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ON ϕ-n-ABSORBING PRIMARY IDEALS OF COMMUTATIVE RINGS

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Abstract. All rings are commutative with $1 \neq 0$ and n is a positive integer. Let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function where $\mathfrak{J}(R)$ denotes the set of all ideals of R. We say that a proper ideal I of R is *ϕ-n-absorbing primary* if whenever $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1 a_2 \cdots a_{n+1} \in I \backslash \phi(I)$, either $a_1 a_2 \cdots a_n \in I$ or the product of a_{n+1} with $(n-1)$ of $a_1, \ldots, a_n$ is in $\sqrt{I}$. The aim of this paper is to investigate the concept of ϕ-n-absorbing primary ideals.

1. Introduction

Throughout this paper R will be a commutative ring with a nonzero identity. In [2], Anderson and Smith called a proper ideal I of a commutative ring R to be *weakly prime* if whenever $a, b \in R$ and $0 \neq ab \in I$, either $a \in I$ or $b \in I$. In [9], Bhatwadekar and Sharma defined a proper ideal I of an integral domain R to be *almost prime* (*resp. m-almost prime*) if for $a, b \in R$ with $ab \in I \backslash I^2$, (resp. $ab \in I \backslash I^m$, $m \geq 3$) either $a \in I$ or $b \in I$. This definition can obviously be made for any commutative ring R. Later, Anderson and Batanieh [1] gave a generalization of prime ideals which covers all the above mentioned definitions. Let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. A proper ideal I of R is said to be *ϕ-prime* if for $a, b \in R$ with $ab \in I \backslash \phi(I)$, $a \in I$ or $b \in I$. Since $I \backslash \phi(I) = I \backslash (I \cap \phi(I))$, without loss of generality we may assume that $\phi(I) \subseteq I$. We henceforth make this assumption. Weakly primary ideals were first introduced and studied by Ebrahimi Atani and Farzalipour in [10]. A proper ideal I of R is called *weakly primary* if for $a, b \in R$ with $0 \neq ab \in I$, either $a \in I$ or $b \in \sqrt{I}$. In [25], Yousefian Darani called a proper ideal I of R to be *ϕ-primary* if for $a, b \in R$ with $ab \in I \backslash \phi(I)$, then either $a \in I$ or $b \in \sqrt{I}$. He defined the map $\phi_\alpha : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ as follows:

(1) $\phi_\emptyset : \phi(I) = \emptyset$ defines primary ideals.
(2) $\phi_0 : \phi(I) = 0$ defines weakly primary ideals.

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(3) $\phi_2 : \phi(I) = I^2$ defines almost primary ideals.
(4) $\phi_m (m \geq 2) : \phi(I) = I^m$ defines m-almost primary ideals.
(5) $\phi_\omega : \phi(I) = \cap_{m=1}^{\infty} I^m$ defines ω-primary ideals.
(6) $\phi_1 : \phi(I) = I$ defines any ideals.

Given two functions $\psi_1, \psi_2 : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(J) \subseteq \psi_2(J)$ for each $J \in \mathfrak{J}(R)$. Note in this case that

$$\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{m+1} \leq \phi_m \leq \cdots \leq \phi_2 \leq \phi_1.$$

Badawi in [4] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal I of R to be a *2-absorbing ideal of* R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [3] generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an *n-absorbing* (resp. *strongly n-absorbing*) *ideal* if whenever $a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the a_i's (resp. n of the I_i's) whose product is in I. Thus a strongly 1-absorbing ideal is just a prime ideal. Clearly a strongly n-absorbing ideal of R is also an n-absorbing ideal of R. Anderson and Badawi conjectured that these two concepts are equivalent, e.g., they proved that an ideal I of a Prüfer domain R is strongly n-absorbing if and only if I is an n-absorbing ideal of R, [3, Corollary 6.9]. They also gave several results relating strongly n-absorbing ideals. The concept of 2-absorbing ideals has another generalization, called weakly 2-absorbing ideals, which has studied in [8]. A proper ideal I of R is a *weakly 2-absorbing ideal of* R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Generally, Mostafanasab et al. [15] called a proper ideal I of R to be a *weakly n-absorbing* (resp. *strongly weakly n-absorbing*) *ideal* if whenever $0 \neq a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$ (resp. $0 \neq I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the a_i's (resp. n of the I_i's) whose product is in I. Clearly a strongly weakly n-absorbing ideal of R is also a weakly n-absorbing ideal of R. Let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. We say that a proper ideal I of R is a *ϕ-n-absorbing* (resp. *strongly ϕ-n-absorbing*) *ideal of* R if $a_1 a_2 \cdots a_{n+1} \in I \backslash \phi(I)$ for $a_1, a_2, \ldots, a_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ and $I_1 \cdots I_{n+1} \nsubseteq \phi(I)$ for ideals $I_1, \ldots, I_{n+1}$ of R) implies that there are n of the a_i's (resp. n of the I_i's) whose product is in I. Notice that ϕ-n-absorbing ideals of a commutative ring R have already been investigated by Ebrahimpour and Nekooei [11] as $(n, n+1)$-ϕ-prime ideals.

Recall from [6] that a proper ideal I of R is said to be a *2-absorbing primary ideal of* R if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. For more studies concerning 2-absorbing primary (submodules) ideals we refer to [16], [17]. Also, recall from [7] that a proper ideal I of R is said to be a *weakly 2-absorbing primary ideal of* R if whenever $a, b, c \in R$ with $0 \neq abc \in I$ implies $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. We call a proper ideal I of R to be a *ϕ-n-absorbing primary* (resp. *strongly ϕ-n-absorbing primary*) *ideal of* R if $a_1 a_2 \cdots a_{n+1} \in I \backslash \phi(I)$ for some elements $a_1, a_2, \ldots, a_{n+1} \in R$ (resp.

$I_1 \cdots I_{n+1} \subseteq I$ and $I_1 \cdots I_{n+1} \nsubseteq \phi(I)$ for ideals $I_1, \ldots, I_{n+1}$ of R) implies that either $a_1 a_2 \cdots a_n \in I$ or the product of a_{n+1} with $(n-1)$ of $a_1, a_2, \ldots, a_n$ is in $\sqrt{I}$ (resp. either $I_1 I_2 \cdots I_n \subseteq I$ or the product of I_{n+1} with $(n-1)$ of $I_1, I_2, \ldots, I_n$ is in $\sqrt{I}$). We can define the map $\phi_\alpha : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ as follows: Let I be a ϕ_α-n-absorbing primary ideal of R. Then

(1) $\phi_\emptyset(I) = \emptyset \;\Rightarrow\; I$ is an n-absorbing primary ideal.
(2) $\phi_0(I) = 0 \;\Rightarrow\; I$ is a weakly n-absorbing primary ideal.
(3) $\phi_2(I) = I^2 \;\Rightarrow\; I$ is an almost n-absorbing primary ideal.
(4) $\phi_m(I) = I^m$ $(m \geq 2) \;\Rightarrow\; I$ is an m-almost n-absorbing primary ideal.
(5) $\phi_\omega(I) = \cap_{m=1}^{\infty} I^m \;\Rightarrow\; I$ is an ω-n-absorbing primary ideal.
(6) $\phi_1(I) = I \;\Rightarrow\; I$ is any ideal.

Some of our results use the $R(+)M$ construction. Let R be a ring and M be an R-module. Then $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$.

In [22], Quartararo et al. said that a commutative ring R is a *u-ring* provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals. They show that every Bézout ring is a u-ring. Moreover, they proved that every Prüfer domain is a u-domain. Also, any ring which contains an infinite field as a subring is a u-ring, [24, Exercise 3.63].

Let R be a ring and $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. In Section 2, we give some basic properties of ϕ-n-absorbing primary ideals. For instance, we prove that if ϕ reverses the inclusion and for every $1 \leq i \leq k$, I_i is a ϕ-n_i-absorbing primary ideal of R such that $\sqrt{I_i}$ is a ϕ-n_i-absorbing ideal of R, respectively, then $I_1 \cap I_2 \cap \cdots \cap I_k$ and $I_1 I_2 \cdots I_k$ are two ϕ-n-absorbing primary ideals of R where $n = n_1 + n_2 + \cdots + n_k$. It is shown that a Noetherian domain R is a Dedekind domain if and only if a nonzero n-absorbing primary ideal of R is in the form of $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct maximal ideals $M_1, M_2, \ldots, M_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$. Moreover, we prove that if I is an ideal of a ring R such that $\sqrt{I} = M_1 \cap M_2 \cap \cdots \cap M_n$ where M_i's are maximal ideals of R, then I is an n-absorbing primary ideal of R. We show that if I is a ϕ-n-absorbing primary ideal of R that is not an n-absorbing primary ideal, then $I^{n+1} \subseteq \phi(I)$.

In Section 3, we investigate ϕ-n-absorbing primary ideals of direct products of commutative rings. For example, it is shown that if R is an indecomposable ring and J is a finitely generated ϕ-n-absorbing primary ideal of R, where $\phi \leq \phi_{n+2}$, then J is weakly n-absorbing primary. Let $n \geq 2$ be a natural number and $R = R_1 \times \cdots \times R_{n+1}$ be a decomposable ring with identity. Then we prove that R is a von Neumann regular ring if and only if every proper ideal of R is an n-almost n-absorbing primary ideal of R if and only if every proper ideal of R is an ω-n-absorbing primary ideal of R.

In Section 4, we study the stability of ϕ-n-absorbing primary ideals with respect to idealization. As a result of this section, we establish that if I is a proper ideal of R and M is an R-module such that $IM = M$, then $I(+)M$ is an n-almost n-absorbing primary ideal of $R(+)M$ if and only if I is an n-almost n-absorbing primary ideal of R.

In Section 5, we prove that over a u-ring R the two concepts of strongly ϕ-n-absorbing primary ideals and of ϕ-n-absorbing primary ideals are coincide. Moreover, if R is a Prüfer domain and I is an ideal of R, then I is an n-absorbing primary ideal of R if and only if $I[X]$ is an n-absorbing primary ideal of $R[X]$.

2. Properties of ϕ-n-absorbing primary ideals

Let n be a positive integer. Consider elements $a_1, \ldots, a_n$ and ideals $I_1, \ldots, I_n$ of a ring R. Throughout this paper we use the following notations:

- $a_1 \cdots \widehat{a_i} \cdots a_n$: *i-th* term is excluded from $a_1 \cdots a_n$.
- $I_1 \cdots \widehat{I_i} \cdots I_n$: *i-th* term is excluded from $I_1 \cdots I_n$.

It is obvious that any n-absorbing primary ideal of a ring R is a ϕ-n-absorbing primary ideal of R. Also it is evident that the zero ideal is a weakly n-absorbing primary ideal of R. Assume that $p_1, p_2, \ldots, p_{n+1}$ are distinct prime numbers. We know that the zero ideal $I = \{0\}$ is a weakly n-absorbing primary ideal of the ring $\mathbb{Z}_{p_1 p_2 \cdots p_{n+1}}$. Notice that $p_1 p_2 \cdots p_{n+1} = 0 \in I$, but neither $p_1 p_2 \cdots p_n \in I$ nor $p_1 \cdots \widehat{p_i} \cdots p_{n+1} \in \sqrt{I} = \mathrm{Nil}(\mathbb{Z}_{p_1 p_2 \cdots p_{n+1}})$ for every $1 \leq i \leq n$. Hence I is not an n-absorbing primary ideal of $\mathbb{Z}_{p_1 p_2 \cdots p_{n+1}}$.

Remark 2.1. Let I be a proper ideal of a ring R and $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function.

(1) I is ϕ-primary if and only if I is ϕ-1-absorbing primary.
(2) If I is ϕ-n-absorbing primary, then it is ϕ-i-absorbing primary for all $i > n$.
(3) If I is ϕ-primary, then it is ϕ-n-absorbing primary for all $n > 1$.
(4) If I is ϕ-n-absorbing primary for some $n \geq 1$, then there exists the least $n_0 \geq 1$ such that I is ϕ-n_0-absorbing primary. In this case, I is ϕ-n-absorbing primary for all $n \geq n_0$ and it is not ϕ-i-absorbing primary for $n_0 > i > 0$.

Remark 2.2. If I is a radical ideal of a ring R, then clearly I is a ϕ-n-absorbing primary (resp. strongly ϕ-n-absorbing primary) ideal if and only if I is a ϕ-n-absorbing (resp. strongly ϕ-n-absorbing) ideal.

Theorem 2.3. *Let R be a ring and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Then the following conditions are equivalent*:

(1) *I is ϕ-n-absorbing primary*;
(2) *For every elements $x_1, \ldots, x_n \in R$ with $x_1 \cdots x_n \notin \sqrt{I}$,*

$$(I :_R x_1 \cdots x_n) \subseteq [\cup_{i=1}^{n-1} (\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_n)] \cup (I :_R x_1 \cdots x_{n-1}) \cup (\phi(I) :_R x_1 \cdots x_n).$$

Proof. (1)⇒(2) Suppose that $x_1, \ldots, x_n \in R$ such that $x_1 \cdots x_n \notin \sqrt{I}$. Let $a \in (I :_R x_1 \cdots x_n)$. So $ax_1 \cdots x_n \in I$. If $ax_1 \cdots x_n \in \phi(I)$, then $a \in (\phi(I) :_R x_1 \cdots x_n)$. Assume that $ax_1 \cdots x_n \notin \phi(I)$. Since $x_1 \cdots x_n \notin \sqrt{I}$, then either $ax_1 \cdots x_{n-1} \in I$, i.e., $a \in (I :_R x_1 \cdots x_{n-1})$ or for some $1 \le i \le n-1$ we have $ax_1 \cdots \widehat{x_i} \cdots x_n \in \sqrt{I}$, i.e., $a \in (\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_n)$. Consequently

$$(I :_R x_1 \cdots x_n) \subseteq [\cup_{i=1}^{n-1}(\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_n)] \cup (I :_R x_1 \cdots x_{n-1}) \cup (\phi(I) :_R x_1 \cdots x_n).$$

(2)⇒(1) Let $a_1 a_2 \cdots a_{n+1} \in I \backslash \phi(I)$ for some $a_1, a_2, \ldots, a_{n+1} \in R$ such that $a_1 a_2 \cdots a_n \notin I$. Then $a_1 \in (I :_R a_2 \cdots a_{n+1})$. If $a_2 \cdots a_{n+1} \in \sqrt{I}$, then we are done. Hence we may assume that $a_2 \cdots a_{n+1} \notin \sqrt{I}$ and so by part (2),

$$(I :_R a_2 \cdots a_{n+1}) \subseteq [\cup_{i=2}^{n}(\sqrt{I} :_R a_2 \cdots \widehat{a_i} \cdots a_{n+1})] \cup (I :_R a_2 \cdots a_n) \cup (\phi(I) :_R a_2 \cdots a_{n+1}).$$

Since $a_1 a_2 \cdots a_{n+1} \notin \phi(I)$ and $a_1 a_2 \cdots a_n \notin I$, the only possibility is that $a_1 \in \cup_{i=2}^{n}(\sqrt{I} :_R a_2 \cdots \widehat{a_i} \cdots a_{n+1})$. Then $a_1 a_2 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $2 \le i \le n$. Consequently I is ϕ-n-absorbing primary. □

Let R be an integral domain with quotient field K. Badawi and Houston [5] defined a proper ideal I of R to be *strongly primary* if, whenever $ab \in I$ with $a, b \in K$, we have $a \in I$ or $b \in \sqrt{I}$. In [25], a proper ideal I of R is called *strongly ϕ-primary* if whenever $ab \in I \backslash \phi(I)$ with $a, b \in K$, we have either $a \in I$ or $b \in \sqrt{I}$. We say that a proper ideal I of R is *quotient ϕ-n-absorbing primary* if whenever $x_1 x_2 \cdots x_{n+1} \in I \backslash \phi(I)$ with $x_1, x_2, \ldots, x_{n+1} \in K$, we have either $x_1 x_2 \cdots x_n \in I$ or $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{I}$ for some $1 \le i \le n$.

Proposition 2.4. *Let V be a valuation domain with the quotient field K, and let $\phi : \mathfrak{J}(V) \to \mathfrak{J}(V) \cup \{\emptyset\}$ be a function. Then every ϕ-n-absorbing primary ideal of V is quotient ϕ-n-absorbing primary.*

Proof. Assume that I is a ϕ-n-absorbing primary ideal of V. Let $x_1 x_2 \cdots x_{n+1} \in I$ for some $x_1, x_2, \ldots, x_{n+1} \in K$ such that $x_1 x_2 \cdots x_n \notin I$. If $x_{n+1} \notin V$, then $x_{n+1}^{-1} \in V$, since V is valuation. So $x_1 \cdots x_n x_{n+1} x_{n+1}^{-1} = x_1 \cdots x_n \in I$, a contradiction. Hence $x_{n+1} \in V$. If $x_i \in V$ for every $1 \le i \le n$, then there is nothing to prove. If $x_i \notin V$ for some $1 \le i \le n$, then $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in I \subseteq \sqrt{I}$. Consequently, I is quotient ϕ-n-absorbing primary. □

Proposition 2.5. *Let R be a von Neumann regular ring and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Then I is a ϕ-n-absorbing primary ideal of R if and only if $e_1 e_2 \cdots e_{n+1} \in I \backslash \phi(I)$ for some idempotent elements $e_1, e_2, \ldots, e_{n+1} \in R$ implies that either $e_1 e_2 \cdots e_n \in I$ or $e_1 \cdots \widehat{e_i} \cdots e_{n+1} \in \sqrt{I}$ for some $1 \le i \le n$.*

Proof. Notice the fact that any finitely generated ideal of a von Neumann regular ring R is generated by an idempotent element. □

Theorem 2.6. *Let R be a ring and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. If I is a ϕ-n-absorbing primary ideal of R such that $\sqrt{\phi(I)} = \phi(\sqrt{I})$, then $\sqrt{I}$ is a ϕ-n-absorbing ideal of R.*

Proof. Let $x_1x_2 \cdots x_{n+1} \in \sqrt{I}\backslash\phi(\sqrt{I})$ for some $x_1, x_2, \ldots, x_{n+1} \in R$ such that $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \notin \sqrt{I}$ for every $1 \leq i \leq n$. Then there is a natural number m such that $x_1^m x_2^m \cdots x_{n+1}^m \in I$. If $x_1^m x_2^m \cdots x_{n+1}^m \in \phi(I)$, then $x_1x_2 \cdots x_{n+1} \in \sqrt{\phi(I)} = \phi(\sqrt{I})$, which is a contradiction. Since I is ϕ-n-absorbing primary, our hypothesis implies that $x_1^m x_2^m \cdots x_n^m \in I$. Hence $x_1x_2 \cdots x_n \in \sqrt{I}$. Therefore $\sqrt{I}$ is a ϕ-n-absorbing ideal of R. □

Corollary 2.7. *Let I be an n-absorbing primary ideal of R. Then $\sqrt{I} = P_1 \cap P_2 \cap \cdots \cap P_i$ where $1 \leq i \leq n$ and P_i's are the only distinct prime ideals of R that are minimal over I.*

Proof. In Theorem 2.6, suppose that $\phi = \phi_\emptyset$. Now apply [3, Theorem 2.5]. □

Theorem 2.8. *Let R be a ring, and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function that reverses the inclusion. Suppose that for every $1 \leq i \leq k$, I_i is a ϕ-n_i-absorbing primary ideal of R such that $\sqrt{I_i} = P_i$ is a ϕ-n_i-absorbing ideal of R, respectively. Set $n := n_1 + n_2 + \cdots + n_k$. The following conditions hold:*

(1) *$I_1 \cap I_2 \cap \cdots \cap I_k$ is a ϕ-n-absorbing primary ideal of R.*
(2) *$I_1 I_2 \cdots I_k$ is a ϕ-n-absorbing primary ideal of R.*

Proof. (1) Set $L = I_1 \cap I_2 \cap \cdots \cap I_k$. Then $\sqrt{L} = P_1 \cap P_2 \cap \cdots \cap P_k$. Suppose that $a_1a_2 \cdots a_{n+1} \in L\backslash\phi(L)$ for some $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin \sqrt{L}$ for every $1 \leq i \leq n$. By, $\sqrt{L} = P_1 \cap P_2 \cap \cdots \cap P_k$ is ϕ-n-absorbing, then $a_1a_2 \cdots a_n \in P_1 \cap P_2 \cap \cdots \cap P_k$. We claim that $a_1a_2 \cdots a_n \in L$. For every $1 \leq i \leq k$, P_i is ϕ-n_i-absorbing and $a_1a_2 \cdots a_n \in P_i\backslash\phi(P_i)$, then there exist elements $1 \leq \beta_1^i, \beta_2^i, \ldots, \beta_{n_i}^i \leq n$ such that $a_{\beta_1^i} a_{\beta_2^i} \cdots a_{\beta_{n_i}^i} \in P_i$. If $\beta_r^l = \beta_s^m$ for two pairs l, r and m, s, then

$$a_{\beta_1^1} a_{\beta_2^1} \cdots a_{\beta_{n_1}^1} \cdots a_{\beta_1^l} a_{\beta_2^l} \cdots a_{\beta_r^l} \cdots a_{\beta_{n_l}^l} \cdots$$
$$a_{\beta_1^m} a_{\beta_2^m} \cdots \widehat{a_{\beta_s^m}} \cdots a_{\beta_{n_m}^m} \cdots a_{\beta_1^k} a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} \in \sqrt{L}.$$

Therefore $a_1 \cdots \widehat{a_{\beta_s^m}} \cdots a_n a_{n+1} \in \sqrt{L}$, a contradiction. So β_j^i's are distinct. Hence

$$\{a_{\beta_1^1}, a_{\beta_2^1}, \ldots, a_{\beta_{n_1}^1}, a_{\beta_1^2}, a_{\beta_2^2}, \ldots, a_{\beta_{n_2}^2}, \ldots, a_{\beta_1^k}, a_{\beta_2^k}, \ldots, a_{\beta_{n_k}^k}\} = \{a_1, a_2, \ldots, a_n\}.$$

If $a_{\beta_1^i} a_{\beta_2^i} \cdots a_{\beta_{n_i}^i} \in I_i$ for every $1 \leq i \leq k$, then

$$a_1a_2 \cdots a_n = a_{\beta_1^1} a_{\beta_2^1} \cdots a_{\beta_{n_1}^1} a_{\beta_1^2} a_{\beta_2^2} \cdots a_{\beta_{n_2}^2} \cdots a_{\beta_1^k} a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} \in L,$$

thus we are done. Therefore we may assume that $a_{\beta_1^1} a_{\beta_2^1} \cdots a_{\beta_{n_1}^1} \notin I_1$. Since I_1 is ϕ-n_1-absorbing primary and

$$a_{\beta_1^1} a_{\beta_2^1} \cdots a_{\beta_{n_1}^1} a_{\beta_1^2} a_{\beta_2^2} \cdots a_{\beta_{n_2}^2} \cdots a_{\beta_1^k} a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} a_{n+1} = a_1 \cdots a_{n+1} \in I_1\backslash\phi(I_1),$$

then we have $a_{\beta_1^1}\cdots\widehat{a_{\beta_t^1}}\cdots a_{\beta_{n_1}^1}a_{\beta_1^2}a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k}a_{\beta_2^k}\cdots a_{\beta_{n_k}^k}a_{n+1}\in P_1$ for some $1\le t\le n_1$. On the other hand

$$a_{\beta_1^1}\cdots\widehat{a_{\beta_t^1}}\cdots a_{\beta_{n_1}^1}a_{\beta_1^2}a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k}a_{\beta_2^k}\cdots a_{\beta_{n_k}^k}a_{n+1}\in P_2\cap\cdots\cap P_k.$$

Consequently $a_{\beta_1^1}\cdots\widehat{a_{\beta_t^1}}\cdots a_{\beta_{n_1}^1}a_{\beta_1^2}a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k}a_{\beta_2^k}\cdots a_{\beta_{n_k}^k}a_{n+1}\in\sqrt{L}$, which is a contradiction. Similarly we deduce that $a_{\beta_1^i}a_{\beta_2^i}\cdots a_{\beta_{n_i}^i}\in I_i$ for every $2\le i\le k$. Then $a_1a_2\cdots a_n\in L$.

(2) The proof is similar to that of part (1). □

Corollary 2.9. *Let R be a ring with $1\neq 0$ and let $P_1,P_2,\ldots,P_n$ be prime ideals of R. Suppose that for every $1\le i\le n$, $P_i^{t_i}$ is a P_i-primary ideal of R where t_i is a positive integer. Then $P_1^{t_1}\cap P_2^{t_2}\cap\cdots\cap P_n^{t_n}$ and $P_1^{t_1}P_2^{t_2}\cdots P_n^{t_n}$ are n-absorbing primary ideals of R. In particular, $P_1\cap P_2\cap\cdots\cap P_n$ and $P_1P_2\cdots P_n$ are n-absorbing primary ideals of R.*

Example 2.10. Let $R=\mathbb{Z}[X_2,X_3,\ldots,X_n]+3X_1\mathbb{Z}[X_2,X_3,\ldots,X_n,X_1]$. Set $P_i:=X_{i+1}R$ for $1\le i\le n-1$ and $P_n:=3X_1\mathbb{Z}[X_2,X_3,\ldots,X_n,X_1]$. Note that for every $1\le i\le n$, P_i is a prime ideal of R. Let $I=P_1P_2\cdots P_{n-1}P_n^2$. Then $3X_1^2.X_2.\cdots.X_n.3=9X_1^2X_2\cdots X_n\in I$ and $3X_1^2.X_2.\cdots.X_n=3X_1^2X_2\cdots X_n\notin I$. On the other hand $X_2.\cdots.X_n.3=3X_2\cdots X_n\notin\sqrt{I}\subseteq P_n$ and $3X_1^2.X_2.\cdots.\widehat{X_i}.\cdots.X_n.3=9X_1^2X_2\cdots\widehat{X_i}\cdots X_n\notin\sqrt{I}\subseteq P_{i-1}$ for every $2\le i\le n$. Hence I is not n-absorbing primary.

In [6, Example 2.7], the authors offered an example to show that if $I\subset J$ such that I is a 2-absorbing primary ideal of R and $\sqrt{I}=\sqrt{J}$, then J need not be a 2-absorbing ideal of R. They considered the ideal $J=\langle XYZ,Y^3,X^3\rangle$ of the ring $R=\mathbb{Z}[X,Y,Z]$ and showed that $\sqrt{J}=\langle XY\rangle$. But $X\in\sqrt{J}$, which is a contradiction. Therefore their example is incorrect. In the following example we show that if $I\subset J$ such that I is a n-absorbing primary ideal of R and $\sqrt{I}=\sqrt{J}$, then J need not be a n-absorbing ideal of R.

Example 2.11. Let $R=K[X_1,X_2,\ldots,X_{n+2}]$ where K is a field. Consider the ideal $J=\langle X_1X_2\cdots X_{n+1},X_1^2X_2\cdots X_n,X_1^2X_{n+2}\rangle$ of R. Then

$$\begin{aligned}\sqrt{J}&=\langle X_1X_2\cdots X_n,X_1X_{n+2}\rangle\\&=\langle X_1\rangle\cap\langle X_2,X_{n+2}\rangle\cap\langle X_3,X_{n+2}\rangle\cap\cdots\cap\langle X_n,X_{n+2}\rangle.\end{aligned}$$

Set $P_1=\langle X_1\rangle$ and $P_i=\langle X_i,X_{n+2}\rangle$ for every $2\le i\le n$. Note that P_i's are prime ideals of R. Let $I=P_1^2P_2\cdots P_n$. Then $I\subset J$ and $\sqrt{I}=\sqrt{J}=\cap_{i=1}^nP_i$. By Corollary 2.9, I is an n-absorbing primary ideal of R, but J is not an n-absorbing primary ideal of R because $X_1X_2\cdots X_{n+1}\in J$, but $X_1X_2\cdots X_n\notin J$ and $X_2\cdots X_{n+1}\notin\sqrt{J}\subseteq\langle X_1\rangle$ and $X_1\cdots\widehat{X_i}\cdots X_{n+1}\notin\sqrt{J}\subseteq\langle X_i,X_{n+2}\rangle$ for every $2\le i\le n$.

Theorem 2.12. *Let R be a ring, and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Suppose that I is an ideal of R such that $\sqrt{\phi(\sqrt{I})} \subseteq \phi(I)$. If $\sqrt{I}$ is a ϕ-$(n-1)$-absorbing ideal of R, then I is a ϕ-n-absorbing primary ideal of R.*

Proof. Let $\sqrt{I}$ be ϕ-$(n-1)$-absorbing. Assume that $a_1a_2\cdots a_{n+1} \in I\backslash\phi(I)$ for some $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1a_2\cdots a_n \notin I$. Hence

$$(a_1a_{n+1})(a_2a_{n+1})\cdots(a_na_{n+1}) = (a_1a_2\cdots a_n)a_{n+1}^n \in I \subseteq \sqrt{I}.$$

Notice that, if $(a_1a_2\cdots a_n)a_{n+1}^n \in \phi(\sqrt{I})$, then $a_1a_2\cdots a_na_{n+1} \in \sqrt{\phi(\sqrt{I})} \subseteq \phi(I)$ which is a contradiction. Therefore

$$(a_1a_{n+1})(a_2a_{n+1})\cdots(a_na_{n+1}) \in \sqrt{I}\backslash\phi(\sqrt{I}).$$

Then for some $1 \leq i \leq n$,

$$(a_1a_{n+1})\cdots\widehat{(a_ia_{n+1})}\cdots(a_na_{n+1}) = (a_1\cdots\widehat{a_i}\cdots a_n)a_{n+1}^{n-1} \in \sqrt{I},$$

and so $a_1\cdots\widehat{a_i}\cdots a_na_{n+1} \in \sqrt{I}$. Consequently I is ϕ-n-absorbing primary. □

The following example gives an ideal J of a ring R where $\sqrt{J}$ is an n-absorbing ideal of R, but J is not an n-absorbing primary ideal of R.

Example 2.13. Let $R = K[X_1, X_2, \ldots, X_{n+2}]$ where K is a field and let $J = \langle X_1X_2\cdots X_{n+1}, X_1^2X_2\cdots X_n, X_1^2X_{n+2}\rangle$. Then

$$\sqrt{J} = \langle X_1\rangle \cap \langle X_2, X_{n+2}\rangle \cap \langle X_3, X_{n+2}\rangle \cap \cdots \cap \langle X_n, X_{n+2}\rangle.$$

By [3, Theorem 2.1(c)], $\sqrt{J}$ is an n-absorbing ideal of R, but J is not an n-absorbing primary ideal of R as it is shown in Example 2.11.

We know that if I is an ideal of a ring R such that $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R.

Theorem 2.14. *Let I be an ideal of a ring R. If $\sqrt{I} = M_1 \cap M_2 \cap \cdots \cap M_n$ where M_i's are maximal ideals of R, then I is an n-absorbing primary ideal of R.*

Proof. Let $a_1a_2\cdots a_{n+1} \in I$ for some $a_1, a_2, \ldots, a_{n+1} \in R$ such that $a_1\cdots\widehat{a_i}\cdots a_{n+1} \notin \sqrt{I}$ for every $1 \leq i \leq n$. If for some $1 \leq i \leq n$, $a_1\cdots\widehat{a_i}\cdots a_{n+1} \in M_j$ (for every $1 \leq j \leq n$), then $a_1\cdots\widehat{a_i}\cdots a_{n+1} \in \sqrt{I}$ and so we are done. Without loss of generality we may assume that for every $1 \leq i \leq n$, $a_1\cdots\widehat{a_i}\cdots a_{n+1} \notin M_i$, respectively. Since M_i's are maximal, then $M_i + R(a_1\cdots\widehat{a_i}\cdots a_{n+1}) = R$ for every $1 \leq i \leq n$. Therefore for every $1 \leq i \leq n$ there are $m_i \in M_i$ and $r_i \in R$ such that $m_i + r_i(a_1\cdots\widehat{a_i}\cdots a_{n+1}) = 1$. So

$$m_1m_2\cdots m_n + \sum_{t=1}^{n} \sum_{\substack{\alpha_1=1 \\ \alpha_1<\alpha_2< \\ \cdots<\alpha_t\leq n}}^{n-t+1} [r_{\alpha_1}r_{\alpha_2}\cdots r_{\alpha_t}(m_1\cdots\widehat{m_{\alpha_1}}\cdots\widehat{m_{\alpha_2}}\cdots\widehat{m_{\alpha_t}}\cdots m_n)$$

$$\prod_{i=1}^{t}(a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1})] = 1.$$

Since $m_1 m_2 \cdots m_n \in \sqrt{I}$, hence $(m_1 m_2 \cdots m_n)^t \in I$ for some $t \geq 1$. Thus

$$(m_1 m_2 \cdots m_n)^t + s\big[\sum_{t=1}^{n} \sum_{\substack{\alpha_1=1 \\ \alpha_1<\alpha_2< \\ \cdots<\alpha_t\leq n}}^{n-t+1} [r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_t} (m_1 \cdots \widehat{m_{\alpha_1}} \cdots \widehat{m_{\alpha_2}} \cdots \widehat{m_{\alpha_t}} \cdots m_n)$$

$$\prod_{i=1}^{t}(a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1})]\big] = 1$$

for some $s \in R$. Multiply $a_1 a_2 \cdots a_n$ on both sides to get

$$a_1 a_2 \cdots a_n = a_1 a_2 \cdots a_n (m_1 m_2 \cdots m_n)^t +$$

$$s\big[\sum_{t=1}^{n} \sum_{\substack{\alpha_1=1 \\ \alpha_1<\alpha_2< \\ \cdots<\alpha_t\leq n}}^{n-t+1} [r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_t} (m_1 \cdots \widehat{m_{\alpha_1}} \cdots \widehat{m_{\alpha_2}} \cdots \widehat{m_{\alpha_t}} \cdots m_n)$$

$$(a_1 a_2 \cdots a_n) \prod_{i=1}^{t}(a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1})]\big] \in I.$$

Hence I is an n-absorbing primary ideal. □

Let R be an integral domain with $1 \neq 0$ and let K be the quotient field of R. A nonzero ideal I of R is said to be *invertible* if $II^{-1} = R$, where $I^{-1} = \{x \in K \mid xI \subseteq R\}$. An integral domain R is said to be a *Dedekind domain* if every nonzero proper ideal of R is invertible.

Theorem 2.15. *Let R be a Noetherian integral domain with $1 \neq 0$ that is not a field . The following conditions are equivalent*:

(1) *R is a Dedekind domain*;

(2) *A nonzero proper ideal I of R is an n-absorbing primary ideal of R if and only if $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct maximal ideals $M_1, M_2, \ldots, M_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$*;

(3) *If I is a nonzero n-absorbing primary ideal of R, then $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct maximal ideals $M_1, M_2, \ldots, M_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$*;

(4) *A nonzero proper ideal I of R is an n-absorbing primary ideal of R if and only if $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct prime ideals $P_1, P_2, \ldots, P_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$*;

(5) *If I is a nonzero n-absorbing primary ideal of R, then $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct prime ideals $P_1, P_2, \ldots, P_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$.*

Proof. (1)⇒(2) Assume that R is a Dedekind domain that is not a field. Then every nonzero prime ideal of R is maximal. Let I be a nonzero n-absorbing primary ideal of R. Since R is a Dedekind domain, then there are distinct maximal ideals $M_1, M_2, \ldots, M_i$ of R ($k \geq 1$) such that $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ in which t_j's are positive integers. Therefore $\sqrt{I} = M_1 \cap M_2 \cap \cdots \cap M_i$. Since I is n-absorbing primary and every prime ideal of R is maximal, then $\sqrt{I}$ is the intersection of at most n maximal ideals of R, by Corollary 2.7. So $i \leq n$.

Conversely, suppose that $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct maximal ideals $M_1, M_2, \ldots, M_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$. Then I is n-absorbing primary, by Corollary 2.9.

(1)⇒(4) The proof is similar to that of (1)⇒(2).

(2)⇒(3), (3)⇒(5) and (4)⇒(5) are evident.

(5)⇒(1) Let M be an arbitrary maximal ideal of R and I be an ideal of R such that $M^2 \subset I \subset M$. Hence $\sqrt{I} = M$ and so I is M-primary. Then I is n-absorbing primary, and thus by part (5) we have that $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct prime ideals $P_1, P_2, \ldots, P_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$. Then $\sqrt{I} = P_1 \cap P_2 \cap \cdots \cap P_i = M$ which shows that I is a power of M, a contradiction. Therefore, there are no ideals properly between M^2 and M. Consequently R is a Dedekind domain, by [13, Theorem 39.2, p. 470]. □

Since every principal ideal domain is a Dedekind domain, we have the following result as a consequence of Theorem 2.15.

Corollary 2.16. *Let R be a principal ideal domain and I be a nonzero proper ideal of R. Then I is an n-absorbing primary ideal of R if and only if $I = R(p_1^{t_1} p_2^{t_2} \cdots p_i^{t_i})$, where p_j's are prime elements of R, $1 \leq i \leq n$ and t_j's are some integers.*

The following example shows that an n-absorbing primary ideal of a ring R need not be of the form $P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$, where P_j's are prime ideals of R, $1 \leq i \leq n$ and t_j's are some integers.

Example 2.17. Let $R = K[X_1, X_2, \ldots, X_n]$ where K is a field and let $I = \langle X_1, X_2, \ldots, X_{n-1}, X_n^2 \rangle$. Since I is $\langle X_1, X_2, \ldots, X_n \rangle$-primary, then I is an n-absorbing primary ideal of R. But I is not in the form of $P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$, where P_j's are prime ideals of R, $1 \leq i \leq n$ and t_j's are some integers.

Theorem 2.18. *Let R be a ring, $a \in R$ a nonunit and $m \geq 2$ a positive integer. If $(0 :_R a) \subseteq \langle a \rangle$, then $\langle a \rangle$ is ϕ-n-absorbing primary for some ϕ with $\phi \leq \phi_m$ if and only if $\langle a \rangle$ is n-absorbing primary.*

Proof. We may assume that $\langle a \rangle$ is ϕ_m-n-absorbing primary. Let $x_1 x_2 \cdots x_{n+1} \in \langle a \rangle$ for some $x_1, x_2, \ldots, x_{n+1} \in R$. If $x_1 x_2 \cdots x_{n+1} \notin \langle a^m \rangle$, then either $x_1 x_2 \cdots x_n \in \langle a \rangle$ or $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{\langle a \rangle}$ for some $1 \leq i \leq n$. Therefore, assume that $x_1 x_2 \cdots x_{n+1} \in \langle a^m \rangle$. Hence $x_1 x_2 \cdots x_n (x_{n+1} + a) \in \langle a \rangle$. If $x_1 x_2 \cdots x_n (x_{n+1} + a) \notin \langle a^m \rangle$, then either $x_1 x_2 \cdots x_n \in \langle a \rangle$ or $x_1 \cdots \widehat{x_i} \cdots x_n (x_{n+1} + a) \in \sqrt{\langle a \rangle}$

for some $1 \leq i \leq n$. So, either $x_1x_2\cdots x_n \in \langle a\rangle$ or $x_1\cdots \widehat{x_i}\cdots x_{n+1} \in \sqrt{\langle a\rangle}$ for some $1 \leq i \leq n$. Hence, suppose that $x_1x_2\cdots x_n(x_{n+1}+a) \in \langle a^m\rangle$. Thus $x_1x_2\cdots x_{n+1} \in \langle a^m\rangle$ implies that $x_1x_2\cdots x_na \in \langle a^m\rangle$. Therefore, there exists $r \in R$ such that $x_1x_2\cdots x_n - ra^{m-1} \in (0 :_R a) \subseteq \langle a\rangle$. Consequently $x_1x_2\cdots x_n \in \langle a\rangle$. □

Corollary 2.19. *Let R be an integral domain, $a \in R$ a nonunit element and $m \geq 2$ a positive integer. Then $\langle a\rangle$ is ϕ-n-absorbing primary for some ϕ with $\phi \leq \phi_m$ if and only if $\langle a\rangle$ is n-absorbing primary.*

Theorem 2.20. *Let V be a valuation domain and n be a natural number. Suppose that I is an ideal of V such that I^{n+1} is not principal. Then I is a ϕ_{n+1}-n-absorbing primary if and only if it is n-absorbing primary.*

Proof. ($\Rightarrow$) Assume that I is ϕ_{n+1}-n-absorbing primary that is not n-absorbing primary. Therefore there are $a_1,\ldots,a_{n+1} \in R$ such that $a_1\cdots a_{n+1} \in I$, but neither $a_1\cdots a_n \in I$ nor $a_1\cdots\widehat{a_i}\cdots a_{n+1} \in \sqrt{I}$ for every $1 \leq i \leq n$. Hence $\langle a_i\rangle \nsubseteq I$ for every $1 \leq i \leq n+1$. Since V is a valuation domain, thus $I \subset \langle a_i\rangle$ for every $1 \leq i \leq n+1$, and so $I^{n+1} \subseteq \langle a_1\cdots a_{n+1}\rangle$. Since I^{n+1} is not principal, then $a_1\cdots a_{n+1} \in I\backslash I^{n+1}$. Therefore I ϕ_{n+1}-n-absorbing primary implies that either $a_1\cdots a_n \in I$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$, which is a contradiction. Consequently I is n-absorbing primary.

($\Leftarrow$) is trivial. □

Let J be an ideal of R and $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R)\cup\{\emptyset\}$ be a function. Define $\phi_J : \mathfrak{J}(R/J) \to \mathfrak{J}(R/J)\cup\{\emptyset\}$ by $\phi_J(I/J) = (\phi(I)+J)/J$ for every ideal $I \in \mathfrak{J}(R)$ with $J \subseteq I$ (and $\phi_J(I/J) = \emptyset$ if $\phi(I) = \emptyset$).

Theorem 2.21. *Let $J \subseteq I$ be proper ideals of a ring R, and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R)\cup\{\emptyset\}$ be a function.*

(1) *If I is a ϕ-n-absorbing primary ideal of R, then I/J is a ϕ_J-n-absorbing primary ideal of R/J.*
(2) *If $J \subseteq \phi(I)$ and I/J is a ϕ_J-n-absorbing primary ideal of R/J, then I is a ϕ-n-absorbing primary ideal of R.*
(3) *If $\phi(I) \subseteq J$ and I is a ϕ-n-absorbing primary ideal of R, then I/J is a weakly n-absorbing primary ideal of R/J.*
(4) *If $\phi(J) \subseteq \phi(I)$, J is a ϕ-n-absorbing primary ideal of R and I/J is a weakly n-absorbing primary ideal of R/J, then I is a ϕ-n-absorbing primary ideal of R.*

Proof. (1) Let $a_1,a_2,\ldots,a_{n+1} \in R$ be such that $(a_1+J)(a_2+J)\cdots(a_{n+1}+J) \in (I/J)\backslash\phi_J(I/J) = (I/J)\backslash(\phi(I)+J)/J$. Then $a_1a_2\cdots a_{n+1} \in I\backslash\phi(I)$ and I ϕ-n-absorbing primary gives either $a_1\cdots a_n \in I$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. Therefore either $(a_1+J)\cdots(a_n+J) \in I/J$ or $(a_1+J)\cdots\widehat{(a_i+J)}\cdots(a_{n+1}+J) \in \sqrt{I}/J = \sqrt{I/J}$ for some $1 \leq i \leq n$. This shows that I/J is ϕ_J-n-absorbing primary.

(2) Suppose that $a_1a_2\cdots a_{n+1}\in I\backslash\phi(I)$ for some $a_1,a_2,\ldots,a_{n+1}\in R$. Then $(a_1+J)(a_2+J)\cdots(a_{n+1}+J)\in(I/J)\backslash(\phi(I)/J)=(I/J)\backslash\phi_J(I/J)$. Since I/J is assumed to be ϕ_J-n-absorbing primary, we get either $(a_1+J)\cdots(a_n+J)\in I/J$ or $(a_1+J)\cdots\widehat{(a_i+J)}\cdots(a_{n+1}+J)\in\sqrt{I/J}=\sqrt{I}/J$ for some $1\leq i\leq n$. Consequently, either $a_1\cdots a_n\in I$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1}\in\sqrt{I}$ for some $1\leq i\leq n$, that I is ϕ-n-absorbing primary.

(3) is a direct consequence of part (1).

(4) Let $a_1\cdots a_{n+1}\in I\backslash\phi(I)$ where $a_1,\ldots,a_{n+1}\in R$. Note that $a_1\cdots a_{n+1}\notin\phi(J)$ because $\phi(J)\subseteq\phi(I)$. If $a_1\cdots a_{n+1}\in J$, then either $a_1\cdots a_n\in J\subseteq I$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1}\in\sqrt{J}\subseteq\sqrt{I}$ for some $1\leq i\leq n$, since J is ϕ-n-absorbing primary. If $a_1\cdots a_{n+1}\notin J$, then $(a_1+I)\cdots(a_{n+1}+I)\in(I/J)\backslash\{0\}$ and so either $(a_1+I)\cdots(a_n+I)\in I/J$ or $(a_1+J)\cdots\widehat{(a_i+J)}\cdots(a_{n+1}+J)\in\sqrt{I/J}=\sqrt{I}/J$ for some $1\leq i\leq n$. Therefore, either $a_1\cdots a_n\in I$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1}\in\sqrt{I}$ for some $1\leq i\leq n$. Consequently I is a ϕ-n-absorbing primary ideal of R. □

Corollary 2.22. *Let R be a ring, and let $\phi:\mathfrak{J}(R)\to\mathfrak{J}(R)\cup\{\emptyset\}$ be a function. An ideal I of R is ϕ-n-absorbing primary if and only if $I/\phi(I)$ is a weakly n-absorbing primary ideal of $R/\phi(I)$.*

Proof. In parts (2) and (3) of Theorem 2.21 set $J=\phi(I)$. □

Corollary 2.23. *Let R be a ring, $\phi:\mathfrak{J}(R)\to\mathfrak{J}(R)\cup\{\emptyset\}$ be a function and L be a proper ideal of R such that $\phi(\langle X\rangle)\subseteq\phi(\langle L,X\rangle)\subseteq\langle X\rangle$. If $\langle L,X\rangle$ is a ϕ-n-absorbing primary ideal of $R[X]$, then L is a weakly n-absorbing primary ideal of R. The converse holds if in addition R is an integral domain.*

Proof. Consider the isomorphism $\langle L,X\rangle/\langle X\rangle\simeq L$ in $R[X]/\langle X\rangle\simeq R$. Set $I:=\langle L,X\rangle$ and $J:=\langle X\rangle$. Assume that $\langle L,X\rangle$ is a ϕ-n-absorbing primary ideal of $R[X]$. So, by part (3) of Theorem 2.21, $I/J\simeq L$ is a weakly n-absorbing primary ideal of $R[X]/J\simeq R$. Now, suppose that R is an integral domain and L is a weakly n-absorbing primary ideal of R. Since $J=\langle X\rangle$ is a prime ideal of $R[X]$, then it is ϕ-n-absorbing primary. On the other hand $I/J\simeq L$ is a weakly n-absorbing primary ideal of $R[X]/J\simeq R$. Hence, part (4) of Theorem 2.21 implies that $I=\langle L,X\rangle$ is a ϕ-n-absorbing primary ideal of $R[X]$. □

Let S be a multiplicatively closed subset of a ring R. Let $\phi:\mathfrak{J}(R)\to\mathfrak{J}(R)\cup\{\emptyset\}$ be a function and define $\phi_S:\mathfrak{J}(R_S)\to\mathfrak{J}(R_S)\cup\{\emptyset\}$ by $\phi_S(J)=(\phi(J\cap R))_S$ (and $\phi_S(J)=\emptyset$ if $\phi(J\cap R)=\emptyset$) for every ideal J of R_S. Note that $\phi_S(J)\subseteq J$. Let M be an R-module. The set of all zero divisors on M is:

$\mathrm{Z}_R(M)=\{r\in R\mid \text{there exists an element } 0\neq x\in M \text{ such that } rx=0\}$.

Proposition 2.24. *Let R be a ring and $\phi:\mathfrak{J}(R)\to\mathfrak{J}(R)\cup\{\emptyset\}$ be a function. Suppose that S is a multiplicatively closed subset of R and I is a proper ideal of R.*

(1) *If I is a ϕ-n-absorbing primary ideal of R with $I \cap S = \emptyset$ and $\phi(I)_S \subseteq \phi_S(I_S)$, then I_S is a ϕ_S-n-absorbing primary ideal of R_S.*

(2) *If I_S is a ϕ_S-n-absorbing primary ideal of R_S with $\phi_S(I_S) \subseteq \phi(I)_S$, $S \cap Z_R(\frac{I}{\phi(I)}) = \emptyset$ and $S \cap Z_R(\frac{R}{I}) = \emptyset$, then I is a ϕ-n-absorbing primary ideal of R.*

Proof. (1) Assume that $\frac{a_1}{s_1}\frac{a_2}{s_2}\cdots\frac{a_{n+1}}{s_{n+1}} \in I_S \backslash \phi_S(I_S)$ for some $\frac{a_1}{s_1}, \frac{a_2}{s_2}, \ldots, \frac{a_{n+1}}{s_{n+1}} \in R_S$ such that $\frac{a_1}{s_1}\frac{a_2}{s_2}\cdots\frac{a_n}{s_n} \notin I_S$. Since $\frac{a_1}{s_1}\frac{a_2}{s_2}\cdots\frac{a_{n+1}}{s_{n+1}} \in I_S$, then there is $s \in S$ such that $sa_1a_2\cdots a_{n+1} \in I$. If $sa_1a_2\cdots a_{n+1} \in \phi(I)$, then $\frac{a_1}{s_1}\frac{a_2}{s_2}\cdots\frac{a_{n+1}}{s_{n+1}} = \frac{sa_1a_2\cdots a_{n+1}}{ss_1s_2\cdots s_{n+1}} \in \phi(I)_S \subseteq \phi_S(I_S)$, a contradiction. Hence $a_1a_2\cdots a_n(sa_{n+1}) \in I\backslash\phi(I)$. As I is ϕ-n-absorbing primary, we get either $a_1a_2\cdots a_n \in I$ or $a_1\cdots\widehat{a_i}\cdots a_n(sa_{n+1}) \in \sqrt{I}$ for some $1 \leq i \leq n$. The first case implies that $\frac{a_1}{s_1}\frac{a_2}{s_2}\cdots\frac{a_n}{s_n} \in I_S$ which is a contradiction, and the second case implies that $\frac{a_1}{s_1}\cdots\frac{\widehat{a_i}}{s_i}\cdots\frac{a_{n+1}}{s_{n+1}} \in (\sqrt{I})_S = \sqrt{I_S}$ for some $1 \leq i \leq n$. Consequently I_S is a ϕ_S-n-absorbing primary ideal of R_S.

(2) Let $a_1a_2\cdots a_{n+1} \in I\backslash\phi(I)$ for some $a_1, a_2, \ldots, a_{n+1} \in R$ and let $a_1a_2\cdots a_n \notin I$. Then $\frac{a_1}{1}\frac{a_2}{1}\cdots\frac{a_{n+1}}{1} \in I_S$. Assume that $\frac{a_1}{1}\frac{a_2}{1}\cdots\frac{a_{n+1}}{1} \in \phi_S(I_S)$. Since $\phi_S(I_S) \subseteq \phi(I)_S$, then there exists a $s \in S$ such that $sa_1a_2\cdots a_{n+1} \in \phi(I)$. Since $S \cap Z_R(\frac{I}{\phi(I)}) = \emptyset$ we have that $a_1a_2\cdots a_{n+1} \in \phi(I)$, which is a contradiction. Therefore $\frac{a_1}{1}\frac{a_2}{1}\cdots\frac{a_{n+1}}{1} \in I_S\backslash\phi_S(I_S)$. Hence, either $\frac{a_1}{1}\frac{a_2}{1}\cdots\frac{a_n}{1} \in I_S$ or $\frac{a_1}{1}\cdots\frac{\widehat{a_i}}{1}\cdots\frac{a_{n+1}}{1} \in \sqrt{I_S} = (\sqrt{I})_S$ for some $1 \leq i \leq n$. If $\frac{a_1}{1}\frac{a_2}{1}\cdots\frac{a_n}{1} \in I_S$, then there exists $u \in S$ such that $ua_1a_2\cdots a_n \in I$ and so the assumption $S \cap Z_R(\frac{R}{I}) = \emptyset$ shows that $a_1a_2\cdots a_n \in I$, a contradiction. Therefore, there is $1 \leq i \leq n$ such that $\frac{a_1}{1}\cdots\frac{\widehat{a_i}}{1}\cdots\frac{a_{n+1}}{1} \in (\sqrt{I})_S$, and thus there is a $t \in S$ such that $ta_1\cdots\widehat{a_i}\cdots a_{n+1} \in \sqrt{I}$. Note that $S \cap Z_R(\frac{R}{I}) = \emptyset$ implies that $S \cap Z_R(\frac{R}{\sqrt{I}}) = \emptyset$, then $a_1\cdots\widehat{a_i}\cdots a_{n+1} \in \sqrt{I}$. Consequently I is a ϕ-n-absorbing primary ideal of R. □

Let $f : R \to T$ be a homomorphism of rings and let $\phi_T : \mathfrak{J}(T) \to \mathfrak{J}(T) \cup \{\emptyset\}$ be a function. Define $\phi_R : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ by $\phi_R(I) = \phi_T(I^e)^c$ (and $\phi_R(I) = \emptyset$ if $\phi_T(I^e) = \emptyset$). We recall that if R is a Prüfer domain or $T = R_S$ for some multiplicatively closed subset S of R, then for every ideal J of T we have $J^{ce} = J$.

Theorem 2.25. *Let $f : R \to T$ be a homomorphism of rings. If J is a ϕ_T-n-absorbing primary ideal of T such that $\phi_T(J) \subseteq \phi_T(J^{ce})$ (e.g. where $J = J^{ce}$), then J^c is a ϕ_R-n-absorbing primary ideal of R.*

Proof. Let $a_1a_2\cdots a_{n+1} \in J^c\backslash\phi_R(J^c)$ for some $a_1, a_2 \ldots, a_{n+1} \in R$. If

$$f(a_1)f(a_2)\cdots f(a_{n+1}) \in \phi_T(J),$$

then $a_1a_2\cdots a_{n+1} \in \phi_T(J)^c \subseteq \phi_T(J^{ce})^c = \phi_R(J^c)$, which is a contradiction. Therefore $f(a_1)f(a_2)\cdots f(a_{n+1}) \in J\backslash\phi_T(J)$. Hence, either $f(a_1)f(a_2)\cdots f(a_n) \in J$ or $f(a_1)\cdots\widehat{f(a_i)}\cdots f(a_{n+1}) \in \sqrt{J}$ for some $1 \leq i \leq n$. Thus, either

$a_1a_2\cdots a_n \in J^c$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1} \in \sqrt{J^c}$ for some $1 \le i \le n$. Consequently J^c is a ϕ_R-n-absorbing primary ideal of R. □

Let R, T be rings and $\psi_R : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Define $\psi_T : \mathfrak{J}(T) \to \mathfrak{J}(T) \cup \{\emptyset\}$ by $\psi_T(J) = \psi_R(J^c)^e$ (and $\psi_T(J) = \emptyset$ if $\psi_R(J^c) = \emptyset$). We recall that if $f : R \to T$ is a faithfully flat homomorphism of rings, then for every ideal I of R we have $I^{ec} = I$.

Theorem 2.26. *Let $f : R \to T$ be a faithfully flat homomorphism of rings.*

(1) *If J is a ψ_T-n-absorbing primary ideal of T, then J^c is a ψ_R-n-absorbing primary ideal of R.*

(2) *If I^e is a ψ_T-n-absorbing primary ideal of T for some ideal I or R, then I is a ψ_R-n-absorbing primary ideal of R.*

Proof. (1) Suppose that J is a ψ_T-n-absorbing primary ideal of T. In Theorem 2.25 get $\phi_T := \psi_T$. Let I be an ideal of R. Then

$$\phi_R(I) = \phi_T(I^e)^c = \psi_T(I^e)^c = \psi_R(I^{ec})^{ec} = \psi_R(I).$$

So $\phi_R = \psi_R$. Moreover, $\psi_T(J) = \psi_R(J^c)^e = \psi_R(J^{cec})^e = \psi_T(J^{ce})$. Therefore J^c is a ψ_R-n-absorbing primary ideal of R.

(2) By part (1). □

Proposition 2.27. *Let I be an ideal of a ring R such that $\phi(I)$ be an n-absorbing primary ideal of R. If I is a ϕ-n-absorbing primary ideal of R, then I is an n-absorbing primary ideal of R.*

Proof. Assume that $a_1a_2\cdots a_{n+1} \in I$ for some elements $a_1, a_2, \ldots, a_{n+1} \in R$ such that $a_1a_2\cdots a_n \notin I$. If $a_1a_2\cdots a_{n+1} \in \phi(I)$, then $\phi(I)$ n-absorbing primary and $a_1a_2\cdots a_n \notin \phi(I)$ implies that $a_1\cdots\widehat{a_i}\cdots a_{n+1} \in \sqrt{\phi(I)} \subseteq \sqrt{I}$ for some $1 \le i \le n$, and so we are done. When $a_1a_2\cdots a_{n+1} \notin \phi(I)$ clearly the result follows. □

We say that a ϕ-prime ideal P of a ring R is a divided ϕ-prime ideal if $P \subset xR$ for every $x \in R\backslash P$; thus a divided ϕ-prime ideal is comparable to every ideal of R.

Theorem 2.28. *Let P be a divided ϕ-prime ideal of a ring R. Suppose that I is a ϕ-n-absorbing ideal of R with $\sqrt{I} = P$ and $\phi(P) \subseteq \phi(I)$. Then I is a ϕ-primary ideal of R.*

Proof. Let $xy \in I\backslash\phi(I)$ for $x, y \in R$ and $y \notin P$. Since $xy \in P\backslash\phi(P)$, then $x \in P$. If $y^{n-1} \in \phi(P)$, then $y \in \sqrt{I} = P$, which is a contradiction. Therefore $y^{n-1} \notin \phi(P)$, and so $y^{n-1} \notin P$. Thus $P \subset y^{n-1}R$, because P is a divided ϕ-prime ideal of R. Hence $x = y^{n-1}z$ for some $z \in R$. As $y^nz = yx \in I\backslash\phi(I)$, $y^n \notin I$, and I is a ϕ-n-absorbing ideal of R, we have $x = y^{n-1}z \in I$. Hence I is a ϕ-primary ideal of R. □

Let I be an ideal of a ring R and $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Assume that I is a ϕ-n-absorbing primary ideal of R and $a_1, \ldots, a_{n+1} \in R$. We say that $(a_1, \ldots, a_{n+1})$ is an ϕ-$(n+1)$-tuple of I if $a_1 \cdots a_{n+1} \in \phi(I)$, $a_1 a_2 \cdots a_n \notin I$ and for each $1 \leq i \leq n$, $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin \sqrt{I}$.

In the following theorem $a_1 \cdots \widehat{a_i} \cdots \widehat{a_j} \cdots a_n$ denotes that a_i and a_j are eliminated from $a_1 \cdots a_n$.

Theorem 2.29. *Let I be a ϕ-n-absorbing primary ideal of a ring R and suppose that $(a_1, \ldots, a_{n+1})$ is a ϕ-$(n+1)$-tuple of I for some $a_1, \ldots, a_{n+1} \in R$. Then for every elements $\alpha_1, \alpha_2, \ldots, \alpha_m \in \{1, 2, \ldots, n+1\}$ which $1 \leq m \leq n$,*

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} I^m \subseteq \phi(I).$$

Proof. We use induction on m. Let $m = 1$ and suppose that $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} x \notin \phi(I)$ for some $x \in I$. Then $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1}(a_{\alpha_1} + x) \notin \phi(I)$. Since I is a ϕ-n-absorbing primary ideal of R and $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} \notin I$, we conclude that $a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots a_{n+1}(a_{\alpha_1} + x) \in \sqrt{I}$, for some $1 \leq \alpha_2 \leq n+1$ different from α_1. Hence $a_1 \cdots \widehat{a_{\alpha_2}} \cdots a_{n+1} \in \sqrt{I}$, a contradiction. Thus $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} I \subseteq \phi(I)$.

Now suppose $m > 1$ and assume that for all integers less than m the claim holds. Let $a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} x_1 x_2 \cdots x_m \notin \phi(I)$ for some $x_1, x_2, \ldots, x_m \in I$. By induction hypothesis, we conclude that there exists $\zeta \in \phi(I)$ such that

$$\begin{aligned} &a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \\ &= \zeta + a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} x_1 x_2 \cdots x_m \notin \phi(I). \end{aligned}$$

Now, we consider two cases.

Case 1. Assume that $\alpha_m < n+1$. Since I is ϕ-n-absorbing primary, then either

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_n(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \in I,$$

or

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots \widehat{a_j} \cdots a_{n+1}(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \in \sqrt{I}$$

for some $j < n+1$ distinct from α_i's; or

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}(a_{\alpha_1} + x_1) \cdots (\widehat{a_{\alpha_i} + x_i}) \cdots (a_{\alpha_m} + x_m) \in \sqrt{I}$$

for some $1 \leq i \leq m$. Thus either $a_1 a_2 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_j} \cdots a_{n+1} \in \sqrt{I}$ or $a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1} \in \sqrt{I}$, which any of these cases has a contradiction.

Case 2. Assume that $\alpha_m = n+1$. Since I is ϕ-n-absorbing primary, then either

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_{m-1}}} \cdots \widehat{a_{n+1}}(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (\widehat{a_{\alpha_m} + x_m}) \in I,$$

or

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_{m-1}}} \cdots \widehat{a_j} \cdots \widehat{a_{n+1}}(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \in \sqrt{I}$$

for some $j < n+1$ different from α_i's; or

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_{m-1}}} \cdots \widehat{a_{n+1}}(a_{\alpha_1} + x_1) \cdots (\widehat{a_{\alpha_i} + x_i}) \cdots (a_{\alpha_m} + x_m) \in \sqrt{I}$$

for some $1 \leq i \leq m-1$. Thus either $a_1 a_2 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_j} \cdots a_{n+1} \in \sqrt{I}$ or $a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1} \in \sqrt{I}$, which any of these cases has a contradiction. Thus

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} I^m \subseteq \phi(I).$$

□

Theorem 2.30. *Let I be an ϕ-n-absorbing primary ideal of R that is not an n-absorbing primary ideal. Then*

1. $I^{n+1} \subseteq \phi(I)$.
2. $\sqrt{I} = \sqrt{\phi(I)}$.

Proof. (1) Since I is not an n-absorbing primary ideal of R, I has an ϕ-$(n+1)$-triple-zero $(a_1, \ldots, a_{n+1})$ for some $a_1, \ldots, a_{n+1} \in R$. Suppose that $x_1 x_2 \cdots x_{n+1} \notin \phi(I)$ for some $x_1, x_2, \ldots, x_{n+1} \in I$. Then by Theorem 2.29, there is $\zeta \in \phi(I)$ such that $(a_1+x_1) \cdots (a_{n+1}+x_{n+1}) = \zeta + x_1 x_2 \cdots x_{n+1} \notin \phi(I)$. Hence either $(a_1 + x_1) \cdots (a_n + x_n) \in I$ or $(a_1 + x_1) \cdots (\widehat{a_i + x_i}) \cdots (a_{n+1} + x_{n+1}) \in \sqrt{I}$ for some $1 \leq i \leq n$. Thus either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$, a contradiction. Hence $I^{n+1} \subseteq \phi(I)$.

(2) Clearly, $\sqrt{\phi(I)} \subseteq \sqrt{I}$. As $I^{n+1} \subseteq \phi(I)$, we get $\sqrt{I} \subseteq \sqrt{\phi(I)}$, as required. □

Corollary 2.31. *Let I be an ideal of a ring R that is not n-absorbing primary.*

1. *If I is weakly n-absorbing primary, then $I^{n+1} = \{0\}$ and $\sqrt{I} = Nil(R)$.*
2. *If I is ϕ-n-absorbing primary where $\phi \leq \phi_{n+2}$, then $I^{n+1} = I^{n+2}$.*

Corollary 2.32. *Let I be a ϕ-n-absorbing primary ideal where $\phi \leq \phi_{n+2}$. Then I is ω-n-absorbing primary.*

Proof. If I is n-absorbing primary, then it is ω-n-absorbing primary. So assume that I is not n-absorbing primary. Then $I^{n+1} = I^{n+2}$ by Corollary 2.31(2). By hypothesis I is ϕ-n-absorbing primary and $\phi \leq \phi_{n+1}$. So I is ϕ_{n+1}-n-absorbing primary. On the other hand $\phi_\omega(I) = I^{n+1} = \phi_{n+1}(I)$. Therefore I is ω-n-absorbing primary. □

Theorem 2.33. *Let R be a ring and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Suppose that $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals of R such that for every $\lambda, \lambda' \in \Lambda$, $\sqrt{\phi(I_\lambda)} = \sqrt{\phi(I_{\lambda'})}$ and $\phi(I_\lambda) \subseteq \phi(I)$ where $I = \bigcap_{\lambda \in \Lambda} I_\lambda$. If for every $\lambda \in \Lambda$, I_λ is a ϕ-n-absorbing primary ideal of R that is not n-absorbing primary, then I is a ϕ-n-absorbing primary ideal of R.*

Proof. Since I_λ's are ϕ-n-absorbing primary but are not n-absorbing primary, then for every $\lambda \in \Lambda$, $\sqrt{I_\lambda} = \sqrt{\phi(I_\lambda)}$, by Theorem 2.30. On the other hand $\phi(I_\lambda) \subseteq \phi(I)$ for every $\lambda \in \Lambda$, and so $\sqrt{\phi(I_\lambda)} \subseteq \sqrt{I}$. Hence $\sqrt{I} = \sqrt{I_\lambda} = \sqrt{\phi(I_\lambda)}$ for every $\lambda \in \Lambda$. Let $a_1 a_2 \cdots a_{n+1} \in I \backslash \phi(I)$ for some $a_1, a_2, \ldots, a_{n+1} \in$

R, and let $a_1a_2\cdots a_n \notin I$. Therefore there is a $\lambda \in \Lambda$ such that $a_1a_2\cdots a_n \notin I_\lambda$. Since I_λ is ϕ-n-absorbing primary and $a_1a_2\cdots a_{n+1} \in I_\lambda\backslash\phi(I_\lambda)$, then $a_1\cdots\widehat{a_i}\cdots a_{n+1} \in \sqrt{I_\lambda} = \sqrt{I}$ for some $1 \leq i \leq n$. Consequently I is a ϕ-n-absorbing primary ideal of R. □

Corollary 2.34. *Let R be a ring, $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function and I be an ideal of R. Suppose that $\sqrt{\phi(I)} = \phi(\sqrt{I})$ that is an n-absorbing ideal of R. If I is a ϕ-n-absorbing primary ideal of R, then $\sqrt{I}$ is an n-absorbing ideal of R.*

Proof. Assume that I is a ϕ-n-absorbing primary ideal of R. If I is an n-absorbing primary ideal of R, then $\sqrt{I}$ is an n-absorbing ideal, by Theorem 2.6. If I is not an n-absorbing primary ideal of R, then by Theorem 2.30 and by our hypothesis, $\sqrt{I} = \sqrt{\phi(I)}$ which is an n-absorbing ideal. □

Theorem 2.35. *Let I be a ϕ-n-absorbing primary ideal of a ring R that is not n-absorbing primary and let J be a ϕ-m-absorbing primary ideal of R that is not m-absorbing primary, and $n \geq m$. Suppose that the two ideals $\phi(I)$ and $\phi(J)$ are not coprime. Then*

1. $\sqrt{I+J} = \sqrt{\phi(I)+\phi(J)}$.
2. *If $\phi(I) \subseteq J$ and $\phi(J) \subseteq \phi(I+J)$, then $I+J$ is a ϕ-n-absorbing primary ideal of R.*

Proof. (1) By Theorem 2.30, we have $\sqrt{I} = \sqrt{\phi(I)}$ and $\sqrt{J} = \sqrt{\phi(J)}$. Now, by [24, 2.25(i)] the result follows.

(2) Assume that $\phi(I) \subseteq J$ and $\phi(J) \subseteq \phi(I+J)$. Since $\phi(I)+\phi(J) \neq R$, then $I+J$ is a proper ideal of R, by part (1). Since $(I+J)/J \simeq I/(I \cap J)$ and I is ϕ-n-absorbing primary, we get that $(I+J)/J$ is a weakly n-absorbing primary ideal of R/J, by Theorem 2.21(3). On the other hand J is also ϕ-n-absorbing primary, by Remark 2.1(6). Now, the assertion follows from Theorem 2.21(4). □

Let R be a ring and M an R-module. A submodule N of M is called a pure submodule if the sequence $0 \to N\otimes_R E \to M\otimes_R E$ is exact for every R-module E.

As another consequence of Theorem 2.30 we have the following corollary.

Corollary 2.36. *Let R be a ring.*

1. *If I is a pure ϕ-n-absorbing primary ideal of R that is not n-absorbing primary, then $I = \phi(I)$.*
2. *If R is von Neumann regular ring, then every ϕ-n-absorbing primary ideal of R that is not n-absorbing primary is of the form $\phi(I)$ for some ideal I of R.*

Proof. Note that every pure ideal is idempotent (see [12]), also every ideal of a von Neumann regular ring is idempotent. □

Theorem 2.37. *Let $n \geq 2$ be a positive integer, R be a ring and $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Let I be a ϕ-$(n-1)$-absorbing primary ideal of R that is not $(n-1)$-absorbing primary, and J be an ideal of R such that $J \subseteq I$ with $\phi(I) \subseteq \phi(J)$. Then J is a ϕ-n-absorbing primary ideal of R.*

Proof. Since I is a ϕ-$(n-1)$-absorbing primary ideal that is not $(n-1)$-absorbing primary we have $\sqrt{I} = \sqrt{\phi(I)}$, by Theorem 2.30. Hence $\sqrt{J} = \sqrt{I} = \sqrt{\phi(I)}$. Let $a_1a_2 \cdots a_{n+1} \in J \backslash \phi(J)$ for some $a_1, a_2, \ldots, a_{n+1} \in R$ such that $a_1a_2 \cdots a_n \notin J$. Since $J \subseteq I$, we have $a_1a_2 \cdots a_{n+1} \in I \backslash \phi(I)$. Consider two cases.

Case 1. Assume that $a_1a_2 \cdots a_n \notin I$. Since I is ϕ-$(n-1)$-absorbing primary, then it is ϕ-n-absorbing primary, by Remark 2.1(6). Hence $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I} = \sqrt{J}$ for some $1 \leq i \leq n$.

Case 2. Assume that $a_1a_2 \cdots a_n \in I$. Since $a_1a_2 \cdots a_{n+1} \in I \backslash \phi(I)$, we have that $a_1a_2 \cdots a_n \in I \backslash \phi(I)$. On the other hand I is a ϕ-$(n-1)$-absorbing primary ideal, so either $a_1a_2 \cdots a_{n-1} \in I \subseteq \sqrt{J}$ or $a_1 \cdots \widehat{a_i} \cdots a_n \in \sqrt{I} = \sqrt{J}$ for some $1 \leq i \leq n-1$. Hence $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{J}$ for some $1 \leq i \leq n$. Consequently J is a ϕ-n-absorbing primary ideal of R. □

3. ϕ-n-absorbing primary ideals in direct products of commutative rings

Theorem 3.1. *Let R_1 and R_2 be rings, and let I be a weakly n-absorbing primary ideal of R_1. Then $J = I \times R_2$ is a ϕ-n-absorbing primary ideal of $R = R_1 \times R_2$ for each ϕ with $\phi_\omega \leq \phi \leq \phi_1$.*

Proof. Suppose that I is a weakly n-absorbing primary ideal of R_1. If I is n-absorbing primary, then J is n-absorbing primary and hence is ϕ-n-absorbing primary, for all ϕ. Assume that I is not n-absorbing primary. Then $I^{n+1} = \{0\}$, Corollary 2.31(1). Hence $J^{n+1} = \{0\} \times R_2$ and hence $\phi_\omega(J) = \{0\} \times R_2$. Therefore, $J \backslash \phi_\omega(J) = (I \backslash \{0\}) \times R_2$. Let $(x_1, y_1)(x_2, y_2) \cdots (x_{n+1}, y_{n+1}) \in J \backslash \phi_\omega(J)$ for some $x_1, x_2, \ldots, x_{n+1} \in R_1$ and $y_1, y_2, \ldots, y_{n+1} \in R_2$. Then clearly $x_1x_2 \cdots x_{n+1} \in I \backslash \{0\}$. Since I is weakly n-absorbing primary, either $x_1 \cdots x_n \in I$ or $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. Therefore, either $(x_1, y_1) \cdots (x_n, y_n) \in J = I \times R_2$ or $(x_1, y_1) \cdots \widehat{(x_i, y_i)} \cdots (x_{n+1}, y_{n+1}) \in \sqrt{J} = \sqrt{I} \times R_2$ for some $1 \leq i \leq n$. Consequently J is a ω-n-absorbing primary and hence ϕ-n-absorbing primary. □

Theorem 3.2. *Let R be a ring and J be a finitely generated proper ideal of R. Suppose that J is ϕ-n-absorbing primary, where $\phi \leq \phi_{n+2}$. Then, either J is weakly n-absorbing primary or $J^{n+1} \neq 0$ is idempotent and R decomposes as $R_1 \times R_2$ where $R_2 = J^{n+1}$ and $J = I \times R_2$, where I is weakly n-absorbing primary.*

Proof. If J is n-absorbing primary, then J is weakly n-absorbing primary. So we can assume that J is not n-absorbing primary. Then by Corollary 2.31(2),

$J^{n+1} = J^{n+2}$ and hence $J^{n+1} = J^{2(n+1)}$. Thus J^{n+1} is idempotent, since J^{n+1} is finitely generated, $J^{n+1} = \langle e \rangle$ for some idempotent element $e \in R$. Suppose $J^{n+1} = 0$. So $\phi(J) = 0$, and hence J is weakly n-absorbing primary. Assume that $J^{n+1} \neq 0$. Put $R_2 = J^{n+1} = Re$ and $R_1 = R(1-e)$; hence $R = R_1 \times R_2$. Let $I = J(1-e)$, so $J = I \times R_2$, where $I^{n+1} = 0$. We show that I is weakly n-absorbing primary. Let $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1 x_2 \cdots x_{n+1} \in I \backslash \{0\}$ such that $x_1 x_2 \cdots x_n \notin I$. So $(x_1, 0)(x_2, 0) \cdots (x_{n+1}, 0) = (x_1 x_2 \cdots x_{n+1}, 0) \in I \times R_2 = J$. Since $J^{n+1} = \{0\} \times R_2$ and $\phi(J) \subseteq J^{n+1}$, then $(x_1, 0)(x_2, 0) \cdots (x_{n+1}, 0) = (x_1 x_2 \cdots x_{n+1}, 0) \in J \backslash \phi(J)$. Since J is ϕ-n-absorbing primary, so either $(x_1, 0)(x_2, 0) \cdots (x_n, 0) = (x_1 x_2 \cdots x_n, 0) \in I \times R_2 = J$ or $(x_1, 0) \cdots \widehat{(x_i, 0)} \cdots (x_{n+1}, 0) = (x_1 \cdots \widehat{x_i} \cdots x_{n+1}, 0) \in \sqrt{I} \times R_2 = \sqrt{J}$ for some $1 \leq i \leq n$. The first case implies that $x_1 x_2 \cdots x_n \in I$, which is a contradiction. The second case implies that $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. Consequently I is weakly n-absorbing primary. □

Corollary 3.3. *Let R be an indecomposable ring and J a finitely generated ϕ-n-absorbing primary ideal of R, where $\phi \leq \phi_{n+2}$. Then J is weakly n-absorbing primary. Furthermore, if R is an integral domain, then J is actually n-absorbing primary.*

Corollary 3.4. *Let R be a Noetherian integral domain. A proper ideal J of R is n-absorbing primary if and only if it is $(n+2)$-almost n-absorbing primary.*

Theorem 3.5. *Let $R = R_1 \times \cdots \times R_s$ be a decomposable ring and $\psi_i : \mathfrak{I}(R_i) \to \mathfrak{I}(R_i) \cup \{\emptyset\}$ be a function for $i = 1, 2, \ldots, s$. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_n$. Suppose that*

$$L = I_1 \times \cdots \times I_{\alpha_1 - 1} \times R_{\alpha_1} \times I_{\alpha_1 + 1} \times \cdots \times I_{\alpha_j - 1} \times R_{\alpha_j} \times I_{\alpha_j + 1} \times \cdots \times I_s$$

be an ideal of R in which $\{\alpha_1, \ldots, \alpha_j\} \subset \{1, \ldots, s\}$. Moreover, suppose that $\psi_{\alpha_i}(R_{\alpha_i}) \neq R_{\alpha_i}$ for some $\alpha_i \in \{\alpha_1, \ldots, \alpha_j\}$. The following conditions are equivalent:

(1) *L is a ϕ-n-absorbing primary ideal of R*;
(2) *L is an n-absorbing primary ideal of R*;
(3) *$L' := I_1 \times \cdots \times I_{\alpha_1 - 1} \times I_{\alpha_1 + 1} \times \cdots \times I_{\alpha_j - 1} \times I_{\alpha_j + 1} \times \cdots \times I_s$ is an n-absorbing primary ideal of*

$$R' := R_1 \times \cdots \times R_{\alpha_1 - 1} \times R_{\alpha_1 + 1} \times \cdots \times R_{\alpha_j - 1} \times R_{\alpha_j + 1} \times \cdots \times R_s.$$

Proof. (1) ⇒ (2) Since $\psi_{\alpha_i}(R_{\alpha_i}) \neq R_{\alpha_i}$ for some $\alpha_i \in \{\alpha_1, \ldots, \alpha_j\}$, then clearly $L \not\subseteq \sqrt{\phi(L)}$. So by Theorem 2.30(2), L is an n-absorbing primary ideal of R.

(2) ⇒ (3) Assume that L is an n-absorbing primary ideal of R and

$$(a_1^{(1)}, \ldots, a_{\alpha_1 - 1}^{(1)}, a_{\alpha_1 + 1}^{(1)}, \ldots, a_{\alpha_j - 1}^{(1)}, a_{\alpha_j + 1}^{(1)}, \ldots, a_s^{(1)})$$
$$\cdots (a_1^{(n+1)}, \ldots, a_{\alpha_1 - 1}^{(n+1)}, a_{\alpha_1 + 1}^{(n+1)}, \ldots, a_{\alpha_j - 1}^{(n+1)}, a_{\alpha_j + 1}^{(n+1)}, \ldots, a_s^{(n+1)}) \in L',$$

in which $a_i^{(t)}$'s are in R_i, respectively. Then

$$(a_1^{(1)},\ldots,a_{\alpha_1-1}^{(1)},1,a_{\alpha_1+1}^{(1)},\ldots,a_{\alpha_j-1}^{(1)},1,a_{\alpha_j+1}^{(1)},\ldots,a_s^{(1)})$$
$$\cdots(a_1^{(n+1)},\ldots,a_{\alpha_1-1}^{(n+1)},1,a_{\alpha_1+1}^{(n+1)},\ldots,a_{\alpha_j-1}^{(n+1)},1,a_{\alpha_j+1}^{(n+1)},\ldots,a_s^{(n+1)})\in L.$$

So, either

$$(a_1^{(1)},\ldots,a_{\alpha_1-1}^{(1)},1,a_{\alpha_1+1}^{(1)},\ldots,a_{\alpha_j-1}^{(1)},1,a_{\alpha_j+1}^{(1)},\ldots,a_s^{(1)})$$
$$\cdots(a_1^{(n)},\ldots,a_{\alpha_1-1}^{(n)},1,a_{\alpha_1+1}^{(n)},\ldots,a_{\alpha_j-1}^{(n)},1,a_{\alpha_j+1}^{(n)},\ldots,a_s^{(n)})\in L,$$

or there exists $1\leq i\leq n$ such that

$$(a_1^{(1)},\ldots,a_{\alpha_1-1}^{(1)},1,a_{\alpha_1+1}^{(1)},\ldots,a_{\alpha_j-1}^{(1)},1,a_{\alpha_j+1}^{(1)},\ldots,a_s^{(1)})$$
$$\cdots(a_1^{(i-1)},\ldots,a_{\alpha_1-1}^{(i-1)},1,a_{\alpha_1+1}^{(i-1)},\ldots,a_{\alpha_j-1}^{(i-1)},1,a_{\alpha_j+1}^{(i-1)},\ldots,a_s^{(i-1)})$$
$$(a_1^{(i+1)},\ldots,a_{\alpha_1-1}^{(i+1)},1,a_{\alpha_1+1}^{(i+1)},\ldots,a_{\alpha_j-1}^{(i+1)},1,a_{\alpha_j+1}^{(i+1)},\ldots,a_s^{(i+1)})$$
$$\cdots(a_1^{(n+1)},\ldots,a_{\alpha_1-1}^{(n+1)},1,a_{\alpha_1+1}^{(n+1)},\ldots,a_{\alpha_j-1}^{(n+1)},1,a_{\alpha_j+1}^{(n+1)},\ldots,a_s^{(n+1)})\in\sqrt{L},$$

because L is an n-absorbing primary ideal of R. Hence, either

$$(a_1^{(1)},\ldots,a_{\alpha_1-1}^{(1)},a_{\alpha_1+1}^{(1)},\ldots,a_{\alpha_j-1}^{(1)},a_{\alpha_j+1}^{(1)},\ldots,a_s^{(1)})$$
$$\cdots(a_1^{(n)},\ldots,a_{\alpha_1-1}^{(n)},a_{\alpha_1+1}^{(n)},\ldots,a_{\alpha_j-1}^{(n)},a_{\alpha_j+1}^{(n)},\ldots,a_s^{(n)})\in L',$$

or there exists $1\leq i\leq n$ such that

$$(a_1^{(1)},\ldots,a_{\alpha_1-1}^{(1)},a_{\alpha_1+1}^{(1)},\ldots,a_{\alpha_j-1}^{(1)},a_{\alpha_j+1}^{(1)},\ldots,a_s^{(1)})$$
$$\cdots(a_1^{(i-1)},\ldots,a_{\alpha_1-1}^{(i-1)},a_{\alpha_1+1}^{(i-1)},\ldots,a_{\alpha_j-1}^{(i-1)},a_{\alpha_j+1}^{(i-1)},\ldots,a_s^{(i-1)})$$
$$(a_1^{(i+1)},\ldots,a_{\alpha_1-1}^{(i+1)},a_{\alpha_1+1}^{(i+1)},\ldots,a_{\alpha_j-1}^{(i+1)},a_{\alpha_j+1}^{(i+1)},\ldots,a_s^{(i+1)})$$
$$\cdots(a_1^{(n+1)},\ldots,a_{\alpha_1-1}^{(n+1)},a_{\alpha_1+1}^{(n+1)},\ldots,a_{\alpha_j-1}^{(n+1)},a_{\alpha_j+1}^{(n+1)},\ldots,a_s^{(n+1)})\in\sqrt{L'}.$$

Consequently, L' is an n-absorbing primary ideal of R'.

(3) $\Rightarrow$ (1) Let L' is an n-absorbing primary ideal of R'. It is routine to see that L is an n-absorbing primary ideal of R. Consequently, L is a ϕ-n-absorbing primary ideal of R. □

Theorem 3.6. *Let $n\geq 2$ be a positive integer, $R=R_1\times\cdots\times R_n$ be a ring with identity and let $\psi_i:\mathfrak{I}(R_i)\to\mathfrak{I}(R_i)\cup\{\emptyset\}$ be a function for $i=1,2,\ldots,n$ such that $\psi_n(R_n)\neq R_n$. Set $\phi=\psi_1\times\psi_2\times\cdots\times\psi_n$. Suppose that $I_1\times I_2\times\cdots\times I_n$ is an ideal of R which $\psi_1(I_1)\neq I_1$, and for some $2\leq j\leq n$, $\psi_j(I_j)\neq I_j$, and I_i is a proper ideal of R_i for each $1\leq i\leq n-1$. The following conditions are equivalent:*

1. *$I_1\times I_2\times\cdots\times I_n$ is a ϕ-n-absorbing primary ideal of R;*
2. *$I_n=R_n$ and $I_1\times I_2\times\cdots\times I_{n-1}$ is an n-absorbing primary ideal of $R_1\times\cdots\times R_{n-1}$ or I_i is a primary ideal of R_i for every $1\leq i\leq n$, respectively;*

(3) $I_1 \times I_2 \times \cdots \times I_n$ *is an n-absorbing primary ideal of R.*

Proof. (1)⇒(2) Suppose that $I_1 \times I_2 \times \cdots \times I_n$ is a ϕ-n-absorbing primary ideal of R. First assume that $I_n = R_n$. Since $\psi_n(R_n) \neq R_n$, then $I_1 \times I_2 \times \cdots \times I_{n-1}$ is an n-absorbing primary ideal of $R_1 \times \cdots \times R_{n-1}$ by Theorem 3.5. Now, suppose that $I_n \neq R_n$. Fix $2 \leq i \leq n$. We show that I_i is a primary ideal of R_i. Suppose that $ab \in I_i$ for some $a, b \in R_i$. Let $x \in I_1 \backslash \psi_1(I_1)$. Then

$$\begin{aligned}&(x,1,\ldots,1)(1,0,1,\ldots,1,\ldots,1)(1,1,0,1,\ldots,1,\ldots,1)\cdots\\ &\quad(1,\ldots,1,0,\overbrace{1}^{i-th},\ldots,1)(1,\ldots,\overbrace{1}^{i-th},0,1,\ldots,1)\cdots(1,\ldots,1,0)\\ &\qquad(1,\ldots,1,\overbrace{a}^{i-th},1,\ldots,1)(1,\ldots,1,\overbrace{b}^{i-th},1,\ldots,1)\\ &\qquad\quad=(x,0,\ldots,0,\overbrace{ab}^{i-th},0,\ldots,0)\in I_1\times\cdots\times I_n\backslash\psi_1(I_1)\times\cdots\times\psi_n(I_n).\end{aligned}$$

Since $I_1 \times I_2 \times \cdots \times I_n$ is ϕ-n-absorbing primary and I_i's are proper, then either

$$\begin{aligned}&(x,1,\ldots,1)(1,0,1,\ldots,1,\ldots,1)(1,1,0,1,\ldots,1,\ldots,1)\cdots\\ &\quad(1,\ldots,1,0,\overbrace{1}^{i-th},\ldots,1)(1,\ldots,\overbrace{1}^{i-th},0,1,\ldots,1)\cdots(1,\ldots,1,0)\\ &\qquad(1,\ldots,1,\overbrace{a}^{i-th},1,\ldots,1)=(x,0,\ldots,0,\overbrace{a}^{i-th},0,\ldots,0)\in I_1\times\cdots\times I_n,\end{aligned}$$

or

$$\begin{aligned}&(x,1,\ldots,1)(1,0,1,\ldots,1,\ldots,1)(1,1,0,1,\ldots,1,\ldots,1)\cdots\\ &\quad(1,\ldots,1,0,\overbrace{1}^{i-th},\ldots,1)(1,\ldots,\overbrace{1}^{i-th},0,1,\ldots,1)\cdots(1,\ldots,1,0)\\ &\qquad(1,\ldots,1,\overbrace{b}^{i-th},1,\ldots,1)=(x,0,\ldots,0,\overbrace{b}^{i-th},0,\ldots,0)\in\sqrt{I_1\times\cdots\times I_n},\end{aligned}$$

and thus either $a \in I_i$ or $b \in \sqrt{I_i}$. Consequently I_i is a primary ideal of R_i. Since for some $2 \leq j \leq n$, $\psi_j(I_j) \neq I_j$, similarly we can show that I_1 is a primary ideal of R_1.

(2)⇒(3) If $I_n = R_n$ and $I_1 \times I_2 \times \cdots \times I_{n-1}$ is an n-absorbing primary ideal of $R_1 \times \cdots \times R_{n-1}$, then $I_1 \times I_2 \times \cdots \times I_n$ is an n-absorbing primary ideal of R, by Theorem 3.5. Now, assume that I_n is a primary ideal of R_n and for each $1 \leq i \leq n-1$, I_i is a primary ideal of R_i. Suppose that

$$\begin{aligned}&(a_1^{(1)},\ldots,a_n^{(1)})(a_1^{(2)},\ldots,a_n^{(2)})\cdots(a_1^{(n+1)},\ldots,a_n^{(n+1)})\\ &\in I_1\times I_2\times\cdots\times I_n\backslash\psi_1(I_1)\times\cdots\times\psi_n(I_n),\end{aligned}$$

in which for every $1 \leq j \leq n+1$, $a_i^{(j)}$'s are in R_i, respectively. Suppose that

$$(a_1^{(1)},\ldots,a_n^{(1)})(a_1^{(2)},\ldots,a_n^{(2)})\cdots(a_1^{(n)},\ldots,a_n^{(n)})\notin I_1\times I_2\times\cdots\times I_n.$$

Without loss of generality we may assume that $a_1^{(1)} \cdots a_n^{(n)} \notin I_1$. Since I_1 is primary, we deduce that $a_1^{(n+1)} \in \sqrt{I_1}$. On the other hand $\sqrt{I_i}$ is a prime ideal, for any $2 \leq i \leq n$, then at least one of the $a_i^{(j)}$'s is in $\sqrt{I_i}$, say $a_i^{(i)} \in \sqrt{I_i}$. Thus $(a_1^{(2)}, \ldots, a_n^{(2)}) \cdots (a_1^{(n+1)}, \ldots, a_n^{(n+1)}) \in \sqrt{I_1 \times I_2 \times \cdots \times I_n}$. Consequently $I_1 \times I_2 \times \cdots \times I_n$ is an n-absorbing primary ideal of R.

(3)⇒(1) is obvious. □

Theorem 3.7. *Let $R = R_1 \times \cdots \times R_n$ be a ring with identity and let $\psi_i : \Im(R_i) \to \Im(R_i) \cup \{\emptyset\}$ be a function for $i = 1, 2, \ldots, n$ such that $\psi_n(R_n) \neq R_n$. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_n$, and suppose that for every $1 \leq i \leq n-1$, I_i is a proper ideal of R_i such that $\psi_1(I_1) \neq I_1$ and I_n is an ideal of R_n. The following conditions are equivalent*:

1. (1) *$I_1 \times \cdots \times I_n$ is a ϕ-n-absorbing primary ideal of R that is not an n-absorbing primary ideal of R.*
2. (2) *I_1 is a ψ_1-primary ideal of R_1 that is not a primary ideal and for every $2 \leq i \leq n$, $I_i = \psi_i(I_i)$ is a primary ideal of R_i, respectively.*

Proof. (1)⇒(2) Assume that $I_1 \times \cdots \times I_n$ is a ϕ-n-absorbing primary ideal of R that is not an n-absorbing primary ideal. If for some $2 \leq i \leq n$ we have $\psi_i(I_i) \neq I_i$, then $I_1 \times \cdots \times I_n$ is an n-absorbing primary ideal of R by Theorem 3.6, which contradicts our assumption. Thus for every $2 \leq i \leq n$, $\psi_i(I_i) = I_i$ and so $I_n \neq R_n$. A proof similar to part (1)⇒(2) of Theorem 3.6 shows that for every $2 \leq i \leq n$, $\psi_i(I_i) = I_i$ is a primary ideal of R_i. Now, we show that I_1 is a ψ_1-primary ideal of R_1. Consider $a, b \in R_1$ such that $ab \in I_1 \backslash \psi_1(I_1)$. Note that

$$\begin{aligned} &(1,0,1,\ldots,1)(1,1,0,1,\ldots,1)\cdots(1,\ldots,1,0)(a,1,\ldots,1)(b,1,\ldots,1) \\ &= (ab,0,\ldots,0) \in (I_1 \times I_2 \times \cdots \times I_n) \backslash (\psi_1(I_1) \times \cdots \times \psi_n(I_n)). \end{aligned}$$

Because I_i's are proper, the product of $(a,1,\ldots,1)(b,1,\ldots,1)$ with $n-2$ of $(1,0,1,\ldots,1), (1,1,0,1,\ldots,1), \ldots, (1,\ldots,1,0)$ is not in $\sqrt{I_1 \times I_2 \times \cdots \times I_n}$. Since $I_1 \times I_2 \times \cdots \times I_n$ is a ϕ-n-absorbing primary ideal of R, we have either

$$\begin{aligned} &(1,0,1,\ldots,1)(1,1,0,1,\ldots,1)\cdots(1,\ldots,1,0)(a,1,\ldots,1) \\ &= (a,0,\ldots,0) \in I_1 \times I_2 \times \cdots \times I_n, \end{aligned}$$

or

$$\begin{aligned} &(1,0,1,\ldots,1)(1,1,0,1,\ldots,1)\cdots(1,\ldots,1,0)(b,1,\ldots,1) \\ &= (b,0,\ldots,0) \in \sqrt{I_1 \times I_2 \times \cdots \times I_n}. \end{aligned}$$

So either $a \in I_1$ or $b \in \sqrt{I_1}$. Thus I_1 is a ψ_1-primary ideal of R_1. Assume I_1 is a primary ideal of R_1, since for every $2 \leq i \leq n$, I_i is a primary ideal of R_i, it is easy to see that $I_1 \times \cdots \times I_n$ is an n-absorbing primary ideal of R, which is a contradiction.

(2) $\Rightarrow$ (1) It is clear that $I_1 \times \cdots \times I_n$ is a ϕ-n-absorbing primary ideal of R. Since I_1 is not a primary ideal of R_1, there exist elements $a, b \in R_1$ such that $ab \in \psi_1(I_1)$, but $a \notin I_1$ and $b \notin \sqrt{I_1}$. Hence

$$\begin{aligned}&(1,0,1,\dots,1)(1,1,0,1,\dots,1)\cdots(1,\dots,1,0)(a,1,\dots,1)(b,1,\dots,1)\\ &= (ab,0,\dots,0) \in \psi_1(I_1)\times\cdots\times\psi_n(I_n),\end{aligned}$$

but neither

$$\begin{aligned}&(1,0,1,\dots,1)(1,1,0,1,\dots,1)\cdots(1,\dots,1,0)(a,1,\dots,1)\\ &= (a,0,\dots,0) \in I_1\times\cdots\times I_n,\end{aligned}$$

nor

$$\begin{aligned}&(1,0,1,\dots,1)(1,1,0,1,\dots,1)\cdots(1,\dots,1,0)(b,1,\dots,1)\\ &= (b,0,\dots,0) \in \sqrt{I_1\times\cdots\times I_n}.\end{aligned}$$

Also the product of $(a,1,\dots,1)(b,1,\dots,1)$ with $n-2$ of elements $(1,0,1,\dots,1)$, $(1,1,0,1,\dots,1),\dots,(1,\dots,1,0)$ is not in $\sqrt{I_1\times\cdots\times I_n}$. Consequently $I_1 \times \cdots \times I_n$ is not an n-absorbing primary ideal of R. □

Theorem 3.8. *Let $R = R_1 \times \cdots \times R_{n+1}$ where R_i's are rings with identity and let for $i = 1, 2, \dots, n+1$, $\psi_i : \mathfrak{I}(R_i) \to \mathfrak{I}(R_i) \cup \{\emptyset\}$ be a function such that $\psi_i(R_i) \neq R_i$. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_{n+1}$.*

(1) *For every ideal I of R, $\phi(I)$ is not an n-absorbing primary ideal of R;*
(2) *If I is a ϕ-n-absorbing primary ideal of R, then either $I = \phi(I)$, or I is an n-absorbing primary ideal of R.*

Proof. Let I be an ideal of R. We know that the ideal I is of the form $I_1 \times \cdots \times I_{n+1}$ where I_i's are ideals of R_i's, for $i = 1, \dots, n+1$.

(1) Suppose that $\phi(I)$ is an n-absorbing primary ideal of R. Since

$$\begin{aligned}&(0,1,\dots,1)(1,0,1,\dots,1)\cdots(1,\dots,1,0) = (0,\dots,0)\\ &\in \phi(I) = \psi_1(I_1)\times\cdots\times\psi_{n+1}(I_{n+1}),\end{aligned}$$

we have that either

$$\begin{aligned}&(0,1,\dots,1)(1,0,1,\dots,1)\cdots(1,\dots,1,0,1) = (0,\dots,0,1)\\ &\in \psi_1(I_1)\times\cdots\times\psi_{n+1}(I_{n+1}),\end{aligned}$$

or the product of $(1,\dots,1,0)$ with $n-1$ of $(0,1,\dots,1)$, $(1,0,1,\dots,1)$, $\dots$, $(1,\dots,1,0,1)$ is in $\sqrt{\phi(I)}$. Hence, for some $1 \le i \le n+1$, $1 \in \psi_i(I_i)$ which implies that $\psi_i(R_i) = R_i$, a contradiction. Consequently $\phi(I)$ is not an n-absorbing primary ideal of R.

(2) Let $I \neq \phi(I)$. So we have $I = I_1 \times \cdots \times I_{n+1} \neq \psi_1(I_1) \times \psi_2(I_2) \times \cdots \times \psi_{n+1}(I_{n+1})$. Hence, there is an element $(a_1, \dots, a_{n+1}) \in I \backslash (\psi_1(I_1) \times \psi_2(I_2) \times \cdots \times \psi_{n+1}(I_{n+1}))$. Then $(a_1,1,\dots,1)(1,a_2,1,\dots,1)\cdots(1,\dots,1,a_{n+1}) \in I\backslash\phi(I)$. Since I is a ϕ-n-absorbing primary ideal of R, then either

$$(a_1,1,\dots,1)(1,a_2,1,\dots,1)\cdots(1,\dots,1,a_n,1) = (a_1,a_2,\dots,a_n,1) \in I,$$

or, for some $1 \leq i \leq n$ we have

$$(a_1, 1, \dots, 1) \cdots (1, \dots, 1, a_{i-1}, 1, \dots, 1)(1, \dots, 1, a_{i+1}, 1, \dots, 1) \cdots (1, \dots, 1, a_{n+1}) = (a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_{n+1}) \in \sqrt{I}.$$

Then $I_i = R_i$, for some $1 \leq i \leq n+1$ and so $I = I_1 \times \cdots I_{i-1} \times R_i \times I_{i+1} \times \cdots I_{n+1}$. If $I \subseteq \sqrt{\phi(I)}$, then $\psi_i(R_i) = R_i$ which is a contradiction. Therefore, by Theorem 2.30, I must be an n-absorbing primary ideal of R. □

Theorem 3.9. *Let $R = R_1 \times \cdots \times R_{n+1}$ where R_i's are rings with identity and let for $i = 1, 2, \dots, n+1$, $\psi_i : \Im(R_i) \to \Im(R_i) \cup \{\emptyset\}$ be a function such that $\psi_i(R_i) \neq R_i$. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_{n+1}$. Let $L = I_1 \times \cdots \times I_{n+1}$ be a proper ideal of R with $L \neq \phi(L)$. The following conditions are equivalent*:

(1) *$L = I_1 \times \cdots \times I_{n+1}$ is a ϕ-n-absorbing primary ideal of R;*
(2) *$L = I_1 \times \cdots \times I_{n+1}$ is an n-absorbing primary ideal of R;*
(3) *$L = I_1 \times \cdots \times I_{i-1} \times R_i \times I_{i+1} \times \cdots \times I_{n+1}$ for some $1 \leq i \leq n+1$ such that for each $1 \leq t \leq n+1$ different from i, I_t is a primary ideal of R_t or $L = I_1 \times \cdots \times I_{\alpha_1 - 1} \times R_{\alpha_1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j - 1} \times R_{\alpha_j} \times I_{\alpha_j+1} \cdots \times I_{n+1}$ in which $\{\alpha_1, \dots, \alpha_j\} \subset \{1, \dots, n+1\}$ and*

$$I_1 \times \cdots \times I_{\alpha_1 - 1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j - 1} \times I_{\alpha_j+1} \cdots \times I_{n+1}$$

is an n-absorbing primary ideal of

$$R_1 \times \cdots \times R_{\alpha_1 - 1} \times R_{\alpha_1+1} \times \cdots \times R_{\alpha_j - 1} \times R_{\alpha_j+1} \times \cdots \times R_{n+1}.$$

Proof. (1)⇒(2) Since L is a ϕ-n-absorbing primary ideal of R and $L \neq \phi(L)$, then L is an n-absorbing primary ideal of R, by Theorem 3.8.

(2)⇒(3) Suppose that L is an n-absorbing primary ideal of R, then for some $1 \leq i \leq n+1$, $I_i = R_i$ by the proof of Theorem 3.8. Assume that $L = I_1 \times \cdots \times I_{i-1} \times R_i \times I_{i+1} \times \cdots \times I_{n+1}$ for $1 \leq i \leq n+1$ such that for each $1 \leq t \leq n+1$ different from i, I_t is a proper ideal of R_t. Fix an I_t different from I_i. We may assume that $t > i$. Let $ab \in I_t$ for some $a, b \in R_t$. In this case

$$(0,1,\dots,1)(1,0,1,\dots,1)\cdots(1,\dots,1,0,\overbrace{1}^{i-th},\dots,1)(1,\dots,\overbrace{1}^{i-th},0,1,\dots,1)\cdots$$
$$(1,\dots,1,0,\overbrace{1}^{t-th},\dots,1)(1,\dots,\overbrace{1}^{t-th},0,1,\dots,1)\cdots(1,\dots,1,0)(1,\dots,1,\overbrace{a}^{t-th},1,\dots,1)$$
$$(1,\dots,1,\overbrace{b}^{t-th},1,\dots,1) = (0,\dots,0,\overbrace{1}^{i-th},0,\dots,0,\overbrace{ab}^{t-th},0,\dots,0) \in L.$$

Since $I_1 \times \cdots \times I_{n+1}$ is n-absorbing primary and I_j's different from I_i are proper, then either

$$(0,1,\dots,1)(1,0,1,\dots,1)\cdots(1,\dots,1,0,\overbrace{1}^{i-th},\dots,1)(1,\dots,\overbrace{1}^{i-th},0,1,\dots,1)\cdots$$
$$(1,\dots,1,0,\overbrace{1}^{t-th},\dots,1)(1,\dots,\overbrace{1}^{t-th},0,1,\dots,1)\cdots(1,\dots,1,0)(1,\dots,1,\overbrace{a}^{t-th},1,\dots,1)$$
$$= (0,\dots,0,\overbrace{1}^{i-th},0,\dots,0,\overbrace{a}^{t-th},0,\dots,0) \in L,$$

or

$$(0,1,\ldots,1)(1,0,1,\ldots,1)\cdots(1,\ldots,1,0,\overbrace{1}^{i-th},\ldots,1)(1,\ldots,\overbrace{1}^{i-th},0,1,\ldots,1)\cdots$$
$$(1,\ldots,1,0,\overbrace{1}^{t-th},\ldots,1)(1,\ldots,\overbrace{1}^{t-th},0,1,\ldots,1)\cdots(1,\ldots,1,0)(1,\ldots,1,\overbrace{b}^{t-th},1,\ldots,1)$$
$$=(0,\ldots,0,\overbrace{1}^{i-th},0,\ldots,0,\overbrace{b}^{t-th},0,\ldots,0)\in\sqrt{L},$$

and thus either $a \in I_t$ or $b \in \sqrt{I_t}$. Consequently I_t is a primary ideal of R_t. Now, assume that

$$L = I_1 \times \cdots \times I_{\alpha_1 - 1} \times R_{\alpha_1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times R_{\alpha_j} \times I_{\alpha_j+1} \times \cdots \times I_{n+1}$$

in which $\{\alpha_1, \ldots, \alpha_j\} \subset \{1, \ldots, n+1\}$. Since L is n-absorbing primary, then $I_1 \times \cdots \times I_{\alpha_1-1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times I_{\alpha_j+1} \cdots \times I_{n+1}$ is an n-absorbing primary ideal of

$$R_1 \times \cdots \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times \cdots \times R_{\alpha_j-1} \times R_{\alpha_j+1} \times \cdots \times R_{n+1}$$

by Theorem 3.5.

(3)⇒(1) If L is in the first form, then similar to the proof of part (2)⇒(3) of Theorem 3.6 we can verify that L is an n-absorbing primary ideal of R, and hence L is a ϕ-n-absorbing primary ideal of R. For the second form apply Theorem 3.5. □

Theorem 3.10. *Let $R = R_1 \times \cdots \times R_{n+1}$ where R_i's are rings with identity and let for $i = 1, 2, \ldots, n+1$, $\psi_i : \Im(R_i) \to \Im(R_i) \cup \{\emptyset\}$ be a function. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_{n+1}$. Then, every proper ideal of R is a ϕ-n-absorbing primary ideal (ϕ-n-absorbing ideal) of R if and only if $I = \psi_i(I)$ for every $1 \leq i \leq n+1$ and every proper ideal I of R_i.*

Proof. Assume that every proper ideal of R is a ϕ-n-absorbing primary ideal (ϕ-n-absorbing ideal) of R. Fix an i and let I be a proper ideal of R_i. Assume that $I \neq \psi_i(I)$, so give an element $x \in I \backslash \psi_i(I)$. Set

$$J := I \times \{0\} \cdots \times \{0\}.$$

Notice that

$$(1,0,1,\ldots,1)(1,1,0,1,\ldots,1)\cdots(1,\ldots,1,0)(x,1,\ldots,1) \in J \backslash \phi(J).$$

Since I is ϕ-n-absorbing primary, then either

$$(1,0,1,\ldots,1)(1,1,0,1,\ldots,1)\cdots(1,\ldots,1,0) \in J,$$

or the product of $(x,1,\ldots,1)$ with $n-1$ of $(1,0,1,\ldots,1), (1,1,0,1,\ldots,1), \ldots,$ $(1,\ldots,1,0)$ is in $\sqrt{J}$ which implies that either $1 \in I$ or $1 \in \{0\}$, a contradiction. Consequently $I = \psi_i(I)$. The converse is obvious. □

Corollary 3.11. *Let $n \geq 2$ be a natural number and $R = R_1 \times \cdots \times R_{n+1}$ be a decomposable ring with identity. The following conditions are equivalent*:

(1) *R is a von Neumann regular ring*;
(2) *Every proper ideal of R is an n-almost n-absorbing primary ideal of R*;
(3) *Every proper ideal of R is an ω-n-absorbing primary ideal of R*;

(4) *Every proper ideal of R is an n-almost n-absorbing ideal of R.*

Proof. (1)⇔(2), (1)⇔(3) and (1)⇔(4): Notice that, $\phi_n(I) = I$ (or $\phi_\omega(I) = I$) if and only if $I = I^2$. By the fact that R is von Neumann regular if and only if $I = I^2$ for every ideal I of R and regarding Theorem 3.10 we have the implications. □

Corollary 3.12. *Let $R_1, R_2, \ldots, R_{n+1}$ be rings and let $R = R_1 \times R_2 \times \cdots \times R_{n+1}$. Then the following conditions are equivalent*:

(1) *$R_1, R_2, \ldots, R_{n+1}$ are fields*;
(2) *Every proper ideal of R is a weakly n-absorbing ideal of R*;
(3) *Every proper ideal of R is a weakly n-absorbing primary ideal of R.*

Proof. (1)⇒(2) By [11, Theorem 1.10].
(2)⇒(3) is clear.
(3)⇒(1) In Theorem 3.10 assume that $\phi = \phi_0$.
□

4. The stability of ϕ-n-absorbing primary ideals with respect to idealization

Let R be a commutative ring and M be an R-module. We recall from [14, Theorem 25.1] that every ideal of $R(+)M$ is in the form of $I(+)N$ in which I is an ideal of R and N is a submodule of M such that $IM \subseteq N$. Moreover, if $I_1(+)N_1$ and $I_2(+)N_2$ are ideals of $R(+)M$, then $(I_1(+)N_1) \cap (I_2(+)N_2) = (I_1 \cap I_2)(+)(N_1 \cap N_2)$.

Theorem 4.1. *Let R be a ring, I a proper ideal of R and M an R-module. Suppose that $\psi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ and $\phi : \mathfrak{I}(R(+)M) \to \mathfrak{I}(R(+)M) \cup \{\emptyset\}$ are two functions such that $\phi(I(+)M) = \psi(I)(+)N$ for some submodule N of M with $\psi(I)M \subseteq N$. Then the following conditions are equivalent*:

(1) *$I(+)M$ is a ϕ-n-absorbing primary ideal of $R(+)M$;*
(2) *I is a ψ-n-absorbing primary ideal of R and if $(a_1, \ldots, a_{n+1})$ is a ψ-$(n+1)$-tuple, then the second component of $(a_1, m_1) \cdots (a_{n+1}, m_{n+1})$ is in N for any elements $m_1, \ldots, m_{n+1} \in M$.*

Proof. (1) ⇒ (2) Assume that $I(+)M$ is a ϕ-n-absorbing primary ideal of $R(+)M$. Let $x_1 \cdots x_{n+1} \in I \backslash \psi(I)$ for some $x_1, \ldots, x_{n+1} \in R$. Therefore

$$(x_1, 0) \cdots (x_{n+1}, 0) = (x_1 \cdots x_{n+1}, 0) \in I(+)M \backslash \phi(I(+)M),$$

because $\phi(I(+)M) = \psi(I)(+)N$. Hence either $(x_1, 0) \cdots (x_n, 0) = (x_1 \cdots x_n, 0) \in I(+)M$ or $(x_1, 0) \cdots \widehat{(x_i, 0)} \cdots (x_{n+1}, 0) = (x_1 \cdots \widehat{x_i} \cdots x_{n+1}, 0) \in \sqrt{I(+)M} = \sqrt{I}(+)M$ for some $1 \leq i \leq n$. So either $x_1 \cdots x_n \in I$ or $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$ which shows that I is ψ-n-absorbing primary. For the second statement suppose that $a_1 \cdots a_{n+1} \in \psi(I)$, $a_1 \cdots a_n \notin I$ and $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin$

$\sqrt{I}$ for all $1 \le i \le n$. If the second component of $(a_1, m_1) \cdots (a_{n+1}, m_{n+1})$ is not in N, then

$$(a_1, m_1) \cdots (a_{n+1}, m_{n+1}) \in I(+)M \backslash \psi(I)(+)N.$$

Thus either $(a_1, m_1) \cdots (a_n, m_n) \in I(+)M$ or

$$(a_1, m_1) \cdots \widehat{(a_i, m_i)} \cdots (a_{n+1}, m_{n+1}) \in \sqrt{I}(+)M$$

for some $1 \le i \le n$. So either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \le i \le n$, which is a contradiction.

(2) $\Rightarrow$ (1) Suppose that $(a_1, m_1) \cdots (a_{n+1}, m_{n+1}) \in I(+)M \backslash \psi(I)(+)N$ for some $a_1, \ldots, a_{n+1} \in R$ and some $m_1, \ldots, m_{n+1} \in M$. Clearly $a_1 \cdots a_{n+1} \in I$. If $a_1 \cdots a_{n+1} \in \psi(I)$, then the second component of $(a_1, m_1) \cdots (a_{n+1}, m_{n+1})$ cannot be in N. Hence either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \le i \le n$. If $a_1 \cdots a_{n+1} \notin \psi(I)$, then I ψ-n-absorbing primary implies that either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \le i \le n$. Therefore we have either $(a_1, m_1) \cdots (a_n, m_n) \in I(+)M$ or $(a_1, m_1) \cdots \widehat{(a_i, m_i)} \cdots (a_{n+1}, m_{n+1}) \in \sqrt{I(+)M}$ for some $1 \le i \le n$. Consequently $I(+)M$ is a ϕ-n-absorbing primary ideal of $R(+)M$. □

Corollary 4.2. *Let R be a ring, I be a proper ideal of R and M be an R-module. The following conditions are equivalent*:

(1) *$I(+)M$ is an n-absorbing primary ideal of $R(+)M$;*
(2) *I is an n-absorbing primary ideal of R.*

Proof. In Theorem 4.1 set $\phi = \phi_\emptyset$, $\psi = \phi_\emptyset$ and $N = M$. □

Corollary 4.3. *Let R be a ring, I be a proper ideal of R and M be an R-module. The following conditions are equivalent*:

(1) *$I(+)M$ is a weakly n-absorbing primary ideal of $R(+)M$;*
(2) *I is a weakly n-absorbing primary ideal of R and if $(a_1, \ldots, a_{n+1})$ is an $(n+1)$-tuple-zero, then the second component of*

$$(a_1, m_1) \cdots (a_{n+1}, m_{n+1})$$

is zero for any elements $m_1, \ldots, m_{n+1} \in M$.

Proof. In Theorem 4.1 set $\phi = \phi_0$, $\psi = \phi_0$ and $N = \{0\}$. □

Corollary 4.4. *Let R be a ring, I be a proper ideal of R and M be an R-module. Then the following conditions are equivalent*:

(1) *$I(+)M$ is an n-almost n-absorbing primary ideal of $R(+)M$;*
(2) *I is an n-almost n-absorbing primary ideal of R and if $(a_1, \ldots, a_{n+1})$ is a ϕ_n-$(n+1)$-tuple, then for any elements $m_1, \ldots, m_{n+1} \in M$ the second component of $(a_1, m_1) \cdots (a_{n+1}, m_{n+1})$ is in $I^{n-1}M$.*

Proof. Notice that $(I(+)M)^n = I^n(+)I^{n-1}M$. In Theorem 4.1 set $\phi = \phi_n$, $\psi = \phi_n$ and $N = I^{n-1}M$. □

Corollary 4.5. *Let R be a ring, I be a proper ideal of R and M be an R-module such that $IM = M$. Then $I(+)M$ is an n-almost n-absorbing primary ideal of $R(+)M$ if and only if I is an n-almost n-absorbing primary ideal of R.*

Corollary 4.6. *Let R be a ring, I be a proper ideal of R and M be an R-module. Then $I(+)M$ is an ω-n-absorbing primary ideal of $R(+)M$ if and only if I is an ω-n-absorbing primary ideal of R.*

5. Strongly ϕ-n-absorbing primary ideals

Proposition 5.1. *Let I be a proper ideal of a ring R. Then the following conditions are equivalent*:

1. *I is strongly ϕ-n-absorbing primary*;
2. *For every ideals $I_1, \ldots, I_{n+1}$ of R such that $I \subseteq I_1$, $I_1 \cdots I_{n+1} \subseteq I \backslash \phi(I)$ implies that either $I_1 \cdots I_n \subseteq I$ or $I_1 \cdots \widehat{I_i} \cdots I_{n+1} \subseteq \sqrt{I}$ for some $1 \leq i \leq n$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let $J, I_2, \ldots, I_{n+1}$ be ideals of R such that $JI_2 \cdots I_{n+1} \subseteq I$ and $JI_2 \cdots I_{n+1} \nsubseteq \phi(I)$. Then we have that

$$(J + I)I_2 \cdots I_{n+1} = (JI_2 \cdots I_{n+1}) + (II_2 \cdots I_{n+1}) \subseteq I.$$

On the other hand

$$(J + I)I_2 \cdots I_{n+1} \nsubseteq \phi(I),$$

since $JI_2 \cdots I_{n+1} \subseteq (J+I)I_2 \cdots I_{n+1}$. Set $I_1 := J+I$. Then, by the hypothesis either $I_1 \cdots I_n \subseteq I$ or $I_2 \cdots I_{n+1} \subseteq \sqrt{I}$ or there exists $2 \leq i \leq n$ such that $(J + I)I_2 \cdots \widehat{I_i} \cdots I_{n+1} \subseteq \sqrt{I}$. Therefore, either $JI_2 \cdots I_n \subseteq I$ or $I_2 \cdots I_{n+1} \subseteq \sqrt{I}$ or there exists $2 \leq i \leq n$ such that $JI_2 \cdots \widehat{I_i} \cdots I_{n+1} \subseteq \sqrt{I}$. So I is strongly ϕ-n-absorbing primary. □

Remark 5.2. Let R be a ring. Notice that $\mathrm{Jac}(R)$ is a radical ideal of R. So $\mathrm{Jac}(R)$ is a strongly n-absorbing ideal of R if and only if I is a strongly n-absorbing primary ideal of R.

Given any set X, one can define a topology on X where every subset of X is an open set. This topology is referred to as the discrete topology on X, and X is a discrete topological space if it is equipped with its discrete topology.

We denote by $\mathrm{Max}(R)$ the set of all maximal ideals of R.

Theorem 5.3. *Let R be a ring and* $\mathrm{Max}(R)$ *be a discrete topological space. Then* $\mathrm{Max}(R)$ *is an infinite set if and only if* $\mathrm{Jac}(R)$ *is not strongly n-absorbing for every natural number n.*

Proof. ($\Leftarrow$) We can verify this implication without any assumption on $\mathrm{Max}(R)$, by [3, Theorem 2.1].

($\Rightarrow$) Notice that $\mathrm{Max}(R)$ is a discrete topological space if and only if the Jacobson radical of R is the irredundant intersection of the maximal ideals of R, [21, Corollary 3.3]. Let $\mathrm{Max}(R)$ be an infinite set. Assume that for some natural number n, $\mathrm{Jac}(R)$ is a strongly n-absorbing ideal. Choose n distinct elements $M_1, M_2, \ldots, M_n$ of $\mathrm{Max}(R)$. Set $\mathcal{M} := \{M_1, M_2, \ldots, M_n\}$, and denote by $\mathcal{M}^c$ the complement of $\mathcal{M}$ in $\mathrm{Max}(R)$. Since $\mathrm{Jac}(R) = M_1 \cap M_2 \cap \cdots \cap M_n \cap (\bigcap_{M\in\mathcal{M}^c} M)$, then either $M_1 \cdots M_{i-1}M_{i+1} \cdots M_n(\bigcap_{M\in\mathcal{M}^c} M) \subseteq \mathrm{Jac}(R)$ for some $1 \leq i \leq n$, or $M_1M_2 \cdots M_n \subseteq \mathrm{Jac}(R)$. In the first case we have $M_1 \cdots M_{i-1}M_{i+1} \cdots M_n(\bigcap_{M\in\mathcal{M}^c} M) \subseteq M_i$ and so $\bigcap_{M\in\mathcal{M}^c} M \subseteq M_i$, a contradiction. If $M_1M_2 \cdots M_n \subseteq \mathrm{Jac}(R)$, then $M_1M_2 \cdots M_n \subseteq M$ for every $M \in \mathcal{M}^c$, and so again we reach a contradiction. Consequently $\mathrm{Jac}(R)$ is not strongly n-absorbing. □

In the next theorem we investigate ϕ-n-absorbing primary ideals over u-rings. Notice that any Bézout ring is a u-ring, [22, Corollary 1.2].

Theorem 5.4. *Let R be a u-ring and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Then the following conditions are equivalent:*

(1) *I is strongly ϕ-n-absorbing primary;*
(2) *I is ϕ-n-absorbing primary;*
(3) *For every elements $x_1, \ldots, x_n \in R$ with $x_1 \cdots x_n \notin \sqrt{I}$ either*
$$(I :_R x_1 \cdots x_n) = (I :_R x_1 \cdots x_{n-1})$$
or $(I :_R x_1 \cdots x_n) \subseteq (\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_n)$ for some $1 \leq i \leq n-1$ or $(I :_R x_1 \cdots x_n) = (\phi(I) :_R x_1 \cdots x_n)$;
(4) *For every t ideals $I_1, \ldots, I_t$, $1 \leq t \leq n-1$, and for every elements $x_1, \ldots, x_{n-t} \in R$ with $x_1 \cdots x_{n-t}I_1 \cdots I_t \nsubseteq \sqrt{I}$,*
$$(I :_R x_1 \cdots x_{n-t}I_1 \cdots I_t) = (I :_R x_1 \cdots x_{n-t-1}I_1 \cdots I_t)$$
or
$$(I :_R x_1 \cdots x_{n-t}I_1 \cdots I_t) \subseteq (\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t}I_1 \cdots I_t)$$
for some $1 \leq i \leq n-t-1$ or
$$(I :_R x_1 \cdots x_{n-t}I_1 \cdots I_t) \subseteq (\sqrt{I} :_R x_1 \cdots x_{n-t}I_1 \cdots \widehat{I_j} \cdots I_t)$$
for some $1 \leq j \leq t$ or
$$(I :_R x_1 \cdots x_{n-t}I_1 \cdots I_t) = (\phi(I) :_R x_1 \cdots x_{n-t}I_1 \cdots I_t).$$
(5) *For every ideals $I_1, I_2, \ldots, I_n$ of R with $I_1I_2 \cdots I_n \nsubseteq I$, either there is $1 \leq i \leq n$ such that $(I :_R I_1 \cdots I_n) \subseteq (\sqrt{I} :_R I_1 \cdots \widehat{I_i} \cdots I_n)$ or $(I :_R I_1 \cdots I_n) = (\phi(I) :_R I_1 \cdots I_n)$.*

Proof. (1)⇒(2) It is clear.

(2)⇒(3) Suppose that $x_1, \ldots, x_n \in R$ such that $x_1 \cdots x_n \notin \sqrt{I}$. By Theorem 2.3,
$$(I :_R x_1 \cdots x_n) \subseteq [\cup_{i=1}^{n-1}(\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_n)] \cup (I :_R x_1 \cdots x_{n-1}) \cup (\phi(I) :_R x_1 \cdots x_n).$$

Since R is a u-ring we have either $(I :_R x_1 \cdots x_n) \subseteq (\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_n)$ for some $1 \leq i \leq n-1$ or $(I :_R x_1 \cdots x_n) = (I :_R x_1 \cdots x_{n-1})$ or $(I :_R x_1 \cdots x_n) = (\phi(I) :_R x_1 \cdots x_n)$.

(3)⇒(4) We use induction on t. For $t = 1$, consider elements $x_1, \dots, x_{n-1} \in R$ and ideal I_1 of R such that $x_1 \cdots x_{n-1} I_1 \nsubseteq \sqrt{I}$. Let $a \in (I :_R x_1 \cdots x_{n-1} I_1)$. Then $I_1 \subseteq (I :_R ax_1 \cdots x_{n-1})$. If $ax_1 \cdots x_{n-1} \in \sqrt{I}$, then $a \in (\sqrt{I} :_R x_1 \cdots x_{n-1})$. If $ax_1 \cdots x_{n-1} \notin \sqrt{I}$, then by part (3), either $I_1 \subseteq (I :_R ax_1 \cdots x_{n-2})$ or $I_1 \subseteq (\sqrt{I} :_R ax_1 \cdots \widehat{x_i} \cdots x_{n-1})$ for some $1 \leq i \leq n-2$ or $I_1 \subseteq (\sqrt{I} :_R x_1 \cdots x_{n-1})$ or $I_1 \subseteq (\phi(I) :_R ax_1 \cdots x_{n-1})$. The first case implies that $a \in (I :_R x_1 \cdots x_{n-2} I_1)$. The second case implies that $a \in (\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_{n-1} I_1)$ for some $1 \leq i \leq n-2$. The third case cannot be happen, because $x_1 \cdots x_{n-1} I_1 \nsubseteq \sqrt{I}$, and the last case implies that $a \in (\phi(I) :_R x_1 \cdots x_{n-1} I_1)$. Hence

$$\begin{aligned}(I :_R x_1 \cdots x_{n-1} I_1) \subseteq\ & \cup_{i=1}^{n-2} (\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_{n-1} I_1) \cup (\sqrt{I} :_R x_1 \cdots x_{n-1}) \\ & \cup (I :_R x_1 \cdots x_{n-2} I_1) \cup (\phi(I) :_R x_1 \cdots x_{n-1} I_1).\end{aligned}$$

Since R is a u-ring, then either $(I :_R x_1 \cdots x_{n-1} I_1) \subseteq (\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_{n-1} I_1)$ for some $1 \leq i \leq n-2$, or $(I :_R x_1 \cdots x_{n-1} I_1) \subseteq (\sqrt{I} :_R x_1 \cdots x_{n-1})$ or $(I :_R x_1 \cdots x_{n-1} I_1) = (I :_R x_1 \cdots x_{n-2} I_1)$ or $(I :_R x_1 \cdots x_{n-1} I_1) = (\phi(I) :_R x_1 \cdots x_{n-1} I_1)$. Now suppose $t > 1$ and assume that for integer $t-1$ the claim holds. Let $x_1, \dots, x_{n-t}$ be elements of R and let $I_1, \dots, I_t$ be ideals of R such that $x_1 \cdots x_{n-t} I_1 \cdots I_t \nsubseteq \sqrt{I}$. Consider element $a \in (I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t)$. Thus $I_t \subseteq (I :_R ax_1 \cdots x_{n-t} I_1 \cdots I_{t-1})$. If $ax_1 \cdots x_{n-t} I_1 \cdots I_{t-1} \subseteq \sqrt{I}$, then $a \in (\sqrt{I} :_R x_1 \cdots x_{n-t} I_1 \cdots I_{t-1})$. If $ax_1 \cdots x_{n-t} I_1 \cdots I_{t-1} \nsubseteq \sqrt{I}$, then by induction hypothesis, either

$$(I :_R ax_1 \cdots x_{n-t} I_1 \cdots I_{t-1}) \subseteq (\sqrt{I} :_R x_1 \cdots x_{n-t} I_1 \cdots I_{t-1})$$

or

$$(I :_R ax_1 \cdots x_{n-t} I_1 \cdots I_{t-1}) \subseteq (\sqrt{I} :_R ax_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_{t-1})$$

for some $1 \leq i \leq n-t-1$ or

$$(I :_R ax_1 \cdots x_{n-t} I_1 \cdots I_{t-1}) \subseteq (\sqrt{I} :_R ax_1 \cdots x_{n-t} I_1 \cdots \widehat{I_j} \cdots I_{t-1})$$

for some $1 \leq j \leq t-1$ or

$$(I :_R ax_1 \cdots x_{n-t} I_1 \cdots I_{t-1}) = (I :_R ax_1 \cdots x_{n-t-1} I_1 \cdots I_{t-1}),$$

or $(I :_R ax_1 \cdots x_{n-t} I_1 \cdots I_{t-1}) = (\phi(I) :_R ax_1 \cdots x_{n-t} I_1 \cdots I_{t-1})$. Since $x_1 \cdots x_{n-t} I_1 \cdots I_t \nsubseteq \sqrt{I}$, then the first case cannot happen. Consequently, either

$$a \in (\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_t)$$

for some $1 \leq i \leq n-t-1$ or $a \in (\sqrt{I} :_R x_1 \cdots x_{n-t} I_1 \cdots \widehat{I_j} \cdots I_t)$ for some $1 \leq j \leq t-1$ or $a \in (I :_R x_1 \cdots x_{n-t-1} I_1 \cdots I_t)$, or $a \in (\phi(I) :_R x_1 \cdots x_{n-t} I_1 \cdots I_t)$. Hence

$$(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) \subseteq [\cup_{i=1}^{n-t-1} (\sqrt{I} :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_t)]$$

$$\cup\,[\cup_{j=1}^{t}(\sqrt{I}:_R x_1\cdots x_{n-t}I_1\cdots\widehat{I_j}\cdots I_t)]$$
$$\cup\,(I:_R x_1\cdots x_{n-t-1}I_1\cdots I_t)$$
$$\cup\,(\phi(I):_R x_1\cdots x_{n-t}I_1\cdots I_t).$$

Now, since R is u-ring we are done.

(4)⇒(5) Let $I_1, I_2, \ldots, I_n$ be ideals of R such that $I_1I_2\cdots I_n \nsubseteq I$. Suppose that $a \in (I:_R I_1I_2\cdots I_n)$. Then $I_n \subseteq (I:_R aI_1I_2\cdots I_{n-1})$. If $aI_1I_2\cdots I_{n-1} \subseteq \sqrt{I}$, then $a \in (\sqrt{I}:_R I_1I_2\cdots I_{n-1})$. If $aI_1I_2\cdots I_{n-1} \nsubseteq \sqrt{I}$, then by part (4) we have either $I_n \subseteq (I:_R I_1I_2\cdots I_{n-1})$ or $I_n \subseteq (\sqrt{I}:_R aI_1\cdots\widehat{I_i}\cdots I_{n-1})$ for some $1 \le i \le n-1$ or $I_n \subseteq (\phi(I):_R aI_1I_2\cdots I_{n-1})$. By hypothesis, the first case is not hold. The second case implies that $a \in (\sqrt{I}:_R I_1\cdots\widehat{I_i}\cdots I_n)$ for some $1 \le i \le n-1$. The third case implies that $a \in (\phi(I):_R I_1I_2\cdots I_n)$. Similarly, since R is u-ring, there is $1 \le i \le n$ such that $(I:_R I_1\cdots I_n) \subseteq (\sqrt{I}:_R I_1\cdots\widehat{I_i}\cdots I_n)$ or $(I:_R I_1\cdots I_n) = (\phi(I):_R I_1\cdots I_n)$.

(5)⇒(1) This implication has an easy verification. □

Remark 5.5. Note that in Theorem 5.4, for the case $n = 2$ and $\phi = \phi_\emptyset$ we can omit the condition u-ring, by the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them. So we conclude that an ideal I of a ring R is 2-absorbing primary if and only if it is strongly 2-absorbing primary.

Let R be a ring with identity. We recall that if $f = a_0 + a_1X + \cdots + a_tX^t$ is a polynomial on the ring R, then *content* of f is defined as the R-ideal, generated by the coefficients of f, i.e. $c(f) = (a_0, a_1, \ldots, a_t)$. Let T be an R-algebra and c the function from T to the ideals of R defined by $c(f) = \cap\{I \mid I$ is an ideal of R and $f \in IT\}$ known as the content of f. Note that the content function c is nothing but the generalization of the content of a polynomial $f \in R[X]$. The R-algebra T is called a *content R-algebra* if the following conditions hold:

(1) For all $f \in T$, $f \in c(f)T$.
(2) (Faithful flatness) For any $r \in R$ and $f \in T$, the equation $c(rf) = rc(f)$ holds and $c(1_T) = R$.
(3) (Dedekind-Mertens content formula) For each $f, g \in T$, there exists a natural number n such that $c(f)^n c(g) = c(f)^{n-1}c(fg)$.

For more information on content algebras and their examples we refer to [19], [20] and [23]. In [18] Nasehpour gave the definition of a Gaussian R-algebra as follows: Let T be an R-algebra such that $f \in c(f)T$ for all $f \in T$. T is said to be a Gaussian R-algebra if $c(fg) = c(f)c(g)$, for all $f, g \in T$.

Example 5.6 ([18])**.** Let T be a content R-algebra such that R is a Prüfer domain. Since every nonzero finitely generated ideal of R is a cancellation ideal of R, the Dedekind-Mertens content formula causes T to be a Gaussian R-algebra.

In the following theorem we use the functions ϕ_R and ϕ_T that defined just prior to Theorem 2.25.

Theorem 5.7. *Let R be a Prüfer domain, T a content R-algebra and I an ideal of R. Then I is a ϕ_R-n-absorbing primary ideal of R if and only if IT is a ϕ_T-n-absorbing primary ideal of T.*

Proof. Assume that I is a ϕ_R-n-absorbing primary ideal of R. Let $f_1f_2\cdots f_{n+1} \in IT\backslash\phi_T(IT)$ for some $f_1, f_2, \ldots, f_{n+1} \in T$ such that $f_1f_2\cdots f_n \notin IT$. Then $c(f_1f_2\cdots f_{n+1}) \subseteq I$. Since R is a Prüfer domain and T is a content R-algebra, then T is a Gaussian R-algebra. Therefore

$$c(f_1f_2\cdots f_{n+1}) = c(f_1)c(f_2)\cdots c(f_{n+1}) \subseteq I.$$

If $c(f_1f_2\cdots f_{n+1}) \subseteq \phi_R(I) = \phi_T(IT) \cap R$, then

$$f_1f_2\cdots f_{n+1} \in c(f_1f_2\cdots f_{n+1})T \subseteq (\phi_T(IT) \cap R)T \subseteq \phi_T(IT),$$

which is a contradiction. Hence $c(f_1)c(f_2)\cdots c(f_{n+1}) \subseteq I$ and

$$c(f_1)c(f_2)\cdots c(f_{n+1}) \nsubseteq \phi_R(I).$$

Since R is a u-domain, I is a strongly ϕ_R-n-absorbing primary ideal of R, by Theorem 5.4, and this implies either $c(f_1)c(f_2)\cdots c(f_n) \subseteq I$ or

$$c(f_1)\cdots\widehat{c(f_i)}\cdots c(f_{n+1}) \subseteq \sqrt{I}$$

for some $1 \leq i \leq n$. In the first case we have $f_1f_2\cdots f_n \in c(f_1f_2\cdots f_n)T \subseteq IT$, which contradicts our hypothesis. In the second case we have $f_1\cdots\widehat{f_i}\cdots f_{n+1} \in (\sqrt{I})T \subseteq \sqrt{IT}$ for some $1 \leq i \leq n$. Consequently IT is a ϕ_T-n-absorbing primary ideal of T.

For the converse, note that since T is a content R-algebra, $IT \cap R = I$ for every ideal I of R. Now, apply Theorem 2.25. □

The algebra of all polynomials over an arbitrary ring with an arbitrary number of indeterminate is an example of content algebras.

Corollary 5.8. *Let R be a Prüfer domain and I be an ideal of R. Then I is a ϕ_R-n-absorbing primary ideal of R if and only if $I[X]$ is a $\phi_{R[X]}$-n-absorbing primary ideal of $R[X]$.*

As two special cases of Corollary 5.8, when $\phi_R = \phi_T = \emptyset$ and $\phi_R = \phi_T = 0$ we have the following result.

Corollary 5.9. *Let R be a Prüfer domain and I be an ideal of R. Then I is an n-absorbing primary ideal of R if and only if $I[X]$ is an n-absorbing primary ideal of $R[X]$.*

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