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SCALE TRANSFORMATIONS FOR PRESENT POSITION-INDEPENDENT CONDITIONAL EXPECTATIONS

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Abstract. Let $C[0,t]$ denote a generalized Wiener space, the space of real-valued continuous functions on the interval $[0,t]$ and define a random vector $Z_n : C[0,t] \to \mathbb{R}^n$ by $Z_n(x) = (\int_0^{t_1} h(s)dx(s), \dots, \int_0^{t_n} h(s)dx(s))$, where $0 < t_1 < \cdots < t_n < t$ is a partition of $[0,t]$ and $h \in L_2[0,t]$ with $h \neq 0$ a.e. In this paper we will introduce a simple formula for a generalized conditional Wiener integral on $C[0,t]$ with the conditioning function Z_n and then evaluate the generalized analytic conditional Wiener and Feynman integrals of the cylinder function $F(x) = f(\int_0^t e(s)dx(s))$ for $x \in C[0,t]$, where $f \in L_p(\mathbb{R})(1 \le p \le \infty)$ and e is a unit element in $L_2[0,t]$. Finally we express the generalized analytic conditional Feynman integral of F as two kinds of limits of non-conditional generalized Wiener integrals of polygonal functions and of cylinder functions using a change of scale transformation for which a normal density is the kernel. The choice of a complete orthonormal subset of $L_2[0,t]$ used in the transformation is independent of e and the conditioning function Z_n does not contain the present positions of the generalized Wiener paths.

1. Introduction

Let $C_0[0,t]$ denote the Wiener space, the space of continuous real-valued functions x on $[0,t]$ with $x(0) = 0$. As mentioned in [1, 2], the Wiener measure and Wiener measurability behave badly under change of scale transformation and under translation. Various kinds of change of scale formulas for Wiener integrals of bounded and unbounded functions were developed on the classical and abstract Wiener spaces [3, 10, 13, 14, 15]. Furthermore the author and his coauthors [5, 8, 11] introduced various kinds of change of scale formulas for the conditional Wiener integrals of functions defined on $C_0[0,t]$, the infinite dimensional Wiener space and $C[0,t]$, an analogue of Wiener space [9] which is the space of real-valued continuous paths on $[0,t]$.

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Let $h \in L_2[0,t]$ with $h \neq 0$ a.e. on $[0,t]$. Define a stochastic process $Z : C[0,t] \times [0,t] \to \mathbb{R}$ by

$$Z(x,s) = \int_0^s h(u)dx(u)$$

for $x \in C[0,t]$ and $s \in [0,t]$, where the integral denotes the Paley-Wiener-Zygmund integral. Let

$$Z_n(x) = (Z(x,t_1), \ldots, Z(x,t_n))$$

and

$$Z_{n+1}(x) = (Z(x,t_1), \ldots, Z(x,t_n), Z(x,t_{n+1}))$$

for $x \in C[0,t]$, where $0 < t_1 < \cdots < t_n < t_{n+1} = t$ is a partition of $[0,t]$. On the space $C[0,t]$ the author [6] derived a simple formula for a generalized conditional Wiener integral given the vector-valued conditioning function Z_{n+1}. Using the formula with Z_{n+1}, Yoo and the author [12] evaluated a generalized analytic conditional Wiener integral of the function G_r having the form

$$G_r(x) = F(x)\Psi\bigg(\int_0^t v_1(s)dx(s), \ldots, \int_0^t v_r(s)dx(s)\bigg)$$

for F in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ [4] and for $\Psi = f + \phi$ which need not be bounded or continuous, where $f \in L_p(\mathbb{R}^r)(1 \le p \le \infty)$, $\{v_1, \ldots, v_r\}$ is an orthonormal subset of $L_2[0,t]$ and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^r$. They then established various kinds of change of scale formulas for the generalized analytic conditional Wiener integral of G_r with the conditioning function Z_{n+1}. Further works were done by the author. In fact he [7] evaluated generalized analytic conditional Wiener and Feynman integrals of the cylinder function G having the form

$$G(x) = f((e,x))\phi((e,x))$$

for $x \in C[0,t]$, where $f \in L_p(\mathbb{R})(1 \le p \le \infty)$, e is a unit element in $L_2[0,t]$ and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}$. He then expressed the generalized analytic conditional Feynman integral of G as limits of non-conditional generalized Wiener integrals using a change of scale transformation. Except for the results in [7, 10], the choices of orthonormal bases of $L_2[0,t]$ in the existing change of scale formulas depend on the orthonormal set $\{v_1, \ldots, v_r\}$ used in the definition of cylinder function and the conditioning function Z_{n+1} contains the present positions of the generalized Wiener paths.

In this paper we will introduce a simple formula for a generalized conditional Wiener integral on $C[0,t]$ with the conditioning function Z_n and then evaluate the generalized analytic conditional Wiener and Feynman integrals of the cylinder function G. Finally we express the generalized analytic conditional Feynman integral of G as two kinds of limits of non-conditional generalized Wiener integrals of polygonal functions and of cylinder functions using a change of scale transformation. In fact, as a function of $\xi_{n+1} \in \mathbb{R}$, the following normal density

$$\left[\frac{\lambda}{2\pi[b(t)-b(t_n)]}\right]^{\frac{1}{2}} \exp\left\{-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2[b(t)-b(t_n)]}\right\}$$

plays a role of the kernel for the transformation, where ξ_n is a real number, λ is a complex number with positive real part and b is a variance function. The choice of a complete orthonormal subset of $L_2[0,t]$ used in the transformation is independent of e and the conditioning function Z_n does not contain the present positions of the generalized Wiener paths. We note that the results of this paper are different from those in [5, 7, 8, 11, 12].

2. A generalized Wiener space

Let $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of complex numbers and complex numbers with positive real parts, respectively. Let $(C[0,t], \mathcal{B}(C[0,t]), w_\varphi)$ be the analogue of Wiener space associated with a probability measure φ on the Borel class of $\mathbb{R}$, where $\mathcal{B}(C[0,t])$ denotes the Borel class of $C[0,t]$. For $v \in L_2[0,t]$ and $x \in C[0,t]$ let $(v,x) = \int_0^t v(s)dx(s)$ denote the Paley-Wiener-Zygmund integral of v according to x [9]. The inner product on the real Hilbert space $L_2[0,t]$ is denoted by $\langle\cdot,\cdot\rangle$.

