J. Korean Math. Soc. **53** (2016), No. 3, pp. 533–547
http://dx.doi.org/10.4134/JKMS.j150134
pISSN: 0304-9914 / eISSN: 2234-3008

OMORI-YAU MAXIMUM PRINCIPLE ON ALEXANDROV SPACES

Hanjin Lee

Abstract. We prove an Omori-Yau maximum principle on Alexandrov spaces which do not have Perelman singular points and satisfy the infinitesimal Bishop-Gromov condