (resp, $c(\widetilde{t},\widetilde{s}) \in \wedge qA$) means that $c(\widetilde{t},\widetilde{s}) \in A$ or $c(\widetilde{t},\widetilde{s})qA$ (resp, $c(\widetilde{t},\widetilde{s}) \in A$ and $c(\widetilde{t},\widetilde{s})qA$) and $c(\widetilde{t},\widetilde{s})\overline{\in \vee q}A$ means that $c(\widetilde{t},\widetilde{s}) \in \vee qA$ does not hold and $m\{\widehat{t}_1,\widehat{t}_2\} = \min\left\{\widehat{t}_1,\widehat{t}_2\right\}$, $M\{\widehat{s}_1,\widehat{s}_2\} = \max\{\widehat{s}_1,\widehat{s}_2\}$.

Definition 3. An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A\rangle$ in S is called an intuitionistic fuzzy subsemigroup of S if the following conditions hold:
$(IF1)$ $\widehat{\mu}_A(xy) \geq \widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y)$,
$(IF2)$ $\widehat{\lambda}_A(xy) \leq \widehat{\lambda}_A(x) \vee \widehat{\lambda}_A(y)$ for all $x, y \in S$.

Definition 4. An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A\rangle$ in S is called an intuitionistic fuzzy right ideal of S if it satisfy $\widehat{\mu}_A(xy) \geq \widehat{\mu}_A(x)$ and $\lambda_A(xy) \leq \lambda_A(x)$ for all $x, y \in S$.

Definition 5. An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A\rangle$ in S is called an intuitionistic fuzzy left ideal of S if it satisfy $\widehat{\mu}_A(xy) \geq \widehat{\mu}_A(y)$ and $\lambda_A(xy) \leq \lambda_A(y)$ for all $x, y \in S$.

Theorem 1 ([13])**.** *An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A\rangle$ in a semigroup S is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) ideal of a semigroup S if and only if the following conditions hold.*
(a) $(\forall x, y \in S)$ $\widehat{\mu}_A(xy) \geq \min\{\widehat{\mu}_A(y), \widehat{0.5}\}$ *and* $\widehat{\lambda}_A(xy) \leq \max\{\widehat{\lambda}_A(y), \widehat{0.5}\}$ *(resp.* $(\forall x, y \in S)$ $\widehat{\mu}_A(xy) \geq \min\{\widehat{\mu}_A(x), \widehat{0.5}\}$ *and* $\widehat{\lambda}_A(xy) \leq \max\{\widehat{\lambda}_A(x), \widehat{0.5}\}$*).*

3. Interval valued (α, β)-Intuitionistic Fuzzy bi-ideals

Definition 6. An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A\rangle$ in a semigroup S is said to be an interval valued (α, β)-intuitionistic fuzzy subsemigroup of a semigroup S if the following condition holds:
$(\forall x, y \in S)\,(\widehat{t}_1, \widehat{t}_2 \in D(0, 0.5]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0.5, 1) or \widehat{t}_1, \widehat{t}_2 \in D(0.5, 1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0, 0.5))\; x(\widehat{t}_1, \widehat{s}_1)\alpha A \; and \; y(\widehat{t}_2, \widehat{s}_2)\beta A \Longrightarrow (xy)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\})\beta A$.

Definition 7. An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A\rangle$ in a semigroup S is said to be an interval valued (α, β)-intuitionistic fuzzy left (resp, right) ideal of semigroup S if $\forall x, y \in S$ and $\forall \widehat{t}_1, \widehat{t}_2 \in D(0, 0.5]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0.5, 1)$ or $\widehat{t}_1, \widehat{t}_2 \in D(0.5, 1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0, 0.5)$, the following hold.
$(IVIFS2)$ $y(\widehat{t}, \widehat{s})\alpha A \Longrightarrow (xy)(\widehat{t}, \widehat{s})\beta A$ (resp. $x(\widehat{t}, \widehat{s})\alpha A \Longrightarrow (xy)(\widehat{t}, \widehat{s})\beta A$.
An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A\rangle$ in a semigroup S is said to be an interval valued (α, β)-intuitionistic fuzzy ideal of a semigroup S, if $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A\rangle$ is an interval valued (α, β)-intuitionistic fuzzy left ideal and interval valued (α, β)-intuitionistic fuzzy right ideal of a semigroup S.

Definition 8. An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ in a semigroup S is said to be an interval valued (α, β)-intuitionistic fuzzy bi-ideal of a semigroup S, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, if for all $x, y, z \in S$, $\widehat{t}_1, \widehat{t}_1, \widehat{t}_2 \in D(0, 0.5]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0.5, 1)$ or $\widehat{t}_1, \widehat{t}_2 \in D(0.5, 1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0, 0.5)$, the following conditions hold:

$(IFB1)$ $x(\widehat{t}_1, \widehat{s}_1)\alpha A$ and $y(\widehat{t}_2, \widehat{s}_2)\alpha A \Longrightarrow (xy)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\})\beta A$,

$(IFB2)$ $x(\widehat{t}_1, \widehat{s}_1)\alpha A$ and $z(\widehat{t}_2, \widehat{s}_2)\alpha A \Longrightarrow (xyz)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\})\beta A$.

Definition 9. An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ in semigroup S is said to be an interval valued (α, β)-intuitionistic fuzzy (1, 2) ideal of semigroup S if for all $x, y, z, a \in S$, $\widehat{t}_1, \widehat{t}_2 \in D(0, 0.5]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0.5, 1)$ or $\widehat{t}_1, \widehat{t}_2 \in D(0.5, 1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0, 0.5)$, the following conditions hold.

$(IVIFS2)$ $x(\widehat{t}_1, \widehat{s}_1)\alpha A$ and $y(\widehat{t}_2, \widehat{s}_2)\alpha A \Rightarrow (xy)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\})\beta A$.

$(IVIFS2)$ $x(\widehat{t}_1, \widehat{s}_1)\alpha A$ and $z(\widehat{t}_2, \widehat{s}_2)\alpha A \Rightarrow (xa(yz))(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\})\beta A$.

Theorem 2. *Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be a non-zero interval valued (α, β)-intuitionistic fuzzy subsemigroup of S. Then, the set $A_{(\widehat{0},\widehat{1})} = \left\{x \in S : \widehat{\mu}_A(x) > \widehat{0} \text{ and } \widehat{\lambda}_A(x) < \widehat{1}\right\}$ is a subsemigroup of S.*

Proof. Let $x, y \in A_{(\widehat{0},\widehat{1})}$. Then, $\widehat{\mu}_A(x) > \widehat{0}$ and $\widehat{\lambda}_A(x) < \widehat{1}$, and $\widehat{\mu}_A(y) > \widehat{0}$ and $\widehat{\lambda}_A(y) < \widehat{1}$. Suppose that $\widehat{\mu}_A(xy) = \widehat{0}$ and $\widehat{\lambda}_A(xy) = \widehat{1}$. If $\alpha \in \{\in, \in \vee q\}$, then

$$x\left(\widehat{\mu}_A(x), \widehat{\lambda}_A(x)\right)\alpha A \text{ and } y\left(\widehat{\mu}_A(y), \widehat{\lambda}_A(y)\right)\alpha A \text{ but}$$

$$\widehat{\mu}_A(xy) = \widehat{0} < rm\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} \text{ and } \widehat{\lambda}_A(xy) = \widehat{1} > M\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\}$$

So, $(xy)\left(m\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\}, M\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\}\right)\overline{\beta}A$ for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Now, let $x\left(\widehat{1}, \widehat{0}\right)qA$ and $y\left(\widehat{1}, \widehat{0}\right)qA$ but $(xy)\left(\widehat{1}, \widehat{0}\right)\overline{\beta}A$ for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contaradiction. Hence, $\widehat{\mu}_A(xy) > 0$ and $\widehat{\lambda}_A(xy) < 1$, that is $xy \in A_{(\widehat{0},\widehat{1})}$. Thus, $A_{(\widehat{0},\widehat{1})}$ is a subsemigroup of S. □

Theorem 3. *Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be a non-zero interval valued (α, β)-intuitionistic fuzzy subsemigroup of S. Then, the set $A_{(\widehat{0},\widehat{1})} = \left\{x \in S : \widehat{\mu}_A(x) > \widehat{0} \text{ and } \widehat{\lambda}_A(x) < \widehat{1}\right\}$ is a subsemigroup of S.*

Proof. The proof follows from Theorem 2. □

Theorem 4. *Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be a non-zero interval valued (α, β)-intuitionistic fuzzy bi-ideal of S. Then, the set $A_{(\widehat{0},\widehat{1})} = \left\{x \in S : \widehat{\mu}_A(x) > \widehat{0} \text{ and } \widehat{\lambda}_A(x) < \widehat{1}\right\}$ is a bi-ideal of S.*

Proof. Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be a non-zero interval valued (α, β)-intuitionistic fuzzy bi-ideal of S. Then, by Theorem 2, $A_{(\widehat{0},\widehat{1})}$ is a subsemigroup of S. Now, let

$x, z \in A_{(\widehat{0},\widehat{1})}$ and $y \in S$. Then, $\widehat{\mu}_A(x) > \widehat{0}$ and $\widehat{\lambda}_A(x) < \widehat{1}$, and $\widehat{\mu}_A(z) > \widehat{0}$ and $\widehat{\lambda}_A(z) < \widehat{1}$. Suppose that $\widehat{\mu}_A(xyz) = \widehat{0}$ and $\widehat{\lambda}_A(xyz) = \widehat{1}$. If $\alpha \in \{\in, \in \vee q\}$, then

$$x\left(\widehat{\mu}_A(x), \widehat{\lambda}_A(x)\right) \alpha A \text{ and } z\left(\widehat{\mu}_A(z), \widehat{\lambda}_A(z)\right) \alpha A \text{ but}$$

$$\widehat{\mu}_A(xyz) \quad = \quad \widehat{0} < m\{\widehat{\mu}_A(x), \widehat{\mu}_A(z)\} \text{ and } \widehat{\lambda}_A(xyz) = \widehat{1} > M\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(z)\right\}$$

which implies that $(xy)\left(m\{\widehat{\mu}_A(x), \widehat{\mu}_A(z)\}, M\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(z)\right\}\right)\overline{\beta}A$ for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Now, let $x\left(\widehat{1}, \widehat{0}\right) qA$ and $z\left(\widehat{1}, \widehat{0}\right) qA$ but $(xyz)\left(\widehat{1}, \widehat{0}\right)\overline{\beta}A$ for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contaradiction. Hence $\widehat{\mu}_A(xyz) > \widehat{0}$ and $\widehat{\lambda}_A(xyz) < \widehat{1}$, that is, $xyz \in A_{(\widehat{0},\widehat{1})}$. Thus, $A_{(\widehat{0},\widehat{1})}$ is a bi-ideal of S. □

Theorem 5. *Let $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ be a non-zero interval valued (α, β)-intuitionistic fuzzy $(1,\ 2)$ ideal of S. Then, the set $A_{(\widehat{0},\widehat{1})} = \left\{x \in S : \widehat{\mu}_A(x) > \widehat{0} \text{ and } \widehat{\lambda}_A(x) < \widehat{1}\right\}$ is a $(1,\ 2)$ ideal of S.*

Proof. Straightforward. □

Theorem 6. *Let L be a left (resp. right) ideal of S and let $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ be an IVIFS such that*

(a) $(\forall x \in S \setminus B)$ $\left(\widehat{\mu}_A(x) = \widehat{0} \text{ and } \widehat{\lambda}_A(x) = \widehat{1}\right)$,

(b) $(\forall x \in B)$ $\left(\widehat{\mu}_A(x) \geq \widehat{0.5} \text{ and } \widehat{\lambda}_A(x) \leq \widehat{0.5}\right)$.

Then, $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ is an interval valued $(q, \in \vee q)$-intuitionistic fuzzy left (resp. right) ideal of S.

Proof. (For $\alpha = q$), let $x, y \in S$ and $\widehat{t} \in D(0,1]$ and $\widehat{s} \in D[0,1)$ be such that $y\left(\widehat{t}, \widehat{s}\right) qA$. Then, $\widehat{\mu}_A(y) + \widehat{t} > \widehat{1}$ and $\widehat{\lambda}_A(y) + \widehat{s} < \widehat{1}$. So, $y \in L$. Therefore, $xy \in L$. Thus, if $\widehat{t} \leq \widehat{0.5}$ and $\widehat{s} \geq \widehat{0.5}$, then $\widehat{\mu}_A(xy) \geq \widehat{0.5} \geq \widehat{t}$ and $\widehat{\lambda}_A(y) \leq \widehat{0.5} \leq \widehat{s}$ and so $(xy)\left(\widehat{t}, \widehat{s}\right) \in A$. If $\widehat{t} > \widehat{0.5}$ and $\widehat{s} < \widehat{0.5}$, then $\widehat{\mu}_A(xy) + \widehat{t} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$ and $\widehat{\lambda}_A(y) + \widehat{s} < \widehat{0.5} + \widehat{0.5} = \widehat{1}$ and so $(xy)\left(\widehat{t}, \widehat{s}\right) qA$. Therefore, $(xy)\left(\widehat{t}, \widehat{s}\right) \in \vee qA$. Since $\widehat{t} + \widehat{s} \leq \widehat{1}$, the case $\widehat{t} > \widehat{0.5}$ and $\widehat{s} \geq \widehat{0.5}$ does not occur. From the fact that $y\left(\widehat{t}, \widehat{s}\right) qA$, it follows that the case $\widehat{t} \leq \widehat{0.5}$ and $\widehat{s} < \widehat{0.5}$ does not occur. Hence, $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ is an interval valued $(q, \in \vee q)$-intuitionistic fuzzy left ideal of S. □

Theorem 7. *Let B be a subsemigroup of S and let $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ be an IVIFS such that*

(a) $(\forall x \in S \setminus B)$ $\left(\widehat{\mu}_A(x) = 0 \text{ and } \widehat{\lambda}_A(x) = 1\right)$,

(b) $(\forall x \in B)$ $\left(\widehat{\mu}_A(x) \geq \widehat{0.5} \text{ and } \widehat{\lambda}_A(x) \leq \widehat{0.5}\right)$.

Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(q, \in \vee q)$-intuitionistic fuzzy subsemigroup of S.

Proof. Straightforward. $\square$

Theorem 8. *Let B be a bi-ideal of a semigroup S and let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an IVIFS of S such that*

(*a*) $(\forall x \in S \setminus B)$ $\left(\widehat{\mu}_A(x) = 0 \text{ and } \widehat{\lambda}_A(x) = 1\right)$,

(*b*) $(\forall x \in B)$ $\left(\widehat{\mu}_A(x) \geq \widehat{0.5} \text{ and } \widehat{\lambda}_A(x) \leq \widehat{0.5}\right)$.

Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(q, \in \vee q)$-intuitionistic fuzzy bi-ideal of S.

Proof. (*i*) (For $\alpha = q$), let $x, y \in S$ and $\widehat{t}_1, \widehat{t}_2 \in D(0,1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0,1)$ be such that $x(\widehat{t}_1, \widehat{s}_1)qA$ and $y(\widehat{t}_2, \widehat{s}_2)qA$. Then, $\widehat{\mu}_A(x) + \widehat{t}_1 > \widehat{1}$ and $\widehat{\lambda}_A(x) + \widehat{s}_1 < \widehat{1}$, and $\widehat{\mu}_A(y) + \widehat{t}_2 > \widehat{1}$ and $\widehat{\lambda}_A(y) + \widehat{s}_2 < \widehat{1}$. Thus, $x, y \in B$. Since B is subsemigroup. So, $xy \in B$. Thus, $\widehat{\mu}_A(xy) \geq \widehat{0.5}$ and $\widehat{\lambda}_A(xy) \leq \widehat{0.5}$. If $m(\widehat{t}_1, \widehat{t}_2) > \widehat{0.5}$ and $M(\widehat{s}_1, \widehat{s}_2) < \widehat{0.5}$, then $\widehat{\mu}_A(xy) + m(\widehat{t}_1, \widehat{t}_2) > \widehat{1}$ and $\widehat{\lambda}_A(xy) + M(\widehat{s}_1, \widehat{s}_2) < \widehat{1}$. So, $(xy)(m(\widehat{t}_1, \widehat{t}_2), M(\widehat{s}_1, \widehat{s}_2))qA$. If $m(\widehat{t}_1, \widehat{t}_2) \leq \widehat{0.5}$ and $M(\widehat{s}_1, \widehat{s}_2) \geq \widehat{0.5}$, then $(xy)(m(\widehat{t}_1, \widehat{t}_2), M(\widehat{s}_1, \widehat{s}_2)) \in A$. Since $\widehat{t}_1 + \widehat{s}_1 \leq \widehat{1}$ and $\widehat{t}_2 + \widehat{s}_2 \leq \widehat{1}$, the case $\begin{cases} m(\widehat{t}_1, \widehat{t}_2) > \widehat{0.5} \\ M(\widehat{s}_1, \widehat{s}_2) \geq \widehat{0.5} \end{cases}$ does not occur. From the fact that $x(\widehat{t}_1, \widehat{s}_1)qA$ and $y(\widehat{t}_2, \widehat{s}_2)qA$, it follows that $\begin{cases} m(\widehat{t}_1, \widehat{t}_2) \leq \widehat{0.5} \\ M(\widehat{s}_1, \widehat{s}_2) < \widehat{0.5} \end{cases}$ does not occur. Hence, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(q, \in \vee q)$-intuitionistic fuzzy subsemigroup of S. Let $x, y, z \in S$ and $\widehat{t}_1, \widehat{t}_2 \in D(0,1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0,1)$ be such that $x(\widehat{t}_1, \widehat{s}_1)qA$ and $z(\widehat{t}_2, \widehat{s}_2)qA$. Then, $\widehat{\mu}_A(x) + \widehat{t}_1 > \widehat{1}$ and $\widehat{\lambda}_A(x) + \widehat{s}_1 < \widehat{1}$, and $\widehat{\mu}_A(z) + \widehat{t}_2 > \widehat{1}$ and $\widehat{\lambda}_A(z) + \widehat{s}_2 < \widehat{1}$. Thus, $x, z \in B$. Since B is a bi-ideal. So, $xyz \in B$. Thus, $\widehat{\mu}_A(xyz) \geq \widehat{0.5}$ and $\widehat{\lambda}_A(xyz) \leq \widehat{0.5}$. If $m(\widehat{t}_1, \widehat{t}_2) > \widehat{0.5}$ and $M(\widehat{s}_1, \widehat{s}_2) < \widehat{0.5}$, then $\widehat{\mu}_A(xy) + m(\widehat{t}_1, \widehat{t}_2) > \widehat{1}$ and $\widehat{\lambda}_A(xy) + M(\widehat{s}_1, \widehat{s}_2) < \widehat{1}$. So, $(xy)(m(\widehat{t}_1, \widehat{t}_2), M(\widehat{s}_1, \widehat{s}_2))qA$. If $m(\widehat{t}_1, \widehat{t}_2) \leq \widehat{0.5}$ and $M(\widehat{s}_1, \widehat{s}_2) \geq \widehat{0.5}$, then $(xy)(m(\widehat{t}_1, \widehat{t}_2), M(\widehat{s}_1, \widehat{s}_2)) \in A$. Since $\widehat{t}_1 + \widehat{s}_1 \leq \widehat{1}$ and $\widehat{t}_2 + \widehat{s}_2 \leq \widehat{1}$, the case $\begin{cases} m(\widehat{t}_1, \widehat{t}_2) > \widehat{0.5} \\ M(\widehat{s}_1, \widehat{s}_2) \geq \widehat{0.5} \end{cases}$ does not occur. From the fact that $x(\widehat{t}_1, \widehat{s}_1)qA$ and $y(\widehat{t}_2, \widehat{s}_2)qA$, it follows that $\begin{cases} m(\widehat{t}_1, \widehat{t}_2) \leq \widehat{0.5} \\ M(\widehat{s}_1, \widehat{s}_2) < \widehat{0.5} \end{cases}$ does not occur. Hence, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(q, \in \vee q)$-intuitionistic fuzzy bi-ideal of S. $\square$

Theorem 9. *Let B be a $(1,2)$ ideal of a semigroup S and let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an IVIFS of S such that*

(*a*) $(\forall x \in S \setminus B)$ $\left(\widehat{\mu}_A(x) = \widehat{0} \text{ and } \widehat{\lambda}_A(x) = \widehat{1}\right)$,

(*b*) $(\forall x \in B)$ $\left(\widehat{\mu}_A(x) \geq \widehat{0.5} \text{ and } \widehat{\lambda}_A(x) \leq \widehat{0.5}\right)$.

Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ *is an interval valued* $(q, \in \vee q)$*-intuitionistic fuzzy* $(1,\ 2)$ *ideal of* S.

Proof. Proof follow from Theorem 8. □

4. Interval Valued Intutionistic Fuzzy Bi-Ideals of type $(\in, \in \vee q)$

Definition 10. An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ in semigroup S is said to be an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of semigroup S if $\forall x, y, a \in S$, $\widehat{t}_1, \widehat{t}_2 \in D(0, 0.5]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0.5, 1)$ or $\widehat{t}_1, \widehat{t}_2 \in D(0.5, 1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0, 0.5)$, the following conditions hold.

$(IFB3)$ $x(\widehat{t}_1, \widehat{s}_1) \in A$ and $y(\widehat{t}_2, \widehat{s}_2) \in A \Longrightarrow (xy)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in \vee qA$.

$(IFB4)$ $x(\widehat{t}_1, \widehat{s}_1) \in A$ and $y(\widehat{t}_2, \widehat{s}_2) \in A \Longrightarrow (xay)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in \vee qA$.

Definition 11. An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ in a semigroup S is said to be an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy $(1, 2)$ ideal of a semigroup S if for all $x, y, z, a \in S$, $\widehat{t}_1, \widehat{t}_2 \in D(0, 0.5]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0.5, 1)$ or $\widehat{t}_1, \widehat{t}_2 \in D(0.5, 1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0, 0.5)$, the following conditions hold:

$(IFB3)$ $x(\widehat{t}_1, \widehat{s}_1) \in A$ and $y(\widehat{t}_2, \widehat{s}_2) \in A \Rightarrow (xy)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in \vee qA$.

$(IFB4)$.$x(\widehat{t}_1, \widehat{s}_1) \in A$ and $y(\widehat{t}_2, \widehat{s}_2) \in A \Rightarrow (xa(yz))(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in \vee qA$.

Proposition 1. *An IVIFS* $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ *of a semigroup* S *is an interval valued intuitionistic fuzzy subsemigroup if and only if it satisfies for all* $x, y \in S$ *and* $\widehat{t}_1, \widehat{t}_2 \in D(0, 0.5]$ *and* $\widehat{s}_1, \widehat{s}_2 \in D[0.5, 1)$ *or* $\widehat{t}_1, \widehat{t}_2 \in D(0.5, 1]$ *and* $\widehat{s}_1, \widehat{s}_2 \in D[0, 0.5)$, $x(\widehat{t}_1, \widehat{s}_1) \in A$ *and* $y(\widehat{t}_2, \widehat{s}_2) \in A \Longrightarrow (xy)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in A$.

Proof. Let us suppose that $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy subsemigroup of S. Let $x, y \in S$, $\widehat{t}_1, \widehat{t}_2 \in D(0, 1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0, 1)$ and let $x(\widehat{t}_1, \widehat{s}_1) \in A$ and $y(\widehat{t}_2, \widehat{s}_2) \in A$. Then, $\widehat{\mu}_A(x) \geq t_1$ and $\widehat{\lambda}_A(x) \leq s_1$, and $\widehat{\mu}_A(y) \geq t_2$ and $\widehat{\lambda}_A(y) \leq s_2$. Since by given condition

$$\begin{aligned} \widehat{\mu}_A(xy) &\geq \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} \text{ and } \widehat{\lambda}_A(xy) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} \\ \widehat{\mu}_A(xy) &\geq m\left\{\widehat{t}_1, \widehat{t}_2\right\} \text{ and } \widehat{\lambda}_A(xy) \leq \{\widehat{s}_1, \widehat{s}_2\}. \end{aligned}$$

So, $(xy)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in A$. Thus, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in)$-intuitionistic fuzzy subsemigroup of S.

Conversely, suppose that $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is satisfied the given condition. We show that $\widehat{\mu}_A(xy) \geq \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\}$ and $\widehat{\lambda}_A(xy) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\}$. On contrary assume that there exist $x, y \in S$ such that $\widehat{\mu}_A(xy) < \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\}$ and $\widehat{\lambda}_A(xy) > \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\}$. Let $t \in D(0, 1]$ and $s \in D[0, 1)$ be such that $\widehat{\mu}_A(xy) < t < \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\}$ and $\widehat{\lambda}_A(xy) > s > \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\}$.

Then, $x(\widehat{t}, \widehat{s}) \in A$ and $y(\widehat{t}, \widehat{s}) \in A$ but $(xy)\,(\widehat{t}, \widehat{s})\overline{\in} A$, which contradicts our hypothesis. Hence, $\widehat{\mu}_A(xy) \geq \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\}$ and $\widehat{\lambda}_A(xy) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\}$. Thus, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ interval valued intuitionistic fuzzy subsemigroup of S. □

Proposition 2. *An IVIFS $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ of a semigroup S is an interval valued intuitionistic fuzzy bi-ideal of S if and only if it satisfy for all $x, y, z \in S$ and $\widehat{t}_1, \widehat{t}_2 \in D(0, 0.5]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0.5, 1)$ or $\widehat{t}_1, \widehat{t}_2 \in D(0.5, 1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0, 0.5)$*

(a) $x(\widehat{t}_1, \widehat{s}_1) \in A$ and $y(\widehat{t}_2, \widehat{s}_2) \in A \Longrightarrow (xy)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in A$,

(b) $x(\widehat{t}_1, \widehat{s}_1) \in A$ and $z(\widehat{t}_2, \widehat{s}_2) \in A \Longrightarrow (xyz)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in A$.

Proof. Proof follows from Proposition 1. □

Theorem 10. *Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an IVIFS in semigroup S. Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of semigroup S if and only if the following condition hold;*

(a) $\widehat{\mu}_A(xy) \geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{\widehat{0.5}}\right\}$ and $\widehat{\lambda}_A(xy) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\}$.

(b) $\widehat{\mu}_A(xay) \geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\}$ and $\widehat{\lambda}_A(xay) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\}$.

Proof. Suppose that $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of semigroup S.

(a) Let $x, y \in S$. We consider the following cases:

(1) $\min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} < \widehat{0.5}$ and $\max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} > \widehat{0.5}$

(2) $\min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} \geq \widehat{0.5}$ and $\max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} \leq \widehat{0.5}$

Case (1) Assume that

$$\widehat{\mu}_A(xy) < \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\} \text{ and } \widehat{\lambda}_A(xy) > \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\}.$$

Then, $\widehat{\mu}_A(xy) < \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\}$ and $\widehat{\lambda}_A(xy) > \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\}$

Choose $\widehat{t} \in D(0, 1]$ and $\widehat{s} \in D[0, 1)$ such that

$$\widehat{\mu}_A(xy) < \widehat{t} < \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} \text{ and } \widehat{\lambda}_A(xy) > \widehat{s} > \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\}.$$

Then, $x(\widehat{t}, \widehat{s}) \in A$ and $y(\widehat{t}, \widehat{s}) \in A$, but $(xy)(\widehat{t}, \widehat{s})\overline{\in \vee q}A$, a contradicts.

Case (2) Assume that $\widehat{\mu}_A(xy) < \widehat{0.5}$ and $\widehat{\lambda}_A(xy) > \widehat{0.5}$. Then, $x(\widehat{0.5}, \widehat{0.5}) \in A$ and $y(\widehat{0.5}, \widehat{0.5}) \in A$, but $(xy)(\widehat{0.5}, \widehat{0.5})\overline{\in \vee q}A$, a contradicts. Therefore, $\widehat{\mu}_A(xy) \geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\}$ and $\widehat{\lambda}_A(xy) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\}$.

(b) Now, let $x, y, a \in S$. We consider the following case's

(1) $\min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} < \widehat{0.5}$ and $\max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} > \widehat{0.5}$

(2) $\min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} \geq \widehat{0.5}$ and $\max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} \leq \widehat{0.5}$

(1) Assume that

$$\widehat{\mu}_A(xay) < \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\} \text{ and } \widehat{\lambda}_A(xay) > \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\}$$

$$\widehat{\mu}_A(xay) < \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} \text{ and } \widehat{\lambda}_A(xay) > \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\}$$

Choose $\widehat{t} \in D(0,1]$ and $\widehat{s} \in D[0,1)$ such that

$$\widehat{\mu}_A(xay) < \widehat{t} < \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} \text{ and } \widehat{\lambda}_A(xay) > \widehat{s} > \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\}.$$

Then, $x(\widehat{t},\widehat{s}) \in A$ and $y(\widehat{t},\widehat{s}) \in A$, but $(xay)(\widehat{t},\widehat{s})\overline{\in \vee q}A$, which is a contradiction.

Case (2) Assume that $\widehat{\mu}_A(xay) < \widehat{0.5}$ and $\widehat{\lambda}_A(xay) > \widehat{0.5}$. Then, $x(\widehat{0.5},\widehat{0.5}) \in A$ and $y(\widehat{0.5},\widehat{0.5}) \in A$, but $(xay)(\widehat{0.5},\widehat{0.5})\overline{\in \vee q}A$, which is a contradiction. Therefore,

$$\widehat{\mu}_A(xay) \geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\} \text{ and } \widehat{\lambda}_A(xay) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\}$$

Conversely, assume that $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ satisfy (a) and (b). Let $x, y \in S$, $\widehat{t}_1, \widehat{t}_2 \in D(0,1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0,1)$, be such that $x(\widehat{t}_1,\widehat{s}_1) \in A$ and $y(\widehat{t}_2,\widehat{s}_2) \in A$. Then, $\widehat{\mu}_A(x) \geq t_1$ and $\widehat{\lambda}_A(x) \leq s_1$, and $\widehat{\mu}_A(y) \geq t_2$ and $\widehat{\lambda}_A(y) \leq s_2$. Now we have

$$\begin{aligned}
\widehat{\mu}_A(xy) &\geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\} \text{ and } \widehat{\lambda}_A(xy) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\} \\
\widehat{\mu}_A(xy) &\geq \min\left\{\widehat{t}_1, \widehat{t}_2, \widehat{0.5}\right\} \text{ and } \widehat{\lambda}_A(xy) \leq \max\left\{\widehat{s}_1, \widehat{s}_2, \widehat{0.5}\right\}
\end{aligned}$$

Then, we have the following cases

(1) $\min\left\{\widehat{t}_1, \widehat{t}_2\right\} \leq \widehat{0.5}$ and $\max\{\widehat{s}_1, \widehat{s}_2\} \geq \widehat{0.5}$.

(2) $\min\left\{\widehat{t}_1, \widehat{t}_2\right\} > \widehat{0.5}$ and $\max\{\widehat{s}_1, \widehat{s}_2\} < \widehat{0.5}$, and other cases does not occur.

Case (1) If $\min\left\{\widehat{t}_1, \widehat{t}_2\right\} \leq \widehat{0.5}$ and $\max\{\widehat{s}_1, \widehat{s}_2\} \geq \widehat{0.5}$, then,$\widehat{\mu}_A(xy) \geq \min\left\{\widehat{t}_1, \widehat{t}_2\right\}$ and $\widehat{\lambda}_A(xy) \leq \max\{\widehat{s}_1, \widehat{s}_2\}$, which implies that $(xy)(m\left\{\widehat{t}_1, \widehat{t}_2\right\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in A$

Case(2) If $\min\left\{\widehat{t}_1, \widehat{t}_2\right\} > \widehat{0.5}$ and $\max\{\widehat{s}_1, \widehat{s}_2\} < \widehat{0.5}$, then $\widehat{\mu}_A(xy) \geq \widehat{0.5}$ and $\widehat{\lambda}_A(xy) \leq \widehat{0.5}$, which implies that $\widehat{\mu}_A(xy) + \min\left\{\widehat{t}_1, \widehat{t}_2\right\} > \widehat{0.5} + \widehat{0.5} = 1$ and $\widehat{\lambda}_A(xy) + \max\{\widehat{s}_1, \widehat{s}_2\} < \widehat{0.5} + \widehat{0.5} = 1$. Therefore, $(xy)(m\left\{\widehat{t}_1, \widehat{t}_2\right\}, M\{\widehat{s}_1, \widehat{s}_2\})qA$. Hence, $(xy)(m\left\{\widehat{t}_1, \widehat{t}_2\right\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in \vee qA$.

Let $x, y, a \in S$ and $\widehat{t}_1, \widehat{t}_2 \in D(0,1]$ and $\widehat{s}_1, \widehat{s}_2 \in D[0,1)$ such that $x(\widehat{t}_1,\widehat{s}_1) \in A$ and $y(\widehat{t}_2,\widehat{s}_2) \in A$. Then, $\widehat{\mu}_A(x) \geq t_1$ and $\widehat{\lambda}_A(x) \leq s_1$, and $\widehat{\mu}_A(y) \geq t_2$ and $\widehat{\lambda}_A(y) \leq s_2$. Now we have

$$\begin{aligned}
\widehat{\mu}_A(xay) &\geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\} \text{ and } \widehat{\lambda}_A(xy) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\} \\
\widehat{\mu}_A(xay) &\geq \min\left\{\widehat{t}_1, \widehat{t}_2, \widehat{0.5}\right\} \text{ and } \widehat{\lambda}_A(xay) \leq \max\left\{\widehat{s}_1, \widehat{s}_2, \widehat{0.5}\right\}
\end{aligned}$$

Then, we have the following case's

(3) $\min\left\{\widehat{t}_1, \widehat{t}_2\right\} \leq \widehat{0.5}$ and $\max\{\widehat{s}_1, \widehat{s}_2\} \geq \widehat{0.5}$

(4) $\min\left\{\widehat{t}_1, \widehat{t}_2\right\} > \widehat{0.5}$ and $\max\{\widehat{s}_1, \widehat{s}_2\} < \widehat{0.5}$

Case (3) If $\min\left\{\widehat{t}_1, \widehat{t}_2\right\} \leq \widehat{0.5}$ and $\max\{\widehat{s}_1, \widehat{s}_2\} \geq \widehat{0.5}$, then $\widehat{\mu}_A(xay) \geq \min\left\{\widehat{t}_1, \widehat{t}_2\right\}$ and $\widehat{\lambda}_A(xay) \leq \max\{\widehat{s}_1, \widehat{s}_2\}$, which implies that $(xay)(m\left\{\widehat{t}_1, \widehat{t}_2\right\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in A$.

Case(4) If $\min\left\{\widehat{t}_1, \widehat{t}_2\right\} > \widehat{0.5}$ and $\max\{\widehat{s}_1, \widehat{s}_2\} < \widehat{0.5}$, then $\widehat{\mu}_A(xay) \geq \widehat{0.5}$ and $\widehat{\lambda}_A(xay) \leq \widehat{0.5}$, which implies that $\widehat{\mu}_A(xay) + \min\left\{\widehat{t}_1, \widehat{t}_2\right\} > \widehat{0.5} + \widehat{0.5} = 1$ and

$\widehat{\lambda}_A(xay)+\max\{\widehat{s}_1, \widehat{s}_2\} < \widehat{0.5}+\widehat{0.5} = 1$. Therefore, $(xay)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\})qA$. Hence, $(xay)(m\{\widehat{t}_1, \widehat{t}_2\}, M\{\widehat{s}_1, \widehat{s}_2\}) \in \vee qA$. This completes the proof. □

Remark 1. Every intuitionistic fuzzy bi-ideal is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of semigroup S. But the converse is not true, see example.

Example 1. Let $S = \{1, 2, 3, 4, 5\}$ be a semigroup defined by the following Cayley table.

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	3	3	3
4	1	1	3	4	5
5	1	1	3	3	5

Let $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ be an IVIFS in a semigroup S defined by $\widehat{\mu}_A(1) = \widehat{\mu}_A(2) = \widehat{\mu}_A(4) = [0.8, 0.85]$, $\widehat{\mu}_A(3) = [0.7, 0.75]$, $\widehat{\mu}_A(5) = [0.6, 0.65]$, and $\widehat{\lambda}_A(1) = \widehat{\lambda}_A(2) = \widehat{\lambda}_A(4) = [0.1, 0.15]$, $\widehat{\lambda}_A(3) = [0.2, 0.25]$, $\widehat{\lambda}_A(5) = [0.3, 0.35]$. Then, by routine calculation $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S but not intuitionistic fuzzy bi-ideal. i.e.,

$$\begin{aligned} \widehat{\mu}_A(4 \cdot 5 \cdot 4) &= \widehat{\mu}_A(3) = [0.7, 0.75] \text{ and } \min(\widehat{\mu}_A(4), \widehat{\mu}_A(4)) = [0.8, 0.85] \\ \widehat{\mu}_A(4 \cdot 5 \cdot 4) &< \min(\widehat{\mu}_A(4), \widehat{\mu}_A(4))\,\widehat{\mu}_A(4). \end{aligned}$$

Remark 2. From above Remark and Example, we can say that interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S is a generalization of an intuitionistic fuzzy bi-ideal of S.

Theorem 11. *Let $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ be an IVIFS in a semigroup S. Then, $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy* (1, 2) *ideal of a semigroup S if and only if the following conditions hold;*

(*a*) $\widehat{\mu}_A(xy) \geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\}$ *and* $\widehat{\lambda}_A(xy) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\}$.

(*b*) $\widehat{\mu}_A(xa(yz)) \geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{\mu}_A(z)\,\widehat{0.5}\right\}$ *and*

$\widehat{\lambda}_A(xa(yz)) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{\lambda}_A(z), \widehat{0.5}\right\}$.

Proof. Straightforward. □

Proposition 3. *Every interval valued $(\in, \in \vee q)$-intuitionistic fuzzy left (right) ideal of S is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S.*

Remark 3. If $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S, then $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ need not to be an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy left (right) ideal of S.

Example 2. Let $S = \{1, 2, 3, 4\}$ be a semigroup with the following Cayley table.

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	1	4	1
3	1	1	1	1
4	1	1	1	1

(1) Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an IVIFS defined as; $\widehat{\mu}_A(1) = \widehat{\mu}_A(2) = [0.3, 0.35]$, $\widehat{\mu}_A(3) = \widehat{\mu}_A(4) = [0.1, 0.15]$ and $\widehat{\lambda}_A(1) = \widehat{\lambda}_A(2) = [0.5, 0.5]$, $\widehat{\lambda}_A(3) = \widehat{\lambda}_A(4) = [0.8, 0.85]$. Then, clearly $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S but not interval valued $(\in, \in \vee q)$-intuitionistic fuzzy right ideal of S because

$$\widehat{\mu}_A(23) \ngeq \min\left\{\widehat{\mu}_A(2), \widehat{0.5}\right\}$$

(1) Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an IVIFS defined as; $\widehat{\mu}_A(1) = \widehat{\mu}_A(3) = [0.4, 0.45]$, $\widehat{\mu}_A(2) = \widehat{\mu}_A(4) = [0, 0.05]$ and $\widehat{\lambda}_A(1) = \widehat{\lambda}_A(3) = [0.5, 0.55]$, $\widehat{\mu}_A(2) = \widehat{\mu}_A(4) = [0.7, 0.75]$. Then, clearly $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S but not interval valued $(\in, \in \vee q)$-intuitionistic fuzzy left ideal of S because

$$\widehat{\mu}_A(23) \ngeq \min\left\{\widehat{\mu}_A(3), \widehat{0.5}\right\}$$

Proposition 4. (1)*Every $(\in \vee q, \in \vee q)$-intuitionistic fuzzy bi-ideal of S is interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal.*

(2) *Every $(\in, \in)$-intuitionistic fuzzy bi-ideal of S is interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal.*

Proof. Straightforward. □

Example 1 shows that the converse of Proposition 4, is not true in general.

Theorem 12. *If $\{A\}_{i\in\Lambda}$ is a family of interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideals of S, then $\bigcap_{i\in\Lambda} A_i$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S, where $\bigcap_{i\in\Lambda} A_i = \langle \bigwedge_{i\in\Lambda} \widehat{\mu}_A, \bigvee_{i\in\Lambda} \widehat{\lambda}_A \rangle$.*

Proof. Strightforward. □

Remark 4. If $\{A\}_{i\in\Lambda}$ is a family of interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideals of S, then $\bigcup_{i\in\Lambda} A_i$ is not an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S, where $\bigcup_{i\in\Lambda} A_i = \langle \bigvee_{i\in\Lambda} \widehat{\mu}_A, \bigwedge_{i\in\Lambda} \widehat{\lambda}_A \rangle$. See example below.

Example 3. Let $S = \{1, 2, 3, 4\}$ be a semigroup defined by the following Cayley table.

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	1	4	1
3	1	1	1	1
4	1	1	1	1

Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ and $B = \langle \widehat{\mu}_B, \widehat{\lambda}_B \rangle$ be IVIFS's of semigroup S defined by $\widehat{\mu}_A(1) = \widehat{\mu}_A(2) = [0.4, 0.45]$, $\widehat{\mu}_A(3) = \widehat{\mu}_A(4) = [0, 0.05]$ and $\widehat{\lambda}_A(1) = \widehat{\lambda}_A(2) = [0.5, 0.55]$, $\widehat{\lambda}_A(3) = \widehat{\lambda}_A(4) = [0.8, 0.85]$ and $\widehat{\mu}_B(1) = \widehat{\mu}_B(3) = [0.4, 0.45]$, $\widehat{\mu}_B(2) = \widehat{\mu}_B(4) = [0, 0.05]$ and $\widehat{\lambda}_B(1) = \widehat{\lambda}_B(3) = [0.5, 0.55]$, $\widehat{\mu}_B(2) = \widehat{\mu}_B(4) = [0.7, 0.75]$.

Then, both $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ and $B = \langle \widehat{\mu}_B, \widehat{\lambda}_B \rangle$ are an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideals of S. But $A \cup B$ is not an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S. i.e.

$$\begin{aligned} (\widehat{\mu}_A \vee \widehat{\mu}_B)(23) &= [0, 0.05], \\ \min\left\{(\widehat{\mu}_A \vee \widehat{\mu}_B)(2), (\widehat{\mu}_A \vee \widehat{\mu}_B)(3), \widehat{0.5}\right\} &= [0.4, 0.45]. \end{aligned}$$

Hence, $(\widehat{\mu}_A \vee \widehat{\mu}_B)(23) \ngeq \min\left\{(\widehat{\mu}_A \vee \widehat{\mu}_B)(2), (\widehat{\mu}_A \vee \widehat{\mu}_B)(3), \widehat{0.5}\right\}$.

Theorem 13. *If $\{A_i\}_{i\in\Lambda}$ is a family of interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideals of S such that $A_i \subseteq A_j$ or $A_j \subseteq A_i$ for all $i, j \in I$, then $\bigcup_{i\in\Lambda} A_i = \langle \bigvee_{i\in\Lambda} \widehat{\mu}_A, \bigwedge_{i\in\Lambda} \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S.*

Proof. For all $x, y \in S$, we have

$$\begin{aligned} \left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(xy) &= \bigvee_{i\in\Lambda} (\widehat{\mu}_A(xy)) \geq \bigvee_{i\in\Lambda} \left[\widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y) \wedge \widehat{0.5}\}\right] \\ &= \left[\bigvee_{i\in\Lambda} \widehat{\mu}_A(x) \wedge \bigvee_{i\in\Lambda} \widehat{\mu}_A(y) \wedge \widehat{0.5}\right] \\ &= \left[\left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(y) \wedge \widehat{0.5}\right] \\ \left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(xy) &\geq \left[\left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(y) \wedge \widehat{0.5}\right]. \end{aligned}$$

It is clear that

$$\bigvee_{i\in\Lambda} \left[\widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y) \wedge \widehat{0.5}\right] \leq \left[\left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(y) \wedge \widehat{0.5}\right].$$

Assume that

$$\bigvee_{i\in\Lambda} \left[\widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y) \wedge \widehat{0.5}\right] \neq \left[\left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(y) \wedge \widehat{0.5}\right].$$

Then, there exists t such that

$$\bigvee_{i\in\Lambda} \left[\widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y) \wedge \widehat{0.5}\right] < \widehat{t_1} < \left[\left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(y) \wedge \widehat{0.5}\right]$$

Since $\widehat{\mu}_A \subseteq \widehat{\mu}_A$ or $\widehat{\mu}_A \subseteq \widehat{\mu}_A$ for all $i,\ j \in I$, so there exists $k \in I$ such that $t < \widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y) \wedge \widehat{0.5}$. On other hand $\widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y) \wedge \widehat{0.5} < t$ for all $i \in I$, a contradiction. Hence,

$$\bigvee_{i\in\Lambda}\left[\widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y) \wedge \widehat{0.5}\right] = \left[\left(\bigvee_{i\in\Lambda}\widehat{\mu}_A\right)(x) \wedge \left(\bigvee_{i\in\Lambda}\widehat{\mu}_A\right)(y) \wedge \widehat{0.5}\right]$$

and

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(xy) &= \bigwedge_{i\in\Lambda}\left(\widehat{\lambda}_A(xy)\right) \leq \bigwedge_{i\in\Lambda}\left[\widehat{\lambda}_A(x) \vee \widehat{\lambda}_A(y) \vee \widehat{0.5}\}\right] \\
&= \left[\bigwedge_{i\in\Lambda}\widehat{\lambda}_A(x) \vee \bigwedge_{i\in\Lambda}\widehat{\lambda}_A(y) \vee \widehat{0.5}\right] \\
&= \left[\left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(x) \vee \left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(y) \vee \widehat{0.5}\right] \\
\left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(xy) &\leq \left[\left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(x) \vee \left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(y) \vee \widehat{0.5}\right].
\end{aligned}$$

It is clear that

$$\bigwedge_{i\in\Lambda}\left[\widehat{\lambda}_A(x) \vee \widehat{\lambda}_A(y) \vee \widehat{0.5}\right] \geq \left[\left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(x) \vee \left(\bigwedge_{i\in\Lambda}\widehat{\mu}_A\right)(y) \vee \widehat{0.5}\right].$$

Assume that

$$\bigwedge_{i\in\Lambda}\left[\widehat{\lambda}_A(x) \vee \widehat{\lambda}_A(y) \vee \widehat{0.5}\right] \neq \left[\left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(x) \vee \left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(y) \vee \widehat{0.5}\right].$$

Then there exists t such that

$$\bigwedge_{i\in\Lambda}\left[\widehat{\lambda}_A(x) \vee \widehat{\lambda}_A(y) \vee \widehat{0.5}\right] > t > \left[\left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(x) \vee \left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(y) \vee \widehat{0.5}\right]$$

Since $\widehat{\lambda}_A \subseteq \widehat{\lambda}_A$ or $\widehat{\lambda}_A \subseteq \widehat{\lambda}_A$ for all $i,\ j \in I$, so there exists $k \in I$ such that $k > \widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y) \wedge \widehat{0.5}$. On the other hand, $\widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y) \wedge \widehat{0.5} > t$ for all $i \in I$, a contradiction. Hence,

$$\bigwedge_{i\in\Lambda}\left[\widehat{\lambda}_A(x) \wedge \widehat{\lambda}_A(y) \wedge \widehat{0.5}\right] = \left[\left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(x) \wedge \left(\bigwedge_{i\in\Lambda}\widehat{\lambda}_A\right)(y) \wedge \widehat{0.5}\right]$$

Let $x, a, y \in S$, we obtain

$$\left(\bigvee_{i\in\Lambda}\widehat{\mu}_A\right)(xay) = \bigvee_{i\in\Lambda}\left(\widehat{\mu}_A(xay)\right) \geq \bigvee_{i\in\Lambda}\left[\widehat{\mu}_A(x) \wedge \widehat{\mu}_A(y) \wedge \widehat{0.5}\}\right]$$

$$\begin{aligned}
&= \left[\bigvee_{i\in\Lambda} \widehat{\mu}_A(x) \wedge \bigvee_{i\in\Lambda} \widehat{\mu}_A(y) \wedge \widehat{0.5}\right] \\
&= \left[\left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(y) \wedge \widehat{0.5}\right] \\
\left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(xay) &\geq \left[\left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \widehat{\mu}_A\right)(y) \wedge \widehat{0.5}\right]
\end{aligned}$$

and

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda} \widehat{\lambda}_A\right)(xay) &= \bigwedge_{i\in\Lambda} \left(\widehat{\lambda}_A(xay)\right) \leq \bigwedge_{i\in\Lambda} \left[\widehat{\lambda}_A(x) \vee \widehat{\lambda}_A(y) \vee \widehat{0.5}\}\right] \\
&= \left[\bigwedge_{i\in\Lambda} \widehat{\lambda}_A(x) \vee \bigwedge_{i\in\Lambda} \widehat{\lambda}_A(y) \vee \widehat{0.5}\right] \\
&= \left[\left(\bigwedge_{i\in\Lambda} \widehat{\lambda}_A\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \widehat{\lambda}_A\right)(y) \vee \widehat{0.5}\right] \\
\left(\bigwedge_{i\in\Lambda} \widehat{\lambda}_A\right)(xay) &\leq \left[\left(\bigwedge_{i\in\Lambda} \widehat{\lambda}_A\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \widehat{\lambda}_A\right)(y) \vee \widehat{0.5}\right]
\end{aligned}$$

Hence, $\bigcup_{i\in\Lambda} A_i = \langle \bigvee_{i\in\Lambda} \widehat{\mu}_A, \bigwedge_{i\in\Lambda} \widehat{\lambda}_A\rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S. □

Definition 12. Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A\rangle$ and $B = \langle \widehat{\mu}_B, \widehat{\lambda}_B\rangle$ be IVIFSs of S. Then, the $\widehat{0.5}$-product of A and B is defined by:

$$\begin{aligned}
A \circ_{\widehat{0.5}} B &= \langle \widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B, \widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\rangle \\
\left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)(x) &= \left\{\begin{array}{ll} \bigvee_{x=yz} \left\{\widehat{\mu}_A(y) \wedge \widehat{\mu}_B(z) \wedge \widehat{0.5}\right\} & \text{if } x = yz \\ 0 & \text{if } x \neq yz \end{array}\right\} \\
\left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right)(x) &= \left\{\begin{array}{ll} \bigwedge_{x=yz} \left\{\widehat{\lambda}_A(y) \vee \widehat{\lambda}_B(z) \vee \widehat{0.5}\right\} & \text{if } x = yz \\ 1 & \text{if } x \neq yz \end{array}\right\} \\
A \cap_{\widehat{0.5}} B &= \langle \widehat{\mu}_A \wedge_{\widehat{0.5}} \widehat{\mu}_B, \widehat{\lambda}_A \vee_{\widehat{0.5}} \widehat{\lambda}_B\rangle \\
\left(\widehat{\mu}_A \wedge_{\widehat{0.5}} \widehat{\mu}_B\right)(x) &= \widehat{\mu}_A(x) \wedge \widehat{\mu}_B(x) \wedge \widehat{0.5} \text{ and} \\
\left(\widehat{\lambda}_A \vee_{\widehat{0.5}} \widehat{\lambda}_B\right)(x) &= \widehat{\lambda}_A(x) \vee \widehat{\lambda}_B(x) \vee \widehat{0.5}.
\end{aligned}$$

Remark 5. If S is a semigroup and A, B, C, D are IVIFSs of S such that $A \subseteq B$ and $C \subseteq D$, then $A \circ_{\widehat{0.5}} B \subseteq C \circ_{\widehat{0.5}} D$.

Proposition 5. *Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ and $B = \langle \widehat{\mu}_B, \widehat{\lambda}_B \rangle$ be interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideals of S. Then, $A \cap_{\widehat{0.5}} B$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S.*

Proof. Straightforward. □

Definition 13. An interval valued $(\in, \in \vee q)$-intuitionistic *fuzzy bi-ideal of* S is called $\widehat{0.5}$-idemoptent if $A \circ_{\widehat{0.5}} A = A$.

Proposition 6. *Let S be a semigroup and A is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy subsemigroup of S. Then, $A \circ_{\widehat{0.5}} A \subseteq A$.*

Proof. Straightforward. □

Lemma 1. *Let S be a semigroup, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ and $B = \langle \widehat{\mu}_B, \widehat{\lambda}_B \rangle$ be interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideals of S. Then, $A \circ_{\widehat{0.5}} B \subseteq 1 \circ_{\widehat{0.5}} B$ (resp. $A \circ_{\widehat{0.5}} B \subseteq A \circ_{\widehat{0.5}} 1$).*

Theorem 14. *Let S be a semigroup, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S. Then, $A \circ_{\widehat{0.5}} \mathcal{S} \circ_{\widehat{0.5}} A \subseteq A$, where $\mathcal{S} = \langle \widehat{1}, \widehat{0} \rangle$, $\widehat{1}(x) = \widehat{1}$ and $\widehat{0}(x) = \widehat{0}$ for all $x \in S$.*

Proof. Let $x \in S$. Then, we have two cases. (1) If $x \neq yz$ for every $y, z \in S$. (2) If $x = yz$ for some $y, z \in S$.

Case 1 : If $x \neq yz$, then clearly

$$\left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{1} \circ_{\widehat{0.5}} \widehat{\mu}_A\right)(x) = 0 \leq \widehat{\mu}_A(x) \wedge \widehat{0.5} \text{ and } \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{0} \circ_{\widehat{0.5}} \widehat{\lambda}_A\right) = \widehat{1} \geq \widehat{\lambda}_A \vee \widehat{0.5}$$

Thus, $A \circ_{\widehat{0.5}} \mathcal{S} \circ_{\widehat{0.5}} A \subseteq A$.

Case 2 :If $x = yz$ for some $y, z \in S$, then we have

$$\begin{aligned}
\left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{1} \circ_{\widehat{0.5}} \widehat{\mu}_A\right)(x) &= \bigvee_{x=yz} \left\{\min\left\{\widehat{\mu}_A(y), \left(\widehat{1} \circ_{\widehat{0.5}} \widehat{\mu}_A\right)(z), \widehat{0.5}\right\}\right\} \\
&= \bigvee_{x=yz} \left\{\min\left\{\widehat{\mu}_A(y), \bigvee_{x=tr} \left\{\min\left\{\widehat{1}(t), \widehat{\mu}_A(r), \widehat{0.5}\right\}\right\}, \widehat{0.5}\right\}\right\} \\
&= \bigvee_{x=yz} \bigvee_{x=tr} \left\{\min\left\{\widehat{\mu}_A(y), \widehat{1}, \widehat{\mu}_A(r), \widehat{0.5}\right\}\right\} \\
&= \bigvee_{x=ytr} \left\{\min\left\{\widehat{\mu}_A(y), \widehat{\mu}_A(r), \widehat{0.5}\right\}\right\}
\end{aligned}$$

Since $x = yz = y(tr) = ytr$ and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S, so we have $\widehat{\mu}_A(ytr) \geq \min\left\{\widehat{\mu}_A(y), \widehat{\mu}_A(r), \widehat{0.5}\right\}$. Thus,

$$\bigvee_{x=ytr} \left\{\min\left\{\widehat{\mu}_A(y), \widehat{\mu}_A(r), \widehat{0.5}\right\}\right\} \leq \bigvee_{x=ytr} \{\widehat{\mu}_A(ytr)\} = \widehat{\mu}_A(x)$$

$$\left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{1} \circ_{\widehat{0.5}} \widehat{\mu}_A\right)(x) \leq \widehat{\mu}_A(x)$$

and

$$\begin{aligned}\left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{0} \circ_{\widehat{0.5}} \widehat{\lambda}_A\right)(x) &= \bigwedge_{x=yz} \left\{\max\left\{\widehat{\lambda}_A(y), \left(\widehat{0} \circ_{\widehat{0.5}} \widehat{\lambda}_A\right)(z), \widehat{0.5}\right\}\right\} \\ &= \bigwedge_{x=yz} \left\{\max\left\{\widehat{\mu}_A(y), \bigwedge_{x=tr} \left\{\max\left\{\widehat{0}(t), \widehat{\lambda}_A(r), \widehat{0.5}\right\}\right\}, \widehat{0.5}\right\}\right\} \\ &= \bigwedge_{x=yz} \bigwedge_{x=tr} \left\{\max\left\{\widehat{\lambda}_A(y), 0, \widehat{\lambda}_A(r), \widehat{0.5}\right\}\right\} \\ &= \bigwedge_{x=ytr} \left\{\max\left\{\widehat{\lambda}_A(y), \widehat{\lambda}_A(r), \widehat{0.5}\right\}\right\}.\end{aligned}$$

Since $x = yz = y(tr) = ytr$ and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S, so we have $\widehat{\lambda}_A(ytr) \leq \max\left\{\widehat{\lambda}_A(y), \widehat{\lambda}_A(r), \widehat{0.5}\right\}$. Thus,

$$\bigwedge_{x=ytr} \left\{\min\left\{\widehat{\lambda}_A(y), \widehat{\lambda}_A(r), \widehat{0.5}, \widehat{0.5}\right\}\right\} \geq \bigwedge_{x=ytr} \left\{\widehat{\lambda}_A(ytr)\right\} = \widehat{\lambda}_A(x)$$

$$\left(\widehat{\lambda}_A \circ_{\widehat{0.5}} 0 \circ_{\widehat{0.5}} \widehat{\lambda}_A\right)(x) \leq \widehat{\lambda}_A(x)$$

Hence, $A \circ_{\widehat{0.5}} \mathcal{S} \circ_{\widehat{0.5}} A \subseteq A$. □

Theorem 15. *Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an IVIFS of S. Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy subsemigroup of S if and only if $A \circ_{\widehat{0.5}} A \subseteq A$.*

Theorem 16. *Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an IVIFS of S. Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S if and only if the following hold:*

(1) $A \circ_{\widehat{0.5}} A \subseteq A$,

(2) $A \circ_{\widehat{0.5}} \mathcal{S} \circ_{\widehat{0.5}} A \subseteq A$.

Proof. Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S. Then, by Proposition 6 and Theorem 14, we have $A \circ_{\widehat{0.5}} A \subseteq A$ and $A \circ_{\widehat{0.5}} \mathcal{S} \circ_{\widehat{0.5}} A \subseteq A$.

Conversely, suppose that the given conditions hold. Now, let $x, y \in S$ such that $a = xy$. Then, we have

$$\begin{aligned}\widehat{\mu}_A(xy) = \widehat{\mu}_A(a) \geq \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_A\right)(a) &= \bigvee_{a=st} \left\{\min\left\{\widehat{\mu}_A(s), \widehat{\mu}_A(t), \widehat{0.5}\right\}\right\} \\ &\geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\}\end{aligned}$$

$$\widehat{\mu}_A(xy) \geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\}$$

and

$$\widehat{\lambda}_A(xy) = \widehat{\lambda}_A(a) \leq \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_A\right)(a) = \bigwedge_{a=st} \left\{\max\left\{\widehat{\lambda}_A(s), \widehat{\lambda}_A(t), \widehat{0.5}\right\}\right\}$$

$$\leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\}$$

$$\widehat{\lambda}_A(xy) \leq \min\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\}.$$

Now, let $x, y, z \in S$ such that $a = xyz$. Then, we have

$$\widehat{\mu}_A(xyz) = \widehat{\mu}_A(a) \geq (\widehat{\mu}_A \circ_{\widehat{0.5}} 1 \circ_{\widehat{0.5}} \widehat{\mu}_A)(a) = \bigvee_{a=st}\left\{\min\left\{\widehat{\mu}_A(s), (1 \circ_{\widehat{0.5}} \widehat{\mu}_A)(t), \widehat{0.5}\right\}\right\}$$

$$= \bigvee_{a=st}\left\{\min\left\{\widehat{\mu}_A(s), \bigvee_{t=pq}\left\{\min\left\{1(p), \widehat{\mu}_A(q), \widehat{0.5}\right\}\right\}, \widehat{0.5}\right\}\right\}$$

$$= \bigvee_{a=st}\left\{\min\left\{\widehat{\mu}_A(s), \bigvee_{t=pq}\left\{\min\left\{1, \widehat{\mu}_A(q), \widehat{0.5}\right\}\right\}, \widehat{0.5}\right\}\right\}$$

$$\geq \bigvee_{a=st}\bigvee_{t=pq}\left\{\min\left\{\widehat{\mu}_A(s), \widehat{\mu}_A(q), \widehat{0.5}\right\}\right\} \geq \bigvee_{a=spq}\left\{\min\left\{\widehat{\mu}_A(s), \widehat{\mu}_A(q), \widehat{0.5}\right\}\right\}$$

$$\geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(z), \widehat{0.5}\right\}$$

$$\widehat{\mu}_A(xyz) \geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(z), \widehat{0.5}\right\}.$$

and

$$\widehat{\lambda}_A(xyz) = \widehat{\lambda}_A(a) \geq \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} 0 \circ_{\widehat{0.5}} \widehat{\lambda}_A\right)(a) = \bigwedge_{a=st}\left\{\max\left\{\widehat{\lambda}_A(s), \left(0 \circ_{\widehat{0.5}} \widehat{\lambda}_A\right)(t), \widehat{0.5}\right\}\right\}$$

$$= \bigwedge_{a=st}\left\{\max\left\{\widehat{\lambda}_A(s), \bigwedge_{t=pq}\left\{\max\left\{0(p), \widehat{\lambda}_A(q), \widehat{0.5}\right\}\right\}, \widehat{0.5}\right\}\right\}$$

$$= \bigwedge_{a=st}\left\{\max\left\{\widehat{\lambda}_A(s), \bigwedge_{t=pq}\left\{\max\left\{0, \widehat{\lambda}_A(q), \widehat{0.5}\right\}\right\}, \widehat{0.5}\right\}\right\}$$

$$\leq \bigwedge_{a=st}\bigwedge_{t=pq}\left\{\max\left\{\widehat{\lambda}_A(s), \widehat{\lambda}_A(q), \widehat{0.5}\right\}\right\} \geq \bigwedge_{a=spq}\left\{\max\left\{\widehat{\lambda}_A(s), \widehat{\lambda}_A(q), \widehat{0.5}\right\}\right\}$$

$$\leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(z), \widehat{0.5}\right\}$$

$$\widehat{\lambda}_A(xyz) \leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(z), \widehat{0.5}\right\}.$$

Hence, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S. □

Theorem 17. *Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an IVIFS of S. Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right, two sided) ideal of S if and only if the following hold: $\mathcal{S} \circ_{\widehat{0.5}} A \subseteq A$ (resp. $A \circ_{\widehat{0.5}} \mathcal{S} \subseteq A$, $A \circ_{\widehat{0.5}} \mathcal{S} \subseteq A$ and $\mathcal{S} \circ_{\widehat{0.5}} A \subseteq A$).*

Proof. Straightforward. □

Theorem 18. *Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ and $B = \langle \widehat{\mu}_B, \widehat{\lambda}_B \rangle$ be interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideals of S. Then, $A \circ_{\widehat{0.5}} B$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S.*

Proof. Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ and $B = \langle \widehat{\mu}_B, \widehat{\lambda}_B \rangle$ be interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideals of S and let $x \in S$. Then, we have two cases (1) If $x \neq yz$ for any $y, z \in S$. (2) If $x = yz$ for some $y, z \in S$.

Case 1 : If $x \neq yz$ for any $y, z \in S$, then

$$\left(\left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right) \circ_{\widehat{0.5}} \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)\right)(x) = 0 \leq \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)(x)$$

and

$$\left(\left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right) \circ_{\widehat{0.5}} \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right)\right)(x) = 1 \geq \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right)(x)$$

Thus, $A \circ_{\widehat{0.5}} A \subseteq A$ in this case.

Case 2 : If $x = yz$ for some $y, z \in S$, then

$$\begin{aligned}
\left(\left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right) \circ_{\widehat{0.5}} \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)\right)(a) &= \bigvee_{x=yz} \left\{ \begin{array}{c} \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)(y) \wedge \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)(z) \\ \wedge \widehat{0.5} \end{array} \right\} \\
&= \bigvee_{x=yz} \left\{ \begin{array}{c} \bigvee_{y=ab} \left\{ \widehat{\mu}_A(a) \wedge \widehat{\mu}_B(b) \wedge \widehat{0.5} \right\} \\ \wedge \bigvee_{z=pq} \left\{ \widehat{\mu}_A(p) \wedge \widehat{\mu}_B(q) \wedge \widehat{0.5} \right\} \end{array} \right\} \\
&= \bigvee_{x=yz} \bigvee_{y=ab} \bigvee_{z=pq} \left\{ \begin{array}{c} \left\{ \widehat{\mu}_A(a) \wedge \widehat{\mu}_B(b) \wedge \widehat{0.5} \right\} \\ \wedge \left\{ \widehat{\mu}_A(p) \wedge \widehat{\mu}_B(q) \wedge \widehat{0.5} \right\} \end{array} \right\} \\
&= \bigvee_{x=yz} \bigvee_{y=ab} \bigvee_{z=pq} \left\{ \begin{array}{c} \{\widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \\ \wedge \widehat{\mu}_B(b) \wedge \widehat{\mu}_B(q) \wedge \widehat{0.5}\} \end{array} \right\} \\
&\leq \bigvee_{x=yz} \bigvee_{y=ab} \bigvee_{z=pq} \left\{ \widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{0.5} \wedge \widehat{\mu}_B(q) \right\}
\end{aligned}$$

Since $x = yz$, $y = ab$ and $z = pq$. So, $x = (ab)(pq) = (abp)\, q$ and we have

$$\begin{aligned}
&\bigvee_{x=yz} \bigvee_{y=ab} \bigvee_{z=pq} \left\{ \widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{0.5} \wedge \widehat{\mu}_B(q) \right\} \\
\leq \; &\bigvee_{x=(pbp)q} \left\{ \widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{0.5} \wedge \widehat{\mu}_B(q) \right\}
\end{aligned}$$

Since $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S we have

$$\widehat{\mu}_A(abp) \geq \widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{0.5}.$$

So,

$$\begin{aligned}
&\bigvee_{x=(abp)q} \left\{ \widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{0.5} \wedge \widehat{\mu}_B(q) \right\} \\
\leq \; &\bigvee_{x=(abp)q} \left\{ \widehat{\mu}_A(abp) \wedge \widehat{\mu}_B(q) \wedge \widehat{0.5} \right\} \\
\leq \; &\bigvee_{x=cd} \left\{ \widehat{\mu}_A(c) \wedge \widehat{\mu}_B(d) \wedge \widehat{0.5} \right\} = \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)(x)
\end{aligned}$$

Therefore, $\left(\left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right) \circ_{\widehat{0.5}} \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)\right)(x) \leq \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)(x)$. Now,

$$
\begin{aligned}
\left(\left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right) \circ_{\widehat{0.5}} \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right)\right)(a) &= \bigwedge_{x=yz} \left\{ \begin{array}{c} \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right)(y) \vee \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right)(z) \\ \vee \widehat{0.5} \end{array} \right\} \\
&= \bigwedge_{x=yz} \left\{ \begin{array}{c} \bigwedge_{y=ab} \left\{\widehat{\lambda}_A(a) \vee \widehat{\lambda}_B(b) \vee \widehat{0.5}\right\} \\ \vee \bigwedge_{z=pq} \left\{\widehat{\lambda}_A(p) \vee \widehat{\lambda}_B(q) \vee \widehat{0.5}\right\} \end{array} \right\} \\
&= \bigwedge_{x=yz} \bigwedge_{y=ab} \bigwedge_{z=pq} \left\{ \begin{array}{c} \left\{\widehat{\lambda}_A(a) \vee \widehat{\lambda}_B(b) \vee \widehat{0.5}\right\} \\ \vee \left\{\widehat{\lambda}_A(p) \vee \widehat{\lambda}_B(q) \vee \widehat{0.5}\right\} \end{array} \right\} \\
&= \bigwedge_{x=yz} \bigwedge_{y=ab} \bigwedge_{z=pq} \left\{ \begin{array}{c} \{\widehat{\lambda}_A(a) \vee \widehat{\lambda}_A(p) \\ \vee \widehat{\lambda}_B(b) \vee \widehat{\lambda}_B(q) \vee \widehat{0.5}\} \end{array} \right\} \\
&\geq \bigwedge_{x=yz} \bigwedge_{y=ab} \bigwedge_{z=pq} \left\{\widehat{\lambda}_A(a) \vee \widehat{\lambda}_A(p) \vee \widehat{0.5} \vee \widehat{\lambda}_B(q)\right\}
\end{aligned}
$$

Since $x = yz$, $y = ab$ and $z = pq$. So, $x = (ab)(pq) = (abp)q$ and we have

$$
\begin{aligned}
&\bigwedge_{x=yz} \bigwedge_{y=ab} \bigwedge_{z=pq} \left\{\widehat{\lambda}_A(a) \vee \widehat{\lambda}_A(p) \vee \widehat{0.5} \vee \widehat{\lambda}_B(q)\right\} \\
\geq &\bigwedge_{x=(abp)q} \left\{\widehat{\lambda}_A(a) \vee \widehat{\lambda}_A(p) \vee \widehat{0.5} \vee \widehat{\lambda}_B(q)\right\}
\end{aligned}
$$

Since $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S we have

$$\widehat{\lambda}_A(abp) \leq \widehat{\lambda}_A(a) \vee \widehat{\lambda}_A(p) \vee \widehat{0.5}.$$

So,

$$
\begin{aligned}
&\bigwedge_{x=(abp)q} \left\{\widehat{\lambda}_A(a) \vee \widehat{\lambda}_A(p) \vee \widehat{0.5} \vee \widehat{\lambda}_B(q)\right\} \\
\leq &\bigwedge_{x=(abp)q} \left\{\widehat{\lambda}_A(abp) \vee \widehat{\lambda}_B(q) \vee \widehat{0.5}\right\} \\
\leq &\bigwedge_{x=cd} \left\{\widehat{\lambda}_A(c) \vee \widehat{\lambda}_B(d) \vee \widehat{0.5}\right\} = \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right)(x).
\end{aligned}
$$

Therefore, $\left(\left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right) \circ_{\widehat{0.5}} \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right)\right)(a) \geq \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B\right)(x)$ and so $A \circ_{\widehat{0.5}} A \subseteq A$. Thus, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy subsemigroup of S.

Now, let $x, y, z \in S$. Then,

$$
\left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)(x) \wedge \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B\right)(z) \wedge \widehat{0.5} = \begin{array}{c} \left[\bigvee_{x=ab} \left\{\widehat{\mu}_A(a) \wedge \widehat{\mu}_B(b) \wedge \widehat{0.5}\right\}\right] \wedge \\ \left[\bigvee_{z=pq} \left\{\widehat{\mu}_A(p) \wedge \widehat{\mu}_B(q) \wedge \widehat{0.5}\right\}\right] \wedge \widehat{0.5} \end{array}
$$

$$= \bigvee_{x=ab} \bigvee_{z=pq} \left[\begin{array}{c} \left\{ \widehat{\mu}_A(a) \wedge \widehat{\mu}_B(b) \wedge \widehat{0.5} \right\} \\ \wedge \left\{ \widehat{\mu}_A(p) \wedge \widehat{\mu}_B(q) \wedge \widehat{0.5} \right\} \\ \wedge \widehat{0.5} \end{array} \right]$$

$$\leq \bigvee_{x=ab} \bigvee_{z=pq} \left[\begin{array}{c} \widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{\mu}_B(b) \\ \wedge \widehat{\mu}_B(q) \wedge \widehat{0.5} \end{array} \right]$$

$$\leq \bigvee_{x=ab} \bigvee_{z=pq} \left[\widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{\mu}_B(q) \wedge \widehat{0.5} \right]$$

Since $x = ab$ and $z = pq$, so $xyz = (ab)\, y\, (pq) = (a\,(by)\,p)\, q$ and we have

$$\bigvee_{x=ab} \bigvee_{z=pq} \left[\widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{\mu}_B(q) \wedge \widehat{0.5} \right]$$
$$\leq \bigvee_{xyz=(a(by)p)q} \left[\left\{ \widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{0.5} \right\} \wedge \widehat{\mu}_B(q) \right]$$

Since $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S we have

$$\widehat{\mu}_A(a\,(by)\,p) \geq \widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{0.5}.$$

So,

$$\bigvee_{xyz=(a(by)p)q} \left[\left\{ \widehat{\mu}_A(a) \wedge \widehat{\mu}_A(p) \wedge \widehat{0.5} \right\} \wedge \widehat{\mu}_B(q) \right]$$
$$\leq \bigvee_{xyz=(a(by)p)q} \left[\widehat{\mu}_A(a\,(by)\,p) \wedge \widehat{\mu}_B(q) \right] = \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B \right)(xyz).$$

Thus,

$$\left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B \right)(xyz) \geq \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B \right)(x) \wedge \left(\widehat{\mu}_A \circ_{\widehat{0.5}} \widehat{\mu}_B \right)(z) \wedge \widehat{0.5}$$

and

$$\left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B \right)(x) \vee \left(\widehat{\lambda}_A \circ_{\widehat{0.5}} \widehat{\lambda}_B \right)(z) \vee \widehat{0.5} = \begin{array}{c} \left[\bigwedge_{x=ab} \left\{ \widehat{\lambda}_A(a) \vee \widehat{\lambda}_B(b) \vee \widehat{0.5} \right\} \right] \vee \\ \left[\bigwedge_{z=pq} \left\{ \widehat{\lambda}_A(p) \vee \widehat{\lambda}_B(q) \vee \widehat{0.5} \right\} \right] \\ \vee \widehat{0.5} \end{array}$$

$$= \bigwedge_{x=ab} \bigwedge_{z=pq} \left[\begin{array}{c} \left\{ \widehat{\lambda}_A(a) \vee \widehat{\lambda}_B(b) \vee \widehat{0.5} \right\} \\ \vee \left\{ \widehat{\lambda}_A(p) \vee \widehat{\lambda}_B(q) \vee \widehat{0.5} \right\} \\ \vee \widehat{0.5} \end{array} \right]$$

$$\geq \bigwedge_{x=ab} \bigwedge_{z=pq} \left[\begin{array}{c} \widehat{\lambda}_A(a) \vee \widehat{\lambda}_A(p) \vee \widehat{\lambda}_B(b) \\ \vee \widehat{\lambda}_B(q) \vee \widehat{0.5} \end{array} \right]$$

$$\geq \bigwedge_{x=ab} \bigwedge_{z=pq} \left[\widehat{\lambda}_A(a) \vee \widehat{\lambda}_A(p) \vee \widehat{\lambda}_B(q) \vee \widehat{0.5} \right]$$

Since $x = ab$ and $z = pq$, so $xyz = (ab)\,y\,(pq) = (a\,(by)\,p)\,q$ and we have

$$\begin{aligned}
&\bigwedge_{x=ab}\bigwedge_{z=pq}\left[\widehat{\lambda}_A(a)\vee\widehat{\lambda}_A(p)\vee\widehat{\lambda}_B(q)\vee\widehat{0.5}\right]\\
\geq &\bigwedge_{xyz=(a(by)p)q}\left[\left\{\widehat{\lambda}_A(a)\vee\widehat{\lambda}_A(p)\vee\widehat{0.5}\right\}\vee\widehat{\lambda}_B(q)\right]
\end{aligned}$$

Since $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ is an interval valued $(\in, \in\vee q)$-intuitionistic fuzzy bi-ideal of S we have

$$\widehat{\lambda}_A(a\,(by)\,p) \geq \widehat{\lambda}_A(a)\vee\widehat{\lambda}_A(p)\vee\widehat{0.5}.$$

So,

$$\begin{aligned}
&\bigwedge_{xyz=(a(by)p)q}\left[\left\{\widehat{\lambda}_A(a)\vee\widehat{\lambda}_A(p)\vee\widehat{0.5}\right\}\vee\widehat{\lambda}_B(q)\right]\\
\geq &\bigwedge_{xyz=(a(by)p)q}\left[\widehat{\lambda}_A(a\,(by)\,p)\vee\widehat{\lambda}_B(q)\right] = \left(\widehat{\lambda}_A\circ_{\widehat{0.5}}\widehat{\lambda}_B\right)(xyz).
\end{aligned}$$

Thus,

$$\left(\widehat{\lambda}_A\circ_{\widehat{0.5}}\widehat{\lambda}_B\right)(xyz) \leq \left(\widehat{\lambda}_A\circ_{\widehat{0.5}}\widehat{\lambda}_B\right)(x)\vee\left(\widehat{\lambda}_A\circ_{\widehat{0.5}}\widehat{\lambda}_B\right)(z)\vee\widehat{0.5}.$$

Hence, $A\circ_{\widehat{0.5}}\widehat{B}$ is an interval valued $(\in, \in\vee q)$-intuitionistic fuzzy bi-ideal of S. □

For any interval valued intuitionistic fuzzy set $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ in S and $\widehat{t} \in D(0,1]$, $s \in D[0,1)$, we denote $A_{(\widehat{t},\widehat{s})} = \{x \in S : x\,(t,s)\,qA\}$ and $[A]_{(\widehat{t},\widehat{s})} = \{x \in S : x\,(t,s) \in \vee qA\}$.

Obviously, $[A]_{(\widehat{t},\widehat{s})} = A_{(\widehat{t},\widehat{s})} \cup U_{(\widehat{t},\widehat{s})}$, where $U_{(\widehat{t},\widehat{s})}$, $A_{(\widehat{t},\widehat{s})}$ and $[A]_{(\widehat{t},\widehat{s})}$ are called $\in$-level set, q-level set and $\in\vee q$-level set of $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$, respectively.

Theorem 19. *Let S be a semigroup and $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ an IVIFS of S. Then, $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ is an interval valued $(\in, \in\vee q)$-intuitionistic fuzzy left (resp. right) ideal of S if and only if for all $\widehat{t} \in D(0, 0.5]$ and $\widehat{s} \in D[0.5, 1)$, the set $U_{(\widehat{t},\widehat{s})} \neq \phi$ is a left (resp. right) ideal of S.*

Proof. Let $A = \langle\widehat{\mu}_A, \widehat{\lambda}_A\rangle$ be an interval valued $(\in, \in\vee q)$-intuitionistic fuzzy left ideal of S and $U_{(\widehat{t},\widehat{s})} \neq \phi$ for any $\widehat{t} \in D(0, 0.5]$ and $\widehat{s} \in D[0.5, 1)$. Let $y \in U_{(\widehat{t},\widehat{s})} \neq \phi$ and $x \in S$. Then, $\widehat{\mu}_A(x) \geq \widehat{t}$ and $\widehat{\lambda}_A(x) \leq s$. Since

$$\begin{aligned}
\widehat{\mu}_A(xy) &\geq \widehat{\mu}_A(y)\wedge\widehat{0.5} \geq \widehat{t}\wedge\widehat{0.5} \geq \widehat{t} \text{ and}\\
\widehat{\lambda}_A(xy) &\leq \widehat{\lambda}_A(y)\vee\widehat{0.5} \leq \widehat{s}\vee\widehat{0.5} \leq \widehat{s}\\
\widehat{\mu}_A(xy) &\geq \widehat{t} \text{ and } \widehat{\lambda}_A(xy) \leq \widehat{s}.
\end{aligned}$$

So, $xy \in U_{(\widehat{t},\widehat{s})}$. Hence, $xy \in U_{(\widehat{t},\widehat{s})}$ is a left ideal of S.

Conversely, Let us suppose that $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an IVIFS of S such that $U_{(\widehat{t},\widehat{s})} \neq \phi$ is a left ideal of S. Suppose on contrary there exist $x, y \in S$ such that

$$\begin{aligned} \widehat{\mu}_A(xy) &< \widehat{\mu}_A(y) \wedge \widehat{0.5} \\ \text{and } \widehat{\lambda}_A(xy) &> \widehat{\lambda}_A(y) \wedge \widehat{0.5}. \end{aligned}$$

Let us choose $t \in D(0, 0.5]$ and $s \in D[0.5, 1)$. Then,

$$\begin{aligned} \widehat{\mu}_A(xy) &< \widehat{t} < \widehat{\mu}_A(y) \wedge \widehat{0.5} \\ \text{and } \widehat{\lambda}_A(xy) &> \widehat{s} > \widehat{\lambda}_A(y) \wedge \widehat{0.5}. \end{aligned}$$

Thus, $y \in U_{(\widehat{t},\widehat{s})}$ but $xy \notin U_{(\widehat{t},\widehat{s})}$, which is a contradiction. Hence,

$$\begin{aligned} \widehat{\mu}_A(xy) &\geq \widehat{\mu}_A(y) \wedge \widehat{0.5} \\ \text{and } \widehat{\lambda}_A(xy) &\leq \widehat{\lambda}_A(y) \wedge \widehat{0.5}. \end{aligned}$$

This completes the proof. □

Theorem 20. *Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ an IVIFS of S. Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S if and only if for all $\widehat{t} \in D(0, 0.5]$ and $\widehat{s} \in D[0.5, 1)$, the set $U_{(\widehat{t},\widehat{s})} \neq \phi$ is a b-ideal of S.*

Proof. The proof follows from Theorem 19. □

Theorem 21. *Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ an IVIFS of S. Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy $(1, 2)$-ideal of S if and only if for all $\widehat{t} \in D(0, 0.5]$ and $\widehat{s} \in D[0.5, 1)$, the set $U_{(\widehat{t},\widehat{s})} \neq \phi$ is a $(1, 2)$-ideal of S.*

Proof. The proof follows from Theorem 19. □

Theorem 22. *Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ an IVIFS of S. Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy subsemigroup ideal of S if and only if for all $\widehat{t} \in D(0, 0.5]$ and $\widehat{s} \in D[0.5, 1)$ or $\widehat{t} \in D(0.5, 1]$ and $\widehat{s} \in D[0, 0.5)$, the set $[A]_{(\widehat{t},\widehat{s})} \neq \phi$ is a subsemigroup of S.*

Proof. Let $x, y \in [A]_{(\widehat{t},\widehat{s})}$. Then, $\widehat{\mu}_A(x) \geq \widehat{t}$ and $\widehat{\lambda}_A(x) \geq s$ or $\widehat{\mu}_A(x) + \widehat{t} > 1$ and $\widehat{\lambda}_A(x) + s < 1$, and $\widehat{\mu}_A(y) \geq t$ and $\widehat{\lambda}_A(y) \geq \widehat{s}$ or $\widehat{\mu}_A(y) + \widehat{t} > 1$ and $\widehat{\lambda}_A(y) + \widehat{s} < 1$. We can consider four cases:

(*i*) $\widehat{\mu}_A(x) \geq \widehat{t}$ and $\widehat{\lambda}_A(x) \leq \widehat{s}$, and $\widehat{\mu}_A(y) \geq \widehat{t}$ and $\widehat{\lambda}_A(y) \leq \widehat{s}$,

(*ii*) $\widehat{\mu}_A(x) \geq \widehat{t}$ and $\widehat{\lambda}_A(x) \leq \widehat{s}$, and $\widehat{\mu}_A(y) + \widehat{t} > 1$ and $\widehat{\lambda}_A(y) + \widehat{s} < 1$,

(*iii*) $\widehat{\mu}_A(x) + \widehat{t} > 1$ and $\widehat{\lambda}_A(x) + \widehat{s} < 1$, and $\widehat{\mu}_A(y) \geq \widehat{t}$ and $\widehat{\lambda}_A(y) \leq \widehat{s}s$,

(*iv*) $\widehat{\mu}_A(x) + \widehat{t} > 1$ and $\widehat{\lambda}_A(x) + \widehat{s} < 1$, and $\widehat{\mu}_A(y) + \widehat{t} > 1$ and $\widehat{\lambda}_A(y) + \widehat{s} < 1$.

For the first case, by Theorem 10 (a), implies that

$$\widehat{\mu}_A(xy) \geq \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\} = \min\{\widehat{t}, \widehat{0.5}\} = \begin{cases} \widehat{0.5} & \text{if } \widehat{t} > \widehat{0.5} \\ \widehat{t} & \text{if } \widehat{t} \leq \widehat{0.5} \end{cases}$$

and

$$\widehat{\lambda}_A(xy) \leq \max\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\} = \max\{\widehat{s}, \widehat{0.5}\} = \begin{cases} \widehat{0.5} & \text{if } \widehat{s} < \widehat{0.5} \\ \widehat{s} & \text{if } \widehat{s} \geq \widehat{0.5} \end{cases}$$

and so $\widehat{\mu}_A(xy) + \widehat{t} > \widehat{0.5} + \widehat{0.5} = 1$ and $\widehat{\lambda}_A(xy) + \widehat{s} < \widehat{0.5} + \widehat{0.5} = 1$, *i.e.*, $(xy)\left(s, \widehat{t}\right) qA$, or $xy \in A_{\left(\widehat{t},\widehat{s}\right)}$. Therefore, $xy \in U_{\left(\widehat{t},\widehat{s}\right)} \cup A_{\left(\widehat{t},\widehat{s}\right)} = [A]_{\left(\widehat{t},\widehat{s}\right)}$. For the case (ii), assume that $\widehat{t} > \widehat{0.5}$ and $s < \widehat{0.5}$. Then, $1 - \widehat{t} < \widehat{0.5}$ and $1 - \widehat{s} > \widehat{0.5}$. If $\min\{\widehat{\mu}_A(y), \widehat{0.5}\} \leq \widehat{\mu}_A(x)$ and $\max\{\widehat{\lambda}_A(y), \widehat{0.5}\} \geq \widehat{\lambda}_A(x)$, then

$$\begin{aligned} \widehat{\mu}_A(xy) &\geq \min\{\widehat{\mu}_A(y), \widehat{0.5}\} > 1 - \widehat{t} \text{ and} \\ \widehat{\lambda}_A(xy) &\leq \max\{\widehat{\lambda}_A(y), \widehat{0.5}\} < 1 - \widehat{s} \end{aligned}$$

and if $\min\{\widehat{\mu}_A(y), \widehat{0.5}\} > \widehat{\mu}_A(x)$ and $\max\{\widehat{\lambda}_A(y), \widehat{0.5}\} < \widehat{\lambda}_A(x)$, then $\widehat{\mu}_A(xy) \geq \widehat{\mu}_A(x) \geq \widehat{t}$ and $\widehat{\lambda}_A(xy) \leq \widehat{\lambda}_A(x) \leq s$. Hence, $xy \in U_{\left(\widehat{t},\widehat{s}\right)} \cup A_{\left(\widehat{t},\widehat{s}\right)} = [A]_{\left(\widehat{t},\widehat{s}\right)}$ for $\widehat{t} > \widehat{0.5}$ and $s < \widehat{0.5}$. Suppose that $\widehat{t} \leq \widehat{0.5}$ and $s \geq \widehat{0.5}$. Then, $1 - \widehat{t} \geq \widehat{0.5}$ and $1 - s \leq \widehat{0.5}$. If $\min\{\widehat{\mu}_A(x), \widehat{0.5}\} \leq \widehat{\mu}_A(y)$ and $\max\{\widehat{\lambda}_A(x), \widehat{0.5}\} \geq \widehat{\lambda}_A(y)$, then

$$\begin{aligned} \widehat{\mu}_A(xy) &\geq \min\{\widehat{\mu}_A(x), \widehat{0.5}\} \geq \widehat{t} \text{ and} \\ \widehat{\lambda}_A(xy) &\leq \max\{\widehat{\lambda}_A(x), \widehat{0.5}\} \leq s \end{aligned}$$

and if $\min\{\widehat{\mu}_A(x), \widehat{0.5}\} > \widehat{\mu}_A(y)$ and $\max\{\widehat{\lambda}_A(x), \widehat{0.5}\} < \widehat{\lambda}_A(y)$, then $\widehat{\mu}_A(xy) \geq \widehat{\mu}_A(y) > 1 - \widehat{t}$ and $\widehat{\lambda}_A(xy) \leq \widehat{\lambda}_A(y) < 1 - \widehat{s}$. Thus, $xy \in U_{\left(\widehat{t},\widehat{s}\right)} \cup A_{\left(\widehat{t},\widehat{s}\right)} = [A]_{\left(\widehat{t},\widehat{s}\right)}$ for $\widehat{t} \leq \widehat{0.5}$ and $s \geq \widehat{0.5}$. We have similar result for the case (iii). For final case, if $\widehat{t} > \widehat{0.5}$ and $s < \widehat{0.5}$, then $1 - \widehat{t} < \widehat{0.5}$ and $1 - \widehat{s} > \widehat{0.5}$. Hence,

$$\begin{aligned} \widehat{\mu}_A(xy) &\geq \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\} \\ &= \begin{cases} \widehat{0.5} > 1 - \widehat{t} & \text{if } \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} \geq \widehat{0.5}, \\ \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} > 1 - \widehat{t} & \text{if } \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} < \widehat{0.5}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \widehat{\lambda}_A(xy) &\leq \max\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\} \\ &= \begin{cases} \widehat{0.5} < 1 - \widehat{s} & \text{if } \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} \leq \widehat{0.5}, \\ \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} < 1 - \widehat{s} & \text{if } \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} > \widehat{0.5}, \end{cases} \end{aligned}$$

and so $xy \in A_{\left(\widehat{t},\widehat{s}\right)} \subseteq [A]_{\left(\widehat{t},\widehat{s}\right)}$. If $\widehat{t} \leq \widehat{0.5}$ and $s \geq \widehat{0.5}$, then $1 - \widehat{t} \geq \widehat{0.5}$ and $1 - s \leq \widehat{0.5}$. Thus,

$$\widehat{\mu}_A(xy) \geq \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\}$$

$$= \begin{cases} \widehat{0.5} \geq \widehat{t} & \text{if } \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} \geq \widehat{0.5}, \\ \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} > 1 - \widehat{t} & \text{if } \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y)\} < \widehat{0.5}, \end{cases}$$

and

$$\begin{aligned} \widehat{\lambda}_A(x+y) &\leq \max\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\} \\ &= \begin{cases} \widehat{0.5} \leq \widehat{s} & \text{if } \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} \leq \widehat{0.5}, \\ \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} < 1 - \widehat{s} & \text{if } \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y)\right\} > \widehat{0.5}, \end{cases} \end{aligned}$$

which implies that $xy \in U_{(\widehat{t},\widehat{s})} \cup A_{(\widehat{t},\widehat{s})} = [A]_{(\widehat{t},\widehat{s})}$.

Conversely, suppose that $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an IVIFS in S such that $[A]_{(\widehat{t},\widehat{s})}$ is a subsemigroup of S. Suppose that $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy subsemigroup of S. Then, there exist $x, y \in S$ such that

$$\widehat{\mu}_A(xy) < \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\} \text{ and } \widehat{\lambda}_A(xy) > \max\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\}.$$

Let

$$\begin{aligned} \widehat{t} &= \frac{1}{2}\left[\widehat{\mu}_A(xy) + \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\}\right] \text{ and} \\ \widehat{s} &= \frac{1}{2}\left[\widehat{\lambda}_A(xy) + \max\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\}\right]. \end{aligned}$$

Then,

$$\begin{aligned} \widehat{\mu}_A(xy) &< \widehat{t} < \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\} \text{ and} \\ \widehat{\lambda}_A(xy) &> \widehat{s} > \max\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\}. \end{aligned}$$

This implies that $x, y \in [A]_{(\widehat{t},\widehat{s})}$ and $(xy) \in [A]_{(\widehat{t},\widehat{s})}$. Hence, $\widehat{\mu}_A(xy) \geq t$ and $\widehat{\lambda}_A(xy) \leq \widehat{s}$ or $\widehat{\mu}_A(xy) + \widehat{t} > 1$ and $\widehat{\lambda}_A(xy) + \widehat{s} < 1$, which is a contradiction. Therefore, we have

$$\widehat{\mu}_A(xy) \geq \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\} \text{ and } \widehat{\lambda}_A(xy) \leq \max\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\}.$$

Thus, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy subsemigroup of S. □

Theorem 23. *Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ an IVIFS of S. Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) ideal of S if and only if for all $t \in D(0,1]$ and $s \in D[0,1)$, the set $[A]_{(\widehat{t},\widehat{s})} \neq \phi$ is a left (resp. right) ideal of S.*

Proof. The proof follows from Theorem 22. □

Theorem 24. *Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ an IVIFS of S. Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S if and only if for all $\widehat{t} \in D(0, 0.5]$ and $\widehat{s} \in D[0.5, 1)$ or $\widehat{t} \in D(0.5, 1]$ and $\widehat{s} \in D[0, 0.5)$, the set $[A]_{(\widehat{t},\widehat{s})} \neq \phi$ is a subsemigroup of S.*

Theorem 25. *Let S be a semigroup and $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ an IVIFS of S. Then, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy $(1,2)$-ideal of S if and only if for all $\widehat{t} \in D(0, 0.5]$ and $\widehat{s} \in D[0.5, 1)$ or $\widehat{t} \in D(0.5, 1]$ and $\widehat{s} \in D[0, 0.5)$, the set $[A]_{(\widehat{t},\widehat{s})} \neq \phi$ is a $(1,2)$-ideal of S.*

Theorem 26. *Every interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of semigroup S is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy $(1,\ 2)$ ideal of semigroup S.*

Proof. Straightforward. □

Theorem 27. *If S is a regular semigroup, then every interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy right (left) ideal of a S.*

Proof. Since S is regular, so every bi-ideal in S is a right (left) ideal. Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S. Let $x, y \in S$, xSx is bi-ideal of S, then xSx is right ideal of S. Since S is regular. We have $xy \in (xSx)S \subseteq xSx$ which implies that $xy = xyx$ for some $y \in S$ and since $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S, it follow that

$$\begin{aligned}
\widehat{\mu}_A(xy) &= \widehat{\mu}_A(xyx) \geq \min\{\widehat{\mu}_A(x), \widehat{\mu}_A(x), \widehat{0.5}\} \geq \min\{\widehat{\mu}_A(x), \widehat{0.5}\} \\
\widehat{\lambda}_A(xy) &= \widehat{\lambda}_A(xyx) \leq \max\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(x), \widehat{0.5}\} \geq \min\{\widehat{\lambda}_A(x), \widehat{0.5}\}
\end{aligned}$$

Therefore, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy right-ideal of S. □

Theorem 28. *If S is a regular semigroup, then every interval valued $(\in, \in \vee q)$-intuitionistic fuzzy $(1,2)$ ideal of S is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S.*

Proof. Assume that S is regular. Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy $(1,2)$ ideal of a semigroup S. Let $x, y, a \in S$. Since S is regular, we get $xa \in (xSx)S \subseteq xSx$, which implies that $xa = xsx$ for some $s \in S$. Thus

$$\begin{aligned}
\widehat{\mu}_A(xay) &= \widehat{\mu}_A((xsx)y) = \widehat{\mu}_A(xs(xy)) \\
&\geq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\} \\
\widehat{\mu}_A(xay) &\leq \min\left\{\widehat{\mu}_A(x), \widehat{\mu}_A(y), \widehat{0.5}\right\} \\
\text{And } \widehat{\lambda}_A(xay) &= \widehat{\lambda}_A((xsx)y) = \widehat{\lambda}_A(xs(xy)) \\
&\leq \max\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\} \\
\widehat{\lambda}_A(xay) &\leq \min\left\{\widehat{\lambda}_A(x), \widehat{\lambda}_A(y), \widehat{0.5}\right\}
\end{aligned}$$

Hence, $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ is an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of semigroup S. □

Theorem 29. *Let $A = \langle \widehat{\mu}_A, \widehat{\lambda}_A \rangle$ be an interval valued $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of a semigroup S. If S is completely regular and $\widehat{\mu}_A < \widehat{0.5}$ and $\widehat{\lambda}_A > \widehat{0.5}$ for all $x \in S$, then $A(x) = A(x^2)$ for all $x \in S$.*

5. Conclusion

It is well known that semigroups are basic structures in many applied branches like automata and formal languages, coding theory, finite state machines and others. Due to these possiblities of applications, semigroups are presently extensively investigated in fuzzy setting. An intuitionistic fuzzy set is more material and concise to describe the essence of fuzziness, and the intuitionistic fuzzy set theory may be more suitable than the fuzzy set theory for dealing with imperfect knowledge in many problems. In study the structure of semigroup, we notice that intuitionistic fuzzy ideals with special properties always play an important role. The intuitionistic fuzzy point of a semigroup S are key tools to describe the algebraic subsystems of S. So, we combined the above concepts and introduced new types of intuitionistic fuzzy bi-ideals and (1,2)-ideals of semigroups which are called interval valued (α, β)-intuitionistic fuzzy bi-ideal and (1,2)-ideal. The results in the paper are generalizations of results about ordinary intuitionistic fuzzy ideals in semigroups. In future, we will focus on the following topics:

(1) Characterizations of regular semigroups by the properties of interval valued (α, β)-intuitionistic fuzzy ideals

(2) We will define (α, β)-intuitionistic fuzzy (interior, prime, generalized bi, prime bi) ideals of a semigroup and characterize different classes of semigroups by the properties of (α, β)-intuitionistic-fuzzy ideals. We will extend to our study to other algebraic structures.

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