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L^2 HARMONIC 1-FORMS ON SUBMANIFOLDS WITH WEIGHTED POINCARÉ INEQUALITY

Xiaoli Chao and Yusha Lv

Abstract. In the present note, we deal with L^2 harmonic 1-forms on complete submanifolds with weighted Poincaré inequality. By supposing submanifold is stable or has sufficiently small total curvature, we establish two vanishing theorems for L^2 harmonic 1-forms, which are some extension of the results of Kim and Yun, Sang and Thanh, Cavalcante Mirandola and Vitório.

1. Introduction

It is an interesting problem in geometry and topology to find sufficient conditions on the manifold for the space of harmonic k-forms to be trivial.

In case of complete orientable stable minimal hypersurfaces, several results on the nonexistence of L^2 harmonic forms are well-known. Recall that a minimal hypersurface M in a Riemannian manifold N is said to be stable, if for any $\eta \in C_0^\infty(M)$,

$$\int_M |\nabla \eta|^2 \geq \int_M (\overline{\mathrm{Ric}}(\nu, \nu) + |A|^2)\eta^2, \quad \eta \in C_0^\infty(M), \tag{1.1}$$

where ν is a unite normal vector field of M, A is the second fundamental form of M, $\overline{\mathrm{Ric}}$ is the Ricci curvature of N. On the other hand, let M be an immersed hypersurface in Riemannian manifold N, if M satisfies (1.1), we say M has *stability condition.* In this case, Palmer [17] proved that the space of L^2 harmonic 1-forms on complete minimal hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ is trivial. Thereafter, using Bochners vanishing technique, Miyaoka [16] showed that a complete orientable noncompact stable minimal hypersurface in a Riemannnian manifold with nonnegative sectional curvature has no nontrivial L^2 harmonic 1-forms. Tanno [22] extended Miyaoka's result to ambient spaces with nonnegative BiRic curvature. Given a Riemannian manifold M, recall that the Bi-Ricci curvature is defined by $\mathrm{BiRic}(X, Y) = \mathrm{Ric}(X, X) + \mathrm{Ric}(Y, Y) - K(X, Y)$ for any orthonormal vector fields X and Y on M, where K is curvature

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operator. Later, Cheng [4] extended Tanno's result and showed that a complete noncompact strongly stable hypersurface with constant mean curvature H in Riemannian manifold N^{n+1} has no nontrivial L^2 harmonic 1-forms under the assumption BiRic $\geq \frac{(n-5)n^2}{4}H^2$. Without the assumption of minimality nor the condition of constant mean curvature of hypersurface, Kim and Yun [10] proved that a complete oriented noncompact immersed hypersurface $M^n (2 \leq n \leq 4)$ in a complete Riemannian manifold N^{n+1} with nonnegative sectional curvature has no nontrivial L^2 harmonic 1-forms, if M satisfies the stability condition (1.1).

Moreover, it turned out that these vanishing theorems hold for more general Riemannian manifold with property $(\mathcal{P}_\rho)$. First let us recall the definition of property $(\mathcal{P}_\rho)$.

Definition. Let M be an n-dimensional complete Riemannian manifold. We say that M has property $(\mathcal{P}_\rho)$, if a weighted Poincaré inequality is valid on M with some nonnegative weight function ρ, namely

$$\int_M \rho(x)\eta^2 \leq \int_M |\nabla \eta|^2, \quad \forall \eta \in C_0^\infty(M).$$

Moreover, the ρ-metric, defined by $ds_\rho^2 = \rho ds_M^2$ is complete.

In particular, if $\lambda_1(M)$ is assumed to be positive, then obviously M possesses property $(\mathcal{P}_\rho)$ with $\rho = \lambda_1(M)$. So, the notion of property $(\mathcal{P}_\rho)$ may be viewed as a generalization of the assumption $\lambda_1(M) > 0$.

In the case of complete Riemannian manifolds satisfying weighted Poincaré type property, some results on the nonexistence of nontrivial L^2 harmonic 1-forms are well-known. Li and Wang [14] proved that complete Riemannian manifold with $\lambda_1 > 0$ has no nontrivial L^2 harmonic 1-form, if Ric $\geq -\frac{n}{n-1}\lambda_1 + \tau$ for some $\tau > 0$. Lam [11] generalized Li and Wang's theory to manifolds satisfying a weighted Poincaré inequality, and proved that a complete Riemannian manifold satisfying weighted Poincaré inequality has no nontrivial L^2 harmonic 1-form, if Ric $\geq -\frac{n}{n-1}\rho + \tau$, and $\rho(x) = o(r^{2-\alpha}), 0 < \alpha < 2, \tau > 0$. This result was generalized by Matheus [15] removing the restrictions on the sign and growth rate of the weight function.

Recently, Seo [20] proved that complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with $\lambda_1 > (2n-1)(n-1)$ has no nontrivial L^2 harmonic 1-forms. Later, this result was generalized by Dung and Seo [5] to a complete stable minimal hypersurface in a Riemannnian manifold with sectional curvature bounded below by a nonpositive constant, and proved that complete noncompact stable non-totally geodesic minimal hypersurface in Riemannian manifold N with $K \leq K_N (K \leq 0)$ has no nontrivial L^2 harmonic 1-form under the assumption of $\lambda_1(M) > -K(2n-1)(n-1)$. Later, without the assumption of non-totally geodesic, this result was extended by Sang and Thanh [18] to hypersurfaces satisfying weighted Poincaré inequality, and proved the following theorem.

Theorem 1.1 ([18])**.** *Let N be an $(n+1)$-dimensional Riemannian manifold, and M be a complete noncompact stable minimal hypersurface in N with $(\mathcal{P}_\rho)$ property for some nonnegative weighted function ρ defined on M. Assume further that*

$$K_N(x) \geq -\frac{(1-\tau)\rho(x)}{(2n-1)(n-1)}, \quad \forall x \in M$$

for some $0 < \tau \leq 1$. If $\rho = o(r^{2-\alpha})$ for some $0 < \alpha < 2$, then there is no nontrivial L^2 harmonic 1-form on M.

In the first part of this paper, inspired by all above results, we consider the nonexistence of nontrivial L^2 harmonic 1-forms on complete noncompact hypersurface satisfying a weighted Poincaré inequality with a nonnegative weighted function ρ and stability condition (1.1) in a Riemannian manifold N with sectional curvature bounded below by a nonpositive function. More precisely, we have the following vanishing theorem.

Theorem 1.2. *Let N^{n+1} be an $(n+1)$-dimensional Riemannian manifold, and $M^n (2 \leq n \leq 4)$ be a complete noncompact hypersurface satisfying a weighted Poincaré inequality with a nonnegative weighted function ρ and stability condition (1.1) in N. Assume further that*

$$K_N(x) \geq -\frac{(1-\tau)\rho(x)}{(2n-1)(n-1)}, \quad \forall x \in M$$

for some $0 < \tau \leq 1$. Then there is no nontrivial L^2 harmonic 1-form on M.

Remark 1.3. (i) We do not assume the minimality of hypersurface nor the constant mean curvature condition, and dimension restriction aries in estimating the Ricci curvature. When $H \equiv 0$ in Theorem 1.2, we obtain the main Theorem 1.1, and in the proof of Theorem 1.2, we know that the restriction on the growth rate of the weight function is not needed.

(ii) When $\rho(x) \equiv 0$, i.e., $K_N \geq 0$, Theorem 1.2 is duo to Theorem 3.3 in [10].

If we choose $\rho(x) = \lambda_1(M)$ in Theorem 1.2, we can get the following corollary which is an extension of Theorem 8 in [5] without the assumption of non-totally geodesic and minimality of a hypersurface, and dimension restriction arises in the estimating of the Ricci curvature.

Corollary 1.4. *Let N^{n+1} be a $(n+1)$-dimensional Riemannian manifold with sectional curvature $K_N \geq K$ where $K \leq 0$ is a constant. Let $M^n (2 \leq n \leq 4)$ be a complete noncompact hypersurface satisfying stability condition (1.1) in N. Assume further that*

$$\lambda_1(M) \geq -(2n-1)(n-1)K + \tau$$

for some $\tau > 0$. Then there is no nontrivial L^2 harmonic 1-form on M.

On the other hand, without the assumption of stability, some vanishing theorems about L^2 harmonic 1-forms have also been obtained. In [24], Yun proved that if $M \subset \mathbb{R}^{n+1}$ is a complete minimal hypersurface with sufficiently small total scalar curvature $||A||^2_{L^n}$, then there is no nontrivial L^2 harmonic 1-form on M. Later, Seo [19] proved this result is valid for complete minimal hypersurface in hyperbolic space. Thereafter, it turned out that these vanishing theorems hold for more general submanifolds. Carron [2] proved that if $M \subset \mathbb{R}^{n+p}$ is a complete minimal submanifold with sufficiently small total scalar curvature $||A||^2_{L^n}$, then there is no nontrivial L^2 harmonic 1-form on M. Given an n-dimensional complete noncompact submanifold with finite total mean curvature $||H||^2_{L^n}$ in Euclidean sapce $\mathbb{R}^{n+p}$, Fu and Li [8] showed that if there exists a positive constant $c(n)$ such that total curvature $||\Phi||^2_{L^n} < c(n)$, then all space of L^2 harmonic forms are trivial. Recently, Cavalcante, Mirandola and Vitório [3] extended Yun's result to ambient spaces with nonpositive sectional curvature, and showed that a complete noncompact submanifold M in a Hadamard manifold N with sectional curvature satisfying $-k^2 \leq K_N \leq 0$ has no nontrivial L^2 harmonic 1-forms, if the total curvature $||\Phi||^2_{L^n}$ is sufficiently small, and with additional assumption $\lambda_1(M) > \frac{(n-1)^2}{n}(k^2 - \inf_M H^2)$ in the case $K_N \not\equiv 0$. After that, Dung and Seo [6] proved a similar vanishing theorem for L^2 harmonic 1-forms on complete noncompact submanifolds under the same assumption as in [3] except that the lower bound of $\lambda_1(M)$ depends on $||\Phi||^2_{L^n}$.

In the second part of this paper, motivated by above results, we consider the nonexistence of nontrivial L^2 harmonic 1-forms on complete noncompact submanifold with property $(\mathcal{P}_\rho)$, assuming that the total curvature is sufficiently small instead of stability condition. More precisely, we have the following theorem which is a generalization of Theorem 1.2 in [3].

Theorem 1.5. *Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold, and M^n be a complete noncompact submanifold with property $(\mathcal{P}_\rho)$ for some nonnegative function ρ in N. Assume that*

$$0 \geq K_N(x) \geq -\frac{n}{(n-1)^2}(1-\tau)\rho(x) - \gamma \inf_M H^2$$

for some $0 < \tau < 1, 0 \leq \gamma < 1$. If there exists a positive constant Λ such that $||\Phi||_{L^n} < \Lambda$, then there is no nontrivial L^2 harmonic 1-form on M.

In particular, if we choose $\rho(x) = \lambda_1(M)$ in Theorem 1.5, we can get the following corollary.

Corollary 1.6. *Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold with $0 \geq K_N \geq K$, where $K \leq 0$ is a constant. Let M^n be a complete noncompact submanifold in N. In the case $K_N \not\equiv 0$, assume further that*

$$\lambda_1(M) \geq \frac{(n-1)^2}{(1-\tau)n}\Big(-K - \gamma \inf_M H^2\Big)$$

for some $0 < \tau < 1, 0 \leq \gamma < 1$. *If there exists a positive constant* Λ *such that* $||\Phi||_{L^n} < \Lambda$, *then there is no nontrivial* L^2 *harmonic* 1*-form on* M.

2. Some lemmas

Let us recall some useful results which will be used in the proof of main theorems. The first two lemma are Bochner-Weitzenböck formula and refined Kato inequality for L^2-harmonic forms.

Lemma 2.1 ([13]). *Given a Riemannian manifold* M^n, *for any* 1*-form* ω *on* M^n, *we have*

$$\triangle|\omega|^2 = 2|\nabla\omega|^2 + 2\langle\triangle\omega, \omega\rangle + 2\mathrm{Ric}(\omega^\sharp, \omega^\sharp),$$

where $\omega^\sharp$ *is the dual vector field of* ω.

Lemma 2.2 ([1]). *Given a Riemannian manifold* M^n, *for any closed and coclosed* k*-form* ω *on* M^n, *we have*

$$|\nabla\omega|^2 \geq C_{n,k}|\nabla|\omega||^2, \quad where \quad C_{n,k} = \begin{cases} \frac{n-k+1}{n-k}, & 1 \leq k \leq \frac{n}{2} \\ \frac{k+1}{k}, & \frac{n}{2} \leq k \leq n-1. \end{cases}$$

What's more, Shiohama and Xu [21] proved the following estimating on the Ricci curvature of submanifold.

Lemma 2.3 ([21]). *Let* M *be an* n*-dimensional complete immersed hypersurface in a Riemannian manifold* N. *If all the sectional curvature of* N *are bounded pointwise from below by a function* k, *then*

$$\mathrm{Ric} \geq (n-1)(H^2 + k) - \frac{n-1}{n}|\Phi|^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi|.$$

We should note in [21], the author assumed that all the sectional curvature of N are bounded below by a constant k. But according to his argument, this assumption was only used in the end of the proof, hence this method can be used to prove the above lemma without any change.

In addition, we will need the conditions for the volume of Riemannian manifold to be infinite. Kim and Yun [10] proved the following important fact.

Lemma 2.4 ([10]). *Let* M *be a complete oriented noncompact immersed hypersurface in a complete Riemannian manifold* N^{n+1} *of nonnegative sectional curvature. If* M *satisfies the stability condition* (1.1), *then the volume of* M *is infinite.*

Besides, we will need the following Hoffman-Spruch inequality.

Lemma 2.5 ([9]). *Let* $x : M^n \to N$ *be an isometric immersion of a complete manifold* M *in a complete simply connected manifold* N *with nonpositive sectional curvature. Then for all* $1 \leq p < n$, *the following inequality holds*:

$$\Big(\int_M h^{\frac{pn}{n-p}} dV\Big)^{\frac{n-p}{n}} \leq S(n,p)\int_M (|\nabla h|^p + (h|H|)^p) dV$$

for all nonnegative C^1-functions $h : M^n \to \mathbb{R}$ with compact support, where $S(n,p)^{\frac{1}{p}} = \frac{2p(n-1)}{n-p}c(n)$ and $c(n)$ is the positive constant, depending only on n.

The last but most important lemma was proved by Matheus [15].

Lemma 2.6 ([15]). *Let M be a complete manifold satisfying a weighted Poincaré inequality with a weight function ρ. Suppose a smooth function u on M satisfies the differential inequality*

$$u\triangle u \geq -a\rho u^2 + b|\nabla u|^2$$

for some constant $0 < a < 1 + b$, and assume

$$\int_M u^2 < \infty.$$

Then the function u is a constant. Moreover, if u is not identically zero, then the volume of M is finite and the weight function ρ is identically zero.

3. Proof of the main theorems

Proof of Theorem 1.2. Let ω be a L^2 harmonic 1-form on M, i.e.,

$$\triangle\omega = 0, \ \int_M |\omega|^2 < \infty.$$

Applying Lemma 2.1 we can deduce that

$$\triangle|\omega|^2 = 2|\nabla\omega|^2 + 2\langle\triangle\omega, \omega\rangle + 2\mathrm{Ric}(\omega^\sharp, \omega^\sharp). \tag{3.1}$$

A simple computation implies

$$\triangle|\omega|^2 = 2|\omega|\triangle|\omega| + 2|\nabla|\omega||^2. \tag{3.2}$$

Combining equation (3.1), (3.2) with Lemma 2.2, we have that

$$|\omega|\triangle|\omega| \geq \frac{1}{n-1}|\nabla|\omega||^2 + \mathrm{Ric}(\omega^\sharp, \omega^\sharp).$$

By Lemma 2.3 and $K_N \geq -\frac{(1-\tau)\rho}{(2n-1)(n-1)}$, we have

$$\begin{aligned} |\omega|\triangle|\omega| \geq{}& \frac{1}{n-1}|\nabla|\omega||^2 + (n-1)H^2|\omega|^2 - \frac{n-1}{n}|\Phi|^2|\omega|^2 \\ & - \frac{(1-\tau)\rho}{2n-1}|\omega|^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi||\omega|^2. \end{aligned} \tag{3.3}$$

The stability condition (1.1) implies that

$$\int_M \Big(|\nabla\eta|^2 - (|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))\eta^2\Big) \geq 0, \quad \forall \eta \in C_0^\infty(M).$$

Since

$$\overline{\mathrm{Ric}}(\nu,\nu) \geq -\frac{n(1-\tau)\rho}{(2n-1)(n-1)},$$

we have

$$\int_M \Big(|\nabla\eta|^2 - \Big(|A|^2 - \frac{n(1-\tau)\rho}{(2n-1)(n-1)}\Big)\eta^2\Big) \geq 0, \quad \forall \eta \in C_0^\infty(M). \tag{3.4}$$

Replacing η by $\eta|\omega|$ in (3.4) and integrating by parts allow us to conclude that

$$\begin{aligned}
0 \leq & -\int_M \eta|\omega|^2\triangle\eta - \int_M \eta^2|\omega|\triangle|\omega| - 2\int_M \eta|\omega|\langle\nabla|\omega|, \nabla\eta\rangle \\
& - \int_M |A|^2|\omega|^2\eta^2 + \int_M \frac{n(1-\tau)\rho}{(2n-1)(n-1)}\eta^2|\omega|^2 \\
= & \int_M \langle\nabla(\eta|\omega|^2), \nabla\eta\rangle - \int_M \eta^2(|\omega|\triangle|\omega| + |\Phi|^2|\omega|^2 + nH^2|\omega|^2) \\
& - 2\int_M \eta|\omega|\langle\nabla|\omega|, \nabla\eta\rangle + \frac{n(1-\tau)}{(2n-1)(n-1)}\int_M \rho\eta^2|\omega|^2 \\
\leq & \int_M |\omega|^2|\nabla\eta|^2 - \frac{1}{n-1}\int_M \eta^2|\nabla|\omega||^2 + \frac{1-\tau}{n-1}\int_M \rho\eta^2|\omega|^2 - \frac{1}{n}\int_M |\Phi|^2|\omega|^2\eta^2 \\
& - (2n-1)\int_M H^2|\omega|^2\eta^2 + \frac{(n-2)\sqrt{n(n-1)}}{n}\int_M |H||\Phi||\omega|^2\eta^2,
\end{aligned} \tag{3.5}$$

where we have used (3.3) in the last inequality. From the assumption on weighted poincaré inequality, it follows

$$\begin{aligned}
\int_M \rho\eta^2|\omega|^2 \leq & \int_M |\nabla(\eta|\omega|)|^2 \\
= & \int_M |\omega|^2|\nabla\eta|^2 + \int_M \eta^2|\nabla|\omega||^2 + 2|\omega|\eta\langle\nabla\eta, \nabla|\omega|\rangle \\
\leq & (1+\frac{1}{\varepsilon})\int_M |\omega|^2|\nabla\eta|^2 + (1+\varepsilon)\int_M \eta^2|\nabla|\omega||^2,
\end{aligned} \tag{3.6}$$

where we have used Schwarz inequality and Young's inequality for any $\varepsilon > 0$ in the last inequality. Combining the inequalities (3.5) with (3.6), we have

$$\begin{aligned}
0 \leq & \Big(1 + \frac{1-\tau}{n-1}(1+\frac{1}{\varepsilon})\Big)\int_M |\omega|^2|\nabla\eta|^2 - (2n-1)\int_M H^2|\omega|^2\eta^2 \\
& + \Big(\frac{(1-\tau)(1+\varepsilon)}{n-1} - \frac{1}{n-1}\Big)\int_M \eta^2|\nabla|\omega||^2 - \frac{1}{n}\int_M |\Phi|^2|\omega|^2\eta^2 \\
& + \frac{(n-2)\sqrt{n(n-1)}}{n}\int_M |H||\Phi||\omega|^2\eta^2.
\end{aligned} \tag{3.7}$$

Using Cauchy-Schwarz inequality allows us to conclude

$$\begin{aligned}
& \frac{(n-2)\sqrt{n(n-1)}}{n}\int_M |H||\Phi||\omega|^2\eta^2 \\
\leq & \frac{1-\frac{\tau}{2}}{n}\int_M |\Phi|^2|\omega|^2\eta^2 + \frac{(n-2)^2(n-1)}{4(1-\frac{\tau}{2})}\int_M H^2|\omega|^2\eta^2.
\end{aligned} \tag{3.8}$$

Inequalities (3.7) and (3.8) imply that

$$\begin{aligned} &\Big(\frac{1}{n-1}-\frac{(1-\tau)(1+\varepsilon)}{n-1}\Big)\int_M \eta^2|\nabla|\omega||^2+\frac{\tau}{2n}\int_M|\Phi|^2|\omega|^2\eta^2 \\ &+F\int_M H^2|\omega|^2\eta^2 \\ &\le\Big(1+\frac{1-\tau}{n-1}(1+\frac{1}{\varepsilon})\Big)\int_M|\omega|^2|\nabla\eta|^2, \end{aligned} \tag{3.9}$$

where $F=(2n-1)-\frac{(n-2)^2(n-1)}{4(1-\frac{\tau}{2})}$. The assumption on n allows us to conclude that $F>0$. Choosing a sufficiently small $\varepsilon>0$, and we have $\frac{1}{n-1}-\frac{(1-\tau)(1+\varepsilon)}{n-1}>0$. For each $r>0$, let B_r denote the geodesic ball of radius r on M centered at some fixed point, and suppose $\eta\in C_0^\infty(M)$ be a smooth function such that

$$\begin{cases}\eta=1 & \text{on } B_r,\\ \eta=0 & \text{on } M\backslash B_{2r}\end{cases}$$

and $|\nabla\eta|\le\frac{1}{r}$ on $B_{2r}\backslash B_r$. Using (3.9) with η implies that

$$\begin{aligned} &\Big(\frac{1}{n-1}-\frac{(1-\tau)(1+\varepsilon)}{n-1}\Big)\int_{B_r}|\nabla|\omega||^2+\frac{\tau}{2n}\int_{B_r}|\Phi|^2|\omega|^2+F\int_{B_r}H^2|\omega|^2 \\ &\le\Big(1+\frac{1-\tau}{n-1}(1+\frac{1}{\varepsilon})\Big)\frac{1}{r^2}\int_{B_{2r}\backslash B_r}|\omega|^2. \end{aligned}$$

Letting $r\to\infty$, using the fact that $\omega\in L^2(M)$, and then letting $\varepsilon\to 0$, we finally obtain that

$$\int_M|\nabla|\omega||^2=\int_M|\Phi|^2|\omega|^2=\int_M H^2|\omega|^2=0,$$

which implies that

$$|\nabla|\omega||^2=|\Phi|^2|\omega|^2=H^2|\omega|^2=0.$$

Therefore, $|\omega|$ is a constant. Consequently, we can get that $\omega\equiv 0$. Otherwise, if $\rho\equiv 0$, i.e., $K_N\ge 0$, from Lemma 2.4, we can conclude that the volume of M is infinite. However, the fact $\omega\in L^2$ infers $\int_M|\omega|^2<\infty$, i.e., the volume of M is finite which is a contradiction. If $\rho\not\equiv 0$, from equation (3.6) we deduce that

$$\int_M\rho|\omega|^2=0,$$

which implies that $\rho\equiv 0$. So the space of L^2 harmonic 1-forms must be trivial. □

Proof of Theorem 1.5. Let ω be a L^2 harmonic 1-form on M. Applying Lemma 2.1 and Lemma 2.2, after a direct computation, we obtain that

$$|\omega|\triangle|\omega|\ge\frac{1}{n-1}|\nabla|\omega||^2+\mathrm{Ric}(\omega^\sharp,\omega^\sharp). \tag{3.10}$$

Under our hypothesis on the sectional curvature of N, we can estimate the Ricci curvature of M by using Lemma 2.3

$$\begin{aligned}\mathrm{Ric}(\omega^\sharp,\omega^\sharp) &\geq (n-1)\Big(H^2-\Big(\frac{n}{(n-1)^2}(1-\tau)\rho+\gamma\inf_M H^2\Big)\Big)|\omega|^2\\ &\quad-\frac{n-1}{n}|\Phi|^2|\omega|^2-\frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi||\omega|^2\\ &=(n-1)(H^2-\gamma\inf_M H^2)|\omega|^2-\frac{n(1-\tau)\rho}{n-1}|\omega|^2\\ &\quad-\frac{n-1}{n}|\Phi|^2|\omega|^2-\frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi||\omega|^2.\end{aligned}$$

Plugging this inequality into (3.10), we have that

$$\begin{aligned}|\omega|\triangle|\omega| &\geq \frac{1}{n-1}|\nabla|\omega||^2+(n-1)(H^2-\gamma\inf_M H^2)|\omega|^2-\frac{n(1-\tau)\rho}{n-1}|\omega|^2\\ &\quad-\frac{n-1}{n}|\Phi|^2|\omega|^2-\frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi||\omega|^2.\end{aligned}\tag{3.11}$$

Let $\eta\in C_0^\infty(M)$ be a smooth function on M with compact support. Multiplying both sides of (3.11) by η^2 and integrating by parts allow us to conclude that

$$\begin{aligned}0\leq\ &-2\int_M\eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle-\frac{n}{n-1}\int_M\eta^2|\nabla|\omega||^2\\ &+\frac{(n-2)\sqrt{n(n-1)}}{n}\int_M\eta^2|H||\Phi||\omega|^2+\frac{n-1}{n}\int_M\eta^2|\Phi|^2|\omega|^2\\ &+\frac{n(1-\tau)}{n-1}\int_M\rho\eta^2|\omega|^2+(n-1)\int_M\Big(\gamma\inf_M H^2-H^2\Big)\eta^2|\omega|^2.\end{aligned}\tag{3.12}$$

For each $a>0$, we apply the Cauchy-Schwarz inequality in (3.12) to obtain

$$\begin{aligned}0\leq\ &-2\int_M\eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle-\frac{n}{n-1}\int_M\eta^2|\nabla|\omega||^2+\frac{n(1-\tau)}{n-1}\int_M\rho\eta^2|\omega|^2\\ &+(n-1)\gamma\inf_M H^2\int_M\eta^2|\omega|^2+C\int_M H^2\eta^2|\omega|^2+B\int_M\eta^2|\Phi|^2|\omega|^2,\end{aligned}\tag{3.13}$$

where

$$\begin{aligned}B&=B(n,a)=\frac{(n-2)\sqrt{n(n-1)}}{2an}+\frac{n-1}{n},\\ C&=C(n,a)=-(n-1)+\frac{a(n-2)\sqrt{n(n-1)}}{2n}.\end{aligned}\tag{3.14}$$

On the other hand, since $n\geq 3$, we use the Hölder inequality and Lemma 2.5 to get

$$\int_M\eta^2|\Phi|^2|\omega|^2\leq||\Phi||^2_{L^n}\Big(\int_M(\eta|\omega|)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}$$

$$
\begin{aligned}
&\leq S||\Phi||^2_{L^n}\int_M (|\nabla(\eta|\omega|)|^2+\eta^2|\omega|^2H^2)\\
&= S||\Phi||^2_{L^n}\int_M \Big(|\omega|^2|\nabla\eta|^2+\eta^2|\nabla|\omega||^2+2\eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle\Big)\\
&\quad + S||\Phi||^2_{L^n}\int_M \eta^2|\omega|^2H^2, \qquad (3.15)
\end{aligned}
$$

where $||\Phi||^2_{L^n}=(\int_{supp(\eta)}|\Phi|^n)^{\frac{2}{n}}$ and $S=S(n,2)$ is a constant in Lemma 2.5. Thus, Combining (3.13) with (3.15) we infer that

$$
\begin{aligned}
&\Big(\frac{n}{n-1}-BS||\Phi||^2_{L^n}\Big)\int_M \eta^2|\nabla|\omega||^2-(C+BS||\Phi||^2_{L^n})\int_M H^2\eta^2|\omega|^2\\
&\leq BS||\Phi||^2_{L^n}\int_M |\omega|^2|\nabla\eta|^2+\frac{n(1-\tau)}{n-1}\int_M\rho\eta^2|\omega|^2\\
&+2\Big(BS||\Phi||^2_{L^n}-1\Big)\int_M\eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle+(n-1)\gamma\inf_M H^2\int_M\eta^2|\omega|^2. \qquad (3.16)
\end{aligned}
$$

From the assumption on weighted poincaré inequality, it follows

$$
\begin{aligned}
\int_M\rho\eta^2|\omega|^2 &\leq \int_M|\nabla(\eta|\omega|)|^2\\
&=\int_M|\omega|^2|\nabla\eta|^2+\int_M\eta^2|\nabla|\omega||^2+2|\omega|\eta\langle\nabla\eta,\nabla|\omega|\rangle. \qquad (3.17)
\end{aligned}
$$

Plugging (3.17) into (3.16) implies that

$$
\begin{aligned}
&D\int_M\eta^2|\nabla|\omega||^2-G\int_M H^2\eta^2|\omega|^2\\
&\leq E\int_M|\omega|^2|\nabla\eta|^2+2F\int_M\eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle+(n-1)\gamma\inf_M H^2\int_M\eta^2|\omega|^2,
\end{aligned}
$$

where

$$
\begin{aligned}
D&=\frac{n}{n-1}-BS||\Phi||^2_{L^n}-\frac{n(1-\tau)}{n-1},\\
E&=BS||\Phi||^2_{L^n}+\frac{n(1-\tau)}{n-1},\\
F&=BS||\Phi||^2_{L^n}+\frac{n(1-\tau)}{n-1}-1,\\
G&=C+BS||\Phi||^2_{L^n}.
\end{aligned}
$$

Moreover, for all $\varepsilon>0$, using the Cauchy-Schwarz inequality

$$
2\Big|\int_M\eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle\Big|\leq\varepsilon\int_M\eta^2|\nabla|\omega||^2+\frac{1}{\varepsilon}\int_M|\omega|^2|\nabla\eta|^2,
$$

we see that

$$
\Big(D-|F|\varepsilon\Big)\int_M\eta^2|\nabla|\omega||^2-G\int_M H^2\eta^2|\omega|^2
$$

$$\leq \Big(E + |F|\frac{1}{\varepsilon}\Big)\int_M |\omega|^2|\nabla\eta|^2 + (n-1)\gamma \inf_M H^2 \int_M \eta^2|\omega|^2. \tag{3.18}$$

Choose $0 < d < \frac{1}{2}$, $a = a(d) > 0$ and $\Lambda = \Lambda(d) > 0$ satisfying:

$$\begin{cases} \frac{a(n-2)\sqrt{n(n-1)}}{2n} < (n-1)d, \\ BS\Lambda^2 < (n-1)d. \end{cases} \tag{3.19}$$

Now we set

$$\begin{aligned} \overline{D} &= \frac{n}{n-1} - BS\Lambda^2 - \frac{n(1-\tau)}{n-1}, \\ \overline{E} &= BS\Lambda^2 + \frac{n(1-\tau)}{n-1}, \\ \overline{F} &= BS\Lambda^2 + \frac{n(1-\tau)}{n-1} - 1, \\ \overline{G} &= C + BS\Lambda^2. \end{aligned} \tag{3.20}$$

Assume that the total curvature of x satisfies $||\Phi||_{L^n(M)} < \Lambda$. Plugging the above choices in (3.18) we obtain

$$\begin{aligned} &\Big(\overline{D} - |\overline{F}|\varepsilon\Big)\int_M \eta^2|\nabla|\omega||^2 - \overline{G}\int_M H^2\eta^2|\omega|^2 \\ &\leq \Big(\overline{E} + |\overline{F}|\frac{1}{\varepsilon}\Big)\int_M |\omega|^2|\nabla\eta|^2 + (n-1)\gamma \inf_M H^2 \int_M \eta^2|\omega|^2. \end{aligned} \tag{3.21}$$

Using (3.14) and (3.19) we get:

$$\begin{aligned} -\overline{G} &= -C - BS\Lambda^2 \\ &> (n-1) - \frac{a(n-2)\sqrt{n(n-1)}}{2n} - (n-1)d \\ &> (n-1)(1-2d) > 0. \end{aligned}$$

Thus (3.21) becomes

$$\begin{aligned} &\Big(\overline{D} - |\overline{F}|\varepsilon\Big)\int_M \eta^2|\nabla|\omega||^2 - (\overline{G} + (n-1)\gamma)\inf_M H^2 \int_M \eta^2|\omega|^2 \\ &\leq \Big(\overline{E} + |\overline{F}|\frac{1}{\varepsilon}\Big)\int_M |\omega|^2|\nabla\eta|^2. \end{aligned} \tag{3.22}$$

Then we can choose d, ε sufficiently small satisfying that

$$\overline{D} - |\overline{F}|\varepsilon = \frac{n}{n-1} - BS\Lambda^2 - \frac{n(1-\tau)}{n-1} - |\overline{F}|\varepsilon > \frac{n\tau}{n-1} - (n-1)d - |\overline{F}|\varepsilon > 0,$$

$$-\overline{G} - (n-1)\gamma > (n-1)(1-\gamma-2d) > 0.$$

For each $r > 0$, let B_r denote the geodesic ball of radius r on M centered at some fixed point and let $\eta \in C_0^\infty(M)$ be a smooth function such that

$$\begin{cases} \eta = 1 & \text{on } B_r, \\ \eta = 0 & \text{on } M \backslash B_{2r} \end{cases}$$

and $|\nabla \eta| \leq \frac{1}{r}$ on $B_{2r} \backslash B_r$. Using (3.22) with η we can get that

$$\begin{aligned} &\Big(\overline{D} - |\overline{F}|\varepsilon\Big) \int_{B_r} \eta^2 |\nabla|\omega||^2 - \Big(\overline{G} + (n-1)\gamma\Big) \inf_M H^2 \int_{B_r} \eta^2 |\omega|^2 \\ &\leq \Big(\overline{E} + |\overline{F}|\frac{1}{\varepsilon}\Big) \int_{B_{2r} \backslash B_r} |\omega|^2 |\nabla \eta|^2. \end{aligned}$$

Using the fact that $\int_M |\omega|^2 < \infty$, and taking $r \to \infty$ allow us to conclude that

$$|\nabla|\omega||^2 = \inf_M H^2 |\omega|^2 = 0, \tag{3.23}$$

which implies $|\omega|$ is a constant. Using (3.21) with η and taking $r \to \infty$ infer that $H^2|\omega|^2 = 0$. From (3.15) and the fact $||\Phi||_{L^n} < \Lambda$, applying same way, we can obtain that $|\Phi|^2|\omega|^2 = 0$. Thus combining (3.11) with Cauchy-Schwarz inequality

$$0 \leq 2|H||\Phi||\omega|^2 \leq (H^2|\omega|^2 + |\Phi|^2|\omega|^2) = 0$$

implies that

$$|\omega| \triangle |\omega| \geq \frac{1}{n-1} |\nabla|\omega||^2 - \frac{n(1-\tau)\rho}{n-1} |\omega|^2. \tag{3.24}$$

Consequently, we conclude that $\omega \equiv 0$. Otherwise, if $|\omega| \not\equiv 0$, Lemma 2.6 implies that $\rho \equiv 0$ and the volume of M is finite, meanwhile (3.23) infers that $\inf_M H^2 = 0$. Thus the condition on K_N becomes $K_N \geq 0$. The conclusion $H^2|\omega|^2 = |\Phi|^2|\omega|^2 = 0$ implies that M is totally geodesic in N. Thus M has nonnegative Ricci curvature, which gives the conclusion that the volume of M is infinite [23], which is a contradiction. So the space of L^2 harmonic 1-forms must be trivial. ☐

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Xiaoli Chao
Department of Mathematics
Southeast University
210096 Nanjing, P. R. China
E-mail address: xlchao@seu.edu.cn

Yusha Lv
Department of Mathematics
Southeast University
210096 Nanjing, P. R. China
E-mail address: lvyushasx@163.com