Let $F : C[0,t] \to \mathbb{C}$ be integrable and let X be a random vector on $C[0,t]$. Then we have the conditional expectation $E[F|X]$ given X from a well-known probability theory. Furthermore there exists a P_X-integrable function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ-a.e. $x \in C[0,t]$, where P_X is the probability distribution of X. The function ψ is called the conditional Wiener w_φ-integral of F given X and it is also denoted by $E[F|X]$.

Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ be a partition of $[0,t]$, where n is a fixed nonnegative integer. Let $h \in L_2[0,t]$ with $h \neq 0$ a.e. on $[0,t]$. For $j = 1, \ldots, n+1$ let

$$\alpha_j = \frac{1}{\|\chi_{(t_{j-1},t_j]}h\|}\chi_{(t_{j-1},t_j]}h$$

and let V be the subspace of $L_2[0,t]$ generated by $\{\alpha_1, \ldots, \alpha_{n+1}\}$. Let $V^\perp$ be the orthogonal complement of V and $\mathcal{P}^\perp : L_2[0,t] \to V^\perp$ be the orthogonal projection. For $x \in C[0,t]$ define a stochastic process $Z : C[0,t] \times [0,t] \to \mathbb{R}$ by

$$Z(x,s) = \int_0^s h(u)dx(u), \quad 0 \le s \le t$$

and let $Z_n : C[0,t] \to \mathbb{R}^n$ be given by

$$Z_n(x) = (Z(x,t_1), \ldots, Z(x,t_n)). \tag{1}$$

Let $b(s) = \int_0^s (h(u))^2 du$ and for $x \in C[0,t]$ define the polygonal function $[Z(x,\cdot)]_b$ of $Z(x,\cdot)$ by

(2) $[Z(x,\cdot)]_b(s)$

$$= \sum_{j=1}^{n+1} \chi_{(t_{j-1},t_j]}(s)\Big[Z(x,t_{j-1}) + \frac{b(s)-b(t_{j-1})}{b(t_j)-b(t_{j-1})}(Z(x,t_j) - Z(x,t_{j-1}))\Big]$$

for $s \in [0,t]$, where $\chi_{(t_{j-1},t_j]}$ denotes the indicator function on the interval $(t_{j-1}, t_j]$. Similarly for $\vec{\xi}_{n+1} = (\xi_1, \ldots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+1}$ the polygonal function $[\vec{\xi}_{n+1}]_b$ of $\vec{\xi}_{n+1}$ is given by (2) replacing $Z(x,t_j)$ by ξ_j ($j = 1, \ldots, n+1$) with $\xi_0 = 0$ and for $\vec{\xi}_n = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ let $[\vec{\xi}_n]_b = \chi_{[0,t_n]}[\vec{\xi}_{n+1}]_b$. For $a, b, u \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ let

$$\Psi(\lambda, u, a, b) = \Big(\frac{\lambda}{2\pi b}\Big)^{\frac{1}{2}} \exp\Big\{-\frac{\lambda}{2b}(u-a)^2\Big\} \text{ with } b \neq 0. \tag{3}$$

For a function $F : C[0,t] \to \mathbb{C}$ let $F_Z(x) = F(Z(x,\cdot))$. If F_Z is integrable over x, then by an application of Theorem 2.12 in [6]

$$\begin{aligned} E[F_Z|Z_n](\vec{\xi}_n) = \int_{\mathbb{R}} &\Psi(1, \xi_{n+1}, \xi_n, b(t) - b(t_n)) \\ &\times E[F(Z(x,\cdot) - [Z(x,\cdot)]_b + [\vec{\xi}_{n+1}]_b)]d\xi_{n+1} \end{aligned} \tag{4}$$

for P_{Z_n}-a.e. $\vec{\xi}_n = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ (for a.e. $\vec{\xi}_n \in \mathbb{R}^n$), where $\vec{\xi}_{n+1} = (\xi_1, \ldots, \xi_n, \xi_{n+1})$ and P_{Z_n} is the probability distribution of Z_n on the Borel class $\mathcal{B}(\mathbb{R}^n)$ of $\mathbb{R}^n$. For $\lambda > 0$ let $F_Z^\lambda(x) = F_Z(\lambda^{-\frac{1}{2}}x)$ and $Z_n^\lambda(x) = Z_n(\lambda^{-\frac{1}{2}}x)$ for $x \in C[0,t]$, where Z_n is given by (1). Suppose that $E[F_Z^\lambda]$ exists. By the definition of the conditional Wiener w_φ-integral and (4)

$$\begin{aligned} E[F_Z^\lambda|Z_n^\lambda](\vec{\xi}_n) = \int_{\mathbb{R}} &\Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n)) \\ &\times E[F(\lambda^{-\frac{1}{2}}(Z(x,\cdot) - [Z(x,\cdot)]_b) + [\vec{\xi}_{n+1}]_b)]d\xi_{n+1} \end{aligned} \tag{5}$$

for $P_{Z_n^\lambda}$-a.e. $\vec{\xi}_n \in \mathbb{R}^n$, where $P_{Z_n^\lambda}$ is the probability distribution of Z_n^λ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Let $I_{F_Z}^\lambda(\vec{\xi}_n)$ be the right hand side of (5). If $I_{F_Z}^\lambda(\vec{\xi}_n)$ has an analytic extension $J_\lambda^*(F_Z)(\vec{\xi}_n)$ on $\mathbb{C}_+$, then it is called the conditional analytic Wiener w_φ-integral of F_Z given Z_n with the parameter λ and denoted by

$$E^{anw_\lambda}[F_Z|Z_n](\vec{\xi}_n) = J_\lambda^*(F_Z)(\vec{\xi}_n)$$

for $\vec{\xi}_n \in \mathbb{R}^n$. Moreover if for nonzero real q, $E^{anw_\lambda}[F_Z|Z_n](\vec{\xi}_n)$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ-integral of F_Z given Z_n with the parameter q and denoted by

$$E^{anf_q}[F_Z|Z_n](\vec{\xi}_n) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F_Z|Z_n](\vec{\xi}_n).$$

Applying Theorem 3.5 in [9] we can easily prove the following theorem.

Theorem 2.1. *Let $\{h_1, h_2, \ldots, h_r\}$ be an orthonormal system of $L_2[0,t]$. For $i = 1, 2, \ldots, r$, let $X_i(x) = (h_i, x)$ on $C[0,t]$. Then $X_1, \ldots, X_r$ are independent* *and each X_i has the standard normal distribution. Moreover if $f : \mathbb{R}^r \to \mathbb{R}$ is Borel measurable, then*

$$\begin{aligned}&\int_{C[0,t]} f(X_1(x), \dots, X_r(x))dw_\varphi(x)\\ &\stackrel{*}{=} \left(\frac{1}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f(u_1, u_2, \dots, u_r) \exp\left\{-\frac{1}{2}\sum_{j=1}^{r} u_j^2\right\} d(u_1, u_2, \dots, u_r),\end{aligned}$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

The following lemma is obvious from Theorem 2.1 in [10].

Lemma 2.2. *Let a and b be positive real numbers. Then for any real u*

$$\int_{\mathbb{R}} \exp\{-av^2 - b(v-u)^2\}dv = \left(\frac{\pi}{a+b}\right)^{\frac{1}{2}} \exp\left\{-\frac{ab}{a+b}u^2\right\}.$$

Lemma 2.3. *Let $v \in L_2[0,t]$, $\vec{\xi}_{n+1} = (\xi_1, \dots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+1}$ and $(v, [\vec{\xi}_n]_b) = \sum_{j=1}^{n} \langle v\alpha_j, \alpha_j \rangle (\xi_j - \xi_{j-1})$, where $\xi_0 = 0$ and $\vec{\xi}_n = (\xi_1, \dots, \xi_n)$. Then*

$$(v, [\vec{\xi}_{n+1}]_b) = \sum_{j=1}^{n+1} \langle v\alpha_j, \alpha_j \rangle (\xi_j - \xi_{j-1}) = (v, [\vec{\xi}_n]_b) + \langle v\alpha_{n+1}, \alpha_{n+1} \rangle (\xi_{n+1} - \xi_n).$$

Proof. By the definition of polygonal function

$$\begin{aligned}(v, [\vec{\xi}_{n+1}]_b) &= \sum_{j=1}^{n+1} \frac{\xi_j - \xi_{j-1}}{b(t_j) - b(t_{j-1})} \int_{t_{j-1}}^{t_j} v(s)db(s)\\ &= \sum_{j=1}^{n+1} \frac{\int_0^t v(s)[\chi_{(t_{j-1},t_j]}(s)h(s)]^2 ds}{\|\chi_{(t_{j-1},t_j]}h\|^2}(\xi_j - \xi_{j-1})\\ &= \sum_{j=1}^{n+1} \langle v\alpha_j, \alpha_j \rangle (\xi_j - \xi_{j-1})\end{aligned}$$

which proves the first equality of the lemma. The second equality is obvious and the proof is now completed. □

3. Generalized analytic conditional Feynman integrals

In this section we establish the analytic conditional Wiener and Feynman integrals of cylinder functions.

Let e be in $L_2[0,t]$ with $\|e\| = 1$. For $1 \le p \le \infty$ let $\mathcal{A}^{(p)}$ be the space of cylinder functions having the following form

$$F(x) = f((e,x)) \tag{6}$$

for w_φ-a.e. $x \in C[0,t]$, where $f \in L_p(\mathbb{R})$. We note that without loss of generality we can take f to be Borel measurable.

Theorem 3.1. *Let $1 \le p \le \infty$. Let Z_n and $F \in \mathcal{A}^{(p)}$ be given by (1) and (6), respectively. Then for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[F_Z|Z_n](\vec{\xi_n})$ exists for a.e. $\vec{\xi_n} \in \mathbb{R}^n$ and it is given by*

$$(7) \qquad E^{anw_\lambda}[F_Z|Z_n](\vec{\xi_n}) = \int_{\mathbb{R}} f(u)\Psi(\lambda, u, (e, [\vec{\xi_n}]_b), \|\mathcal{P}^\perp(eh)\|^2 + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2[b(t) - b(t_n)])du$$

if $\mathcal{P}^\perp(eh) \ne 0$ (i.e., $eh \notin V$) or $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle \ne 0$, where Ψ is given by (3). Furthermore if $p = 1$, then for a nonzero real q, $E^{anf_q}[F_Z|Z_n](\vec{\xi_n})$ is given by the right hand side of (7) replacing λ by $-iq$. If $\mathcal{P}^\perp(eh) = 0$ and $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle = 0$, then for $\lambda \in \mathbb{C}_+$, nonzero real q and a.e. $\vec{\xi_n} \in \mathbb{R}^n$

$$(8) \qquad E^{anw_\lambda}[F_Z|Z_n](\vec{\xi_n}) = E^{anf_q}[F_Z|Z_n](\vec{\xi_n}) = f((e, [\vec{\xi_n}]_b)).$$

Proof. Suppose that $\mathcal{P}^\perp(eh) \ne 0$. For $\lambda > 0$ and a.e. $\vec{\xi_n} = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ we have by Lemma 2.3 and Theorem 2.1 in [7]

$$\begin{aligned}
& I^\lambda_{F_Z}(\vec{\xi_n}) \\
&= \int_{\mathbb{R}} E[F(\lambda^{-\frac{1}{2}}(Z(x,\cdot) - [Z(x,\cdot)]_b) + [\vec{\xi}_{n+1}]_b)]\Psi(\lambda, \xi_{n+1} - \xi_n, 0, b(t) - b(t_n)) \\
&\quad d\xi_{n+1} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)\Psi(\lambda, u, (e, [\vec{\xi}_{n+1}]_b), \|\mathcal{P}^\perp(eh)\|^2)\Psi(\lambda, \xi_{n+1} - \xi_n, 0, b(t) - b(t_n)) \\
&\quad dud\xi_{n+1} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)\Psi(\lambda, u, (e, [\vec{\xi_n}]_b) + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle(\xi_{n+1} - \xi_n), \|\mathcal{P}^\perp(eh)\|^2)\Psi(\lambda, \\
&\quad \xi_{n+1} - \xi_n, 0, b(t) - b(t_n))dud\xi_{n+1},
\end{aligned}$$

where $\vec{\xi}_{n+1} = (\xi_1, \dots, \xi_n, \xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}$. If $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle \ne 0$, then by Lemma 2.2

$$\begin{aligned}
& I^\lambda_{F_Z}(\vec{\xi_n}) \\
&= \left[\frac{\lambda}{2\pi\|\mathcal{P}^\perp(eh)\|^2}\right]^{\frac{1}{2}}\left[\frac{\lambda}{2\pi[b(t) - b(t_n)]}\right]^{\frac{1}{2}}\int_{\mathbb{R}} f(u)\int_{\mathbb{R}}\exp\bigg\{-\frac{\lambda\langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2}{2\|\mathcal{P}^\perp(eh)\|^2} \\
&\quad \times\left(\frac{u - (e, [\vec{\xi_n}]_b)}{\langle e\alpha_{n+1}, \alpha_{n+1}\rangle} - z\right)^2 - \frac{\lambda z^2}{2[b(t) - b(t_n)]}\bigg\}dzdu \\
&= \left[\frac{\lambda}{2\pi[\|\mathcal{P}^\perp(eh)\|^2 + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2[b(t) - b(t_n)]]}\right]^{\frac{1}{2}}\int_{\mathbb{R}} f(u)\exp\bigg\{- \\
&\quad \frac{\lambda}{2[\|\mathcal{P}^\perp(eh)\|^2 + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2[b(t) - b(t_n)]]}(u - (e, [\vec{\xi_n}]_b))^2\bigg\}du \\
&= \int_{\mathbb{R}} f(u)\Psi(\lambda, u, (e, [\vec{\xi_n}]_b), \|\mathcal{P}^\perp(eh)\|^2 + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2[b(t) - b(t_n)])du
\end{aligned}$$

so that we have (7) for $\lambda > 0$. If $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle = 0$, it is not difficult to show

$$\begin{aligned} &I^{\lambda}_{F_Z}(\vec{\xi_n}) \\ &= \int_{\mathbb{R}} f(u) \int_{\mathbb{R}} \Psi(\lambda, u, (e, [\vec{\xi_n}]_b), \|\mathcal{P}^{\perp}(eh)\|^2)\Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n))d\xi_{n+1}du \\ &= \int_{\mathbb{R}} f(u)\Psi(\lambda, u, (e, [\vec{\xi_n}]_b), \|\mathcal{P}^{\perp}(eh)\|^2)du \end{aligned}$$

since Ψ is a normal density so that we have (7) for $\lambda > 0$. Suppose that $\mathcal{P}^{\perp}(eh) = 0$. By Lemma 2.3, Theorem 2.1 in [7] and the change of variable theorem

$$\begin{aligned} I^{\lambda}_{F_Z}(\vec{\xi_n}) &= \int_{\mathbb{R}} f((e, [\vec{\xi}_{n+1}]_b))\Psi(\lambda, \xi_{n+1} - \xi_n, 0, b(t) - b(t_n))d\xi_{n+1} \\ &= \int_{\mathbb{R}} f((e, [\vec{\xi_n}]_b) + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle(\xi_{n+1} - \xi_n))\Psi(\lambda, \xi_{n+1} - \xi_n, \\ &\qquad 0, b(t) - b(t_n))d\xi_{n+1} \\ &= \int_{\mathbb{R}} f(u)\Psi(\lambda, u, (e, [\vec{\xi_n}]_b), \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2[b(t) - b(t_n)])du \end{aligned}$$

if $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle \neq 0$ so that we also have (7) for $\lambda > 0$. Now we have proved (7) for $\lambda > 0$ when $\mathcal{P}^{\perp}(eh) \neq 0$ or $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle \neq 0$. By the Morera's theorem we have (7) for $\lambda \in \mathbb{C}_+$. If $p = 1$, then the existence of $E^{anf_q}[F_Z|Z_n](\vec{\xi_n})$ follows from the dominated convergence theorem.

Finally if $\mathcal{P}^{\perp}(eh) = 0$ and $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle = 0$, then

$$I^{\lambda}_{F_Z}(\vec{\xi_n}) = \int_{\mathbb{R}} f((e, [\vec{\xi_n}]_b))\Psi(\lambda, \xi_{n+1} - \xi_n, 0, b(t) - b(t_n))d\xi_{n+1} = f((e, [\vec{\xi_n}]_b))$$

so that we have (8) trivially. □

Let $\hat{\mathrm{M}}(\mathbb{R})$ be the space of all functions ϕ on $\mathbb{R}$ defined by

$$\phi(u) = \int_{\mathbb{R}} \exp\{iuz\}d\rho(z), \tag{9}$$

where ρ is a complex Borel measure of bounded variation over $\mathbb{R}$. By the boundedness of ϕ and Theorem 3.1 we have the following theorem.

Theorem 3.2. *Let $G(x) = \phi((e, x))F(x)$ for w_{φ}-a.e. $x \in C[0, t]$, where $F \in \mathcal{A}^{(p)}(1 \le p \le \infty)$ and ϕ are given by* (6) *and* (9)*, respectively. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi_n} \in \mathbb{R}^n$*

$$\begin{aligned} E^{anw_\lambda}[G_Z|Z_n](\vec{\xi_n}) = &\int_{\mathbb{R}} f(u)\phi(u)\Psi(\lambda, u, (e, [\vec{\xi_n}]_b), \|\mathcal{P}^{\perp}(eh)\|^2 \\ &+ \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2[b(t) - b(t_n)])du \end{aligned} \tag{10}$$

if $\mathcal{P}^{\perp}(eh) \neq 0$ (i.e., $eh \notin V$) or $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle \neq 0$, where Ψ is given by (3)*. Furthermore if $p = 1$, then for a nonzero real q, $E^{anf_q}[G_Z|Z_n](\vec{\xi_n})$ is given* *by* (10) *replacing* λ *by* $-iq$. *If* $\mathcal{P}^{\perp}(eh) = 0$ *and* $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle = 0$, *then for* $\lambda \in \mathbb{C}_+$, *nonzero real* q *and a.e.* $\vec{\xi_n} \in \mathbb{R}^n$

$$E^{anw_\lambda}[G_Z|Z_n](\vec{\xi_n}) = E^{anf_q}[G_Z|Z_n](\vec{\xi_n}) = f((e,[\vec{\xi_n}]_b))\phi((e,[\vec{\xi_n}]_b)).$$

4. Change of scale formulas using the polygonal function

In this section we derive a change of scale formula for the generalized conditional Wiener integrals of cylinder functions on the analogue of Wiener space using the polygonal function.

Throughout this paper let $\{e_1, e_2, \ldots\}$ be a complete orthonormal basis of $L_2[0,t]$. For $v \in L_2[0,t]$ let

$$c_j(v) = \langle v, e_j\rangle \text{ for } j = 1, 2, \ldots. \tag{11}$$

For $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $x \in C[0,t]$ let

$$K_m(\lambda, x) = \exp\bigg\{\frac{1-\lambda}{2}\sum_{j=1}^m (e_j, x)^2\bigg\}. \tag{12}$$

Lemma 4.1. *Let* m *be a positive integer and* K_m *be given by* (12). *Let* $1 \le p \le \infty$ *and* $F \in \mathcal{A}^{(p)}$ *be given by* (6). *Suppose that* $\mathcal{P}^{\perp}(eh) \neq 0$ (*i.e.*, $eh \notin V$) *or* $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle \neq 0$. *For* $\lambda \in \mathbb{C}_+$ *and* $\vec{\xi_n} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ *let*

$$\begin{aligned}\Gamma(F, \lambda, m, \vec{\xi_n}) = &\int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n)) E[K_m(\lambda, x) \\ &\times F(Z(x,\cdot) - [Z(x,\cdot)]_b + [\vec{\xi}_{n+1}]_b)] d\xi_{n+1},\end{aligned}$$

where $\vec{\xi}_{n+1} = (\xi_1, \ldots, \xi_n, \xi_{n+1})$ *for* $\xi_{n+1} \in \mathbb{R}$ *and* Ψ *is given by* (3). *Then*

$$\begin{aligned}\Gamma(F, \lambda, m, \vec{\xi_n}) = \lambda^{-\frac{m}{2}} &\int_{\mathbb{R}} \Psi(\lambda, u, (e, [\vec{\xi_n}]_b), A(m, \lambda, \mathcal{P}^{\perp}(eh)) \\ &+ \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2 [b(t) - b(t_n)]) f(u) du,\end{aligned} \tag{13}$$

where for $v \in L_2[0,t]$

$$A(m, \lambda, v) = \sum_{j=1}^m [c_j(v)]^2 + \lambda\bigg[\|v\|^2 - \sum_{j=1}^m [c_j(v)]^2\bigg], \tag{14}$$

and the c_j*s are given by* (11).

Proof. Let $\lambda > 0$ and suppose that $\mathcal{P}^{\perp}(eh) \neq 0$. If $\{e_1, \ldots, e_m, \mathcal{P}^{\perp}(eh)\}$ is linearly independent, then by the proof of Lemma 3.1 in [7]

$$\|\mathcal{P}^{\perp}(eh)\|^2 - \sum_{j=1}^m [c_j(\mathcal{P}^{\perp}(eh))]^2 > 0$$

and hence $A(m, \lambda, \mathcal{P}^{\perp}(eh)) > 0$. If $\{e_1, \ldots, e_m, \mathcal{P}^{\perp}(eh)\}$ is linearly dependent, then

$$A(m, \lambda, \mathcal{P}^{\perp}(eh)) = A(m, 0, \mathcal{P}^{\perp}(eh)) \tag{15}$$

$$= \sum_{j=1}^{m} [c_j(\mathcal{P}^\perp(eh))]^2 = \|\mathcal{P}^\perp(eh)\|^2 > 0.$$

Now for $\vec{\xi}_n \in \mathbb{R}^n$

$$\begin{aligned} &\Gamma(F, \lambda, m, \vec{\xi}_n) \\ &= \lambda^{-\frac{m}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)\Psi(\lambda, u, (e, [\vec{\xi}_{n+1}]_b), A(m, \lambda, \mathcal{P}^\perp(eh)))\Psi(\lambda, \xi_{n+1} - \xi_n, 0, b(t) \\ &\quad - b(t_n))dud\xi_{n+1} \end{aligned}$$

by Lemma 3.1 in [7] and Corollary 3.2 in [7]. Using the same process as used in the proof of Theorem 3.1

$$\begin{aligned} \Gamma(F, \lambda, m, \vec{\xi}_n) = \lambda^{-\frac{m}{2}} \int_{\mathbb{R}} f(u)\Psi(\lambda, u, (e, [\vec{\xi}_n]_b), A(m, \lambda, \\ \mathcal{P}^\perp(eh)) + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2 [b(t) - b(t_n)])du \end{aligned}$$

so that we have (13) for $\lambda > 0$. If $\mathcal{P}^\perp(eh) = 0$ and $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle \neq 0$, then $A(m, \lambda, \mathcal{P}^\perp(eh)) = 0$ and hence by Theorem 2.1

$$\begin{aligned} \Gamma(F, \lambda, m, \vec{\xi}_n) &= \left(\frac{1}{2\pi}\right)^{\frac{m}{2}} \int_{\mathbb{R}} f((e, [\vec{\xi}_{n+1}]_b))\Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n)) \\ &\qquad \int_{\mathbb{R}^m} \exp\left\{ -\frac{1}{2}\sum_{j=1}^{m} u_j^2 + \frac{1-\lambda}{2}\sum_{j=1}^{m} u_j^2 \right\} d(u_1, \ldots, u_m)d\xi_{n+1} \\ &= \lambda^{-\frac{m}{2}} \int_{\mathbb{R}} f((e, [\vec{\xi}_{n+1}]_b))\Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n))d\xi_{n+1}. \end{aligned}$$

Using the same process as used in the proof of Theorem 3.1 we have (13) for $\lambda > 0$. Each side of (13) is an analytic function of λ in $\mathbb{C}_+$ so that by the uniqueness of an analytic extension, we have (13) for any $\lambda \in \mathbb{C}_+$. □

Using the same process as used in the proof of Lemma 4.1, we have the following corollary.

Corollary 4.2. *Let* Γ *be as given in Lemma 4.1. Suppose that* $\{e_1, \ldots, e_m, \mathcal{P}^\perp(eh)\}$ *is linearly dependent. If* $\mathcal{P}^\perp(eh) \neq 0$ *or equivalently* $eh \notin V$*, then for* $\lambda \in \mathbb{C}_+$ *and* $\vec{\xi}_n \in \mathbb{R}^n$

$$\begin{aligned} \Gamma(F, \lambda, m, \vec{\xi}_n) &= \lambda^{-\frac{m}{2}} \int_{\mathbb{R}} \Psi(\lambda, u, (e, [\vec{\xi}_n]_b), A(m, 0, \mathcal{P}^\perp(eh)) \\ &\qquad + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2 [b(t) - b(t_n)])f(u)du \\ &= \lambda^{-\frac{m}{2}} \int_{\mathbb{R}} \Psi(\lambda, u, (e, [\vec{\xi}_n]_b), \|P^\perp(eh)\|^2 \\ &\qquad + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2 [b(t) - b(t_n)])f(u)du, \end{aligned} \tag{16}$$

where Ψ *and* A *are given by* (3) *and* (14)*, respectively. Furthermore if* $\mathcal{P}^{\perp}(eh) = 0$ (*i.e.,* $eh \in V$) *and* $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle = 0$*, then for* $\lambda \in \mathbb{C}_+$ *and* $\vec{\xi}_n \in \mathbb{R}^n$

$$\Gamma(F, \lambda, m, \vec{\xi}_n) = \lambda^{-\frac{m}{2}} f((e, [\vec{\xi}_n]_b)).$$

Proof. If $\mathcal{P}^{\perp}(eh) \neq 0$, then (16) immediately follows from (15) and Lemma 4.1. Now suppose that $\mathcal{P}^{\perp}(eh) = 0$ and $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle = 0$. By Lemma 2.3

$$\begin{aligned}\Gamma(F, \lambda, m, \vec{\xi}_n) &= \lambda^{-\frac{m}{2}} \int_{\mathbb{R}} f((e, [\vec{\xi}_{n+1}]_b))\Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n))d\xi_{n+1} \\ &= \lambda^{-\frac{m}{2}} f((e, [\vec{\xi}_n]_b))\end{aligned}$$

which completes the proof of remainder part of the corollary. □

We now have the following theorem by the boundedness of ϕ.

Theorem 4.3. *Let* G *and* Γ *be as given in Theorem 3.2 and Lemma 4.1, respectively.*

(1) *Under the assumptions and notations as given in Lemma 4.1, for* $\lambda \in \mathbb{C}_+$ *and* $\vec{\xi}_n \in \mathbb{R}^n$ $\Gamma(G, \lambda, m, \vec{\xi}_n)$ *is given by* (13) *replacing* f *by* $f\phi$.
(2) *If* $\{e_1, \ldots, e_m, \mathcal{P}^{\perp}(eh)\}$ *is linearly dependent and* $\mathcal{P}^{\perp}(eh) \neq 0$*, then for* $\lambda \in \mathbb{C}_+$ *and* $\vec{\xi}_n \in \mathbb{R}^n$ $\Gamma(G, \lambda, m, \vec{\xi}_n)$ *is given by* (16) *replacing* f *by* $f\phi$.
(3) *If* $\mathcal{P}^{\perp}(eh) = 0$ (*i.e.,* $eh \in V$) *and* $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle = 0$*, then for* $\lambda \in \mathbb{C}_+$ *and* $\vec{\xi}_n \in \mathbb{R}^n$

$$\Gamma(G, \lambda, m, \vec{\xi}_n) = \lambda^{-\frac{m}{2}} f((e, [\vec{\xi}_n]_b))\phi((e, [\vec{\xi}_n]_b)). \tag{17}$$

Theorem 4.4. *Let* G *be as given in Theorem 4.3. Then for* $\lambda \in \mathbb{C}_+$ *and a.e.* $\vec{\xi}_n = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$

$$\begin{aligned}&E^{anw_\lambda}[G_Z|Z_n](\vec{\xi}_n) \\ &= \lim_{m\to\infty} \lambda^{\frac{m}{2}} \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n))E[K_m(\lambda, x)G(Z(x, \cdot) \\ &\quad - [Z(x, \cdot)]_b + [\vec{\xi}_{n+1}]_b)]d\xi_{n+1},\end{aligned} \tag{18}$$

where $\vec{\xi}_{n+1} = (\xi_1, \ldots, \xi_n, \xi_{n+1})$ *for* $\xi_{n+1} \in \mathbb{R}$ *and* Ψ, K_m *are given by* (3), (12)*, respectively. Moreover if* $p = 1$, q *is a nonzero real number and* $\{\lambda_m\}_{m=1}^{\infty}$ *is a sequence in* $\mathbb{C}_+$ *converging to* $-iq$ *as* m *approaches* ∞*, then* $E^{anf_q}[G_Z|Z_n](\vec{\xi}_n)$ *is given by the right hand side of* (18) *replacing* λ *by* λ_m.

Proof. Suppose that $\mathcal{P}^{\perp}(eh) \neq 0$ or $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle \neq 0$. Then for $\lambda \in \mathbb{C}_+$ and $\vec{\xi}_n \in \mathbb{R}^n$

$$\begin{aligned}&\lambda^{\frac{m}{2}} \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n))E[K_m(\lambda, x)G(Z(x, \cdot) - [Z(x, \cdot)]_b \\ &\quad + [\vec{\xi}_{n+1}]_b)]d\xi_{n+1} \\ &= \int_{\mathbb{R}} \Psi(\lambda, u, (e, [\vec{\xi}_n]_b), A(m, \lambda, \mathcal{P}^{\perp}(eh)) + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2[b(t) - b(t_n)])\end{aligned}$$

$$\times f(u)\phi(u)du$$

by Theorem 4.3. By (14) and the Parseval's identity

$$\begin{aligned}&\lim_{m\to\infty} A(m,\lambda,\mathcal{P}^{\perp}(eh))\\ &= \lim_{m\to\infty}\Big[\sum_{j=1}^{m}[c_j(\mathcal{P}^{\perp}(eh))]^2+\lambda[\|\mathcal{P}^{\perp}(eh)\|^2-\sum_{j=1}^{m}[c_j(\mathcal{P}^{\perp}(eh))]^2]\Big]\\ &= \|\mathcal{P}^{\perp}(eh)\|^2+\lambda[\|\mathcal{P}^{\perp}(eh)\|^2-\|\mathcal{P}^{\perp}(eh)\|^2]=\|\mathcal{P}^{\perp}(eh)\|^2\end{aligned}$$

so that we have (18) by Theorem 3.2 and the dominated convergence theorem. If $\mathcal{P}^{\perp}(eh)=0$ and $\langle e\alpha_{n+1},\alpha_{n+1}\rangle=0$, then we have (18) by Theorem 3.2 and (17) in Theorem 4.3. □

The following corollary follows immediately from the proof of Theorem 4.4.

Corollary 4.5. *Let $K_0(\lambda,x)=1$ for $\lambda\in\mathbb{C}_+$ and $x\in C[0,t]$, G be as given in Theorem 4.3 and l be the smallest positive integer such that $\{e_1,\dots,e_l,\mathcal{P}^{\perp}(eh)\}$ is linearly dependent if $\mathcal{P}^{\perp}(eh)\neq 0$. Moreover let $l=0$ if $\mathcal{P}^{\perp}(eh)=0$. Then for any nonnegative integer r with $r\geq l$, for $\lambda\in\mathbb{C}_+$ and a.e. $\vec{\xi_n}=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$*

$$\begin{aligned}E^{anw_\lambda}[G_Z|Z_n](\vec{\xi_n}) = \lambda^{\frac{r}{2}}\int_{\mathbb{R}}\Psi(\lambda,\xi_{n+1},\xi_n,b(t)-b(t_n))E[K_r(\lambda,x)\\ \times G(Z(x,\cdot)-[Z(x,\cdot)]_b+[\vec{\xi}_{n+1}]_b)]d\xi_{n+1},\end{aligned} \tag{19}$$

where $\vec{\xi}_{n+1}=(\xi_1,\dots,\xi_n,\xi_{n+1})$ for $\xi_{n+1}\in\mathbb{R}$ and Ψ, K_r are given by (3), (12), respectively.

Proof. If $\{e_1,\dots,e_l,\mathcal{P}^{\perp}(eh)\}$ is linearly dependent for some positive integer l and $\mathcal{P}^{\perp}(eh)\neq 0$, then for $m\geq l$

$$A(m,\lambda,\mathcal{P}^{\perp}(eh))=A(l,0,\mathcal{P}^{\perp}(eh))=\sum_{j=1}^{l}[c_j(\mathcal{P}^{\perp}(eh))]^2=\|\mathcal{P}^{\perp}(eh)\|^2.$$

If $\mathcal{P}^{\perp}(eh)=0$, then $A(m,\lambda,\mathcal{P}^{\perp}(eh))=0$ for $m\geq 1$. Now the corollary immediately follows from Theorem 3.2. □

Letting $\lambda=\gamma^{-2}$ in (18) and (19) we have the following change of scale formulas for the generalized conditional Wiener integral on the analogue of Wiener space using the polygonal function.

Corollary 4.6. (1) *Under the assumptions as given in Theorem 4.4 we have for $\gamma>0$ and a.e. $\vec{\xi_n}=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$*

$$\begin{aligned}&E[G(\gamma Z(x,\cdot))|\gamma Z_n(x)](\vec{\xi_n})\\ &= \lim_{m\to\infty}\gamma^{-m}\int_{\mathbb{R}}\Psi(1,\xi_{n+1},\xi_n,\gamma^2[b(t)-b(t_n)])E[K_m(\gamma^{-2},x)G(Z(x,\cdot)\end{aligned}$$

$$- [Z(x,\cdot)]_b + [\vec{\xi}_{n+1}]_b)]d\xi_{n+1}.$$

(2) *Under the assumptions as given in Corollary 4.5 we have for any non-negative integer r with $r \geq l$, for $\gamma > 0$ and a.e. $\vec{\xi}_n = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$*

$$\begin{aligned} &E[G(\gamma Z(x,\cdot))|\gamma Z_n(x)](\vec{\xi}_n) \\ &= \gamma^{-r} \int_{\mathbb{R}} \Psi(1, \xi_{n+1}, \xi_n, \gamma^2[b(t) - b(t_n)]) E[K_r(\gamma^{-2}, x) G(Z(x,\cdot) \\ &\quad - [Z(x,\cdot)]_b + [\vec{\xi}_{n+1}]_b)]d\xi_{n+1}. \end{aligned}$$

5. Change of scale formulas using a cylinder function

In this section we derive change of scale formulas for the generalized conditional Wiener integrals of the cylinder functions on the analogue of Wiener space using other cylinder functions.

Theorem 5.1. *Let Z_n be given by* (1) *and F be as given in Lemma 4.1. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi}_n = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$*

$$\begin{aligned} (20) \qquad &E^{anw_\lambda}[F_Z|Z_n](\vec{\xi}_n) \\ &= \lim_{m\to\infty} \lambda^{\frac{m}{2}} \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n)) E[K_m(\lambda,\cdot) f((v,\cdot)\|\mathcal{P}^\perp(eh)\| \\ &\quad + (e, [\vec{\xi}_{n+1}]_b))]d\xi_{n+1} \end{aligned}$$

for any unit $v \in L_2[0,t]$, where $\vec{\xi}_n = (\xi_1, \ldots, \xi_n, \xi_{n+1})$ and Ψ, K_m are given by (3), (12), *respectively. Moreover if $p = 1$, q is a nonzero real number and $\{\lambda_m\}_{m=1}^\infty$ is a sequence in $\mathbb{C}_+$ converging to $-iq$ as m approaches ∞, then $E^{anf_q}[F_Z|Z_n](\vec{\xi}_n)$ is given by the right hand side of* (20) *replacing λ by λ_m.*

Proof. Let

$$\begin{aligned} \Delta(F,\lambda,m,\vec{\xi}_n) = \lambda^{\frac{m}{2}} &\int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n)) \\ &\times E[K_m(\lambda,\cdot) f((v,\cdot)\|\mathcal{P}^\perp(eh)\| + (e, [\vec{\xi}_{n+1}]_b))]d\xi_{n+1}. \end{aligned}$$

Suppose that $\mathcal{P}^\perp(eh) \neq 0$. Using the same process as used in the proof of Lemma 3.1 in [7] we have for $\lambda \in \mathbb{C}_+$ and $\vec{\xi}_n \in \mathbb{R}^n$

$$\begin{aligned} &\Delta(F,\lambda,m,\vec{\xi}_n) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n)) \Psi(\lambda, u, 0, A(m,\lambda,v)) f(u\|\mathcal{P}^\perp(eh)\| \\ &\quad + (e, [\vec{\xi}_{n+1}]_b))d\xi_{n+1}du \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1}, \xi_n, b(t) - b(t_n)) \Psi(\lambda, u, (e, [\vec{\xi}_{n+1}]_b), A(m,\lambda,v)\|\mathcal{P}^\perp(eh)\|^2) \\ &\quad \times f(u)d\xi_{n+1}du \end{aligned}$$

by the change of variable theorem, where A is given by (14). Using the same process as used in the proof of Theorem 3.1

$$\Delta(F,\lambda,m,\vec{\xi_n}) = \int_{\mathbb{R}} f(u)\Psi(\lambda,u,(e,[\vec{\xi_n}]_b),A(m,\lambda,v)\|\mathcal{P}^{\perp}(eh)\|^2 + \langle e\alpha_{n+1},\alpha_{n+1}\rangle^2[b(t)-b(t_n)])du.$$

If $\mathcal{P}^{\perp}(eh)=0$ and $\langle e\alpha_{n+1},\alpha_{n+1}\rangle \neq 0$, then

$$\begin{aligned}&\Delta(F,\lambda,m,\vec{\xi_n})\\ &= \lambda^{\frac{m}{2}}\int_{\mathbb{R}}\Psi(\lambda,\xi_{n+1},\xi_n,b(t)-b(t_n))E[K_m(\lambda,\cdot)f((e,[\vec{\xi}_{n+1}]_b))]d\xi_{n+1}\\ &= \int_{\mathbb{R}}\Psi(\lambda,\xi_{n+1},\xi_n,b(t)-b(t_n))f((e,[\vec{\xi}_{n+1}]_b))d\xi_{n+1}\\ &= \int_{\mathbb{R}} f(u)\Psi(\lambda,u,(e,[\vec{\xi_n}]_b),\langle e\alpha_{n+1},\alpha_{n+1}\rangle^2[b(t)-b(t_n)])du.\end{aligned}$$

If $\mathcal{P}^{\perp}(eh)=0$ and $\langle e\alpha_{n+1},\alpha_{n+1}\rangle = 0$, then $\Delta(F,\lambda,m,\vec{\xi_n}) = f((e,[\vec{\xi_n}]_b))$. We note that

$$A(m,\lambda,v) = \sum_{j=1}^{m}[c_j(v)]^2 + \lambda\left[\|v\|^2 - \sum_{j=1}^{m}[c_j(v)]^2\right] \neq 0$$

so that the above process is justified. Moreover

$$\lim_{m\to\infty} A(m,\lambda,v) = \|v\|^2 = 1.$$

Now letting $m\to\infty$ in each case we have (20) by Theorem 3.1 which completes the proof of the first part of the theorem. The remainder part of the theorem immediately follows from the dominated convergence theorem. □

Now we have the following corollaries by Theorem 5.1.

Corollary 5.2. *Under the assumptions as given in Corollary 4.5 and Theorem 5.1 we have for any nonnegative integer r with $r \geq l$, for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi_n} = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$*

$$E^{anw_\lambda}[F_Z|Z_n](\vec{\xi_n}) = \lambda^{\frac{r}{2}}\int_{\mathbb{R}}\Psi(\lambda,\xi_{n+1},\xi_n,b(t)-b(t_n))E[K_r(\lambda,\cdot) \times f((v,\cdot)\|\mathcal{P}^{\perp}(eh)\| + (e,[\vec{\xi}_{n+1}]_b))]d\xi_{n+1}.$$

Corollary 5.3. (1) *Under the assumptions as given in Theorem 5.1 we have for $\gamma > 0$ and a.e. $\vec{\xi_n} = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$*

$$\begin{aligned}&E[F(\gamma Z(x,\cdot))|\gamma Z_n(x)](\vec{\xi_n})\\ &= \lim_{m\to\infty}\gamma^{-m}\int_{\mathbb{R}}\Psi(1,\xi_{n+1},\xi_n,\gamma^2[b(t)-b(t_n)])E[K_m(\gamma^{-2},\cdot)\\ &\quad\times f((v,\cdot)\|\mathcal{P}^{\perp}(eh)\| + (e,[\vec{\xi}_{n+1}]_b))]d\xi_{n+1}.\end{aligned}$$

(2) *Under the assumptions as given in Corollary 5.2 we have for any non-negative integer r with $r \geq l$, for $\gamma > 0$ and a.e.* $\vec{\xi_n} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$

$$\begin{aligned} &E[F(\gamma Z(x,\cdot))|\gamma Z_n(x)](\vec{\xi_n}) \\ &= \gamma^{-r}\int_{\mathbb{R}} \Psi(1, \xi_{n+1}, \xi_n, \gamma^2[b(t)-b(t_n)])E[K_r(\gamma^{-2}, \cdot) \\ &\quad \times f((v,\cdot)\|\mathcal{P}^{\perp}(eh)\| + (e, [\vec{\xi}_{n+1}]_b))]d\xi_{n+1}. \end{aligned}$$

Remark 5.4. (1) While the complete orthonormal set in [5, 8, 11, 12] contain e used in the definition of the cylinder function, the complete orthonormal set $\{e_1, e_2, \ldots\}$ in this paper and in [7] does not contain e. Furthermore, v in Theorem 5.1, Corollaries 5.2 and 5.3 is independent of both $\{e_1, e_2, \ldots\}$ and e.

(2) Let G be as given in Theorem 3.2. Replacing F and f by G and $f\phi$, respectively, in Theorem 5.1, Corollaries 5.2 and 5.3, the results hold still.

(3) The change of scale formulas in this paper hold still even if $\mathcal{P}^{\perp}(eh) = 0$ and $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle = 0$. Since for $\gamma > 0$ and a.e. $\vec{\xi_n} \in \mathbb{R}^n$

$$E[F(\gamma Z(x,\cdot))|\gamma Z_n(x)](\vec{\xi_n}) = f((e, [\vec{\xi_n}]_b)) = E[F(Z(x,\cdot))|Z_n(x)](\vec{\xi_n})$$

they are surplus in this case.

(4) The conditioning function Z_n does not contain the initial position $Z(x,0)$ of the path $Z(x,\cdot)$ because of $Z(x,0) = 0$. While the conditioning function in [7] contains the position $Z(x,t)$ of the path $Z(x,\cdot)$ at the present time t, the conditioning function Z_n in this paper does not. Furthermore if $h = 1$ a.e., then $Z_n(x) = (x(t_1)-x(0), \ldots, x(t_n)-x(0))$. Hence the formulas in this paper do not extend the existing change of scale formulas in [5, 8, 11] but they do the formulas in [7, 12].

(5) For $\vec{\xi_n} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ it is possible that $[\vec{\xi_n}] \notin C[0,t]$ if $\xi_n \neq 0$. In this case the following symbol $(v, [\vec{\xi_n}]_b)$ does not mean the Paley-Wiener-Zygmund integral of $v \in L_2[0,t]$. It is only the formal expression of $\sum_{j=1}^{n}\langle v\alpha_j, \alpha_j\rangle(\xi_j - \xi_{j-1})$ which is as given in Lemma 2.3.

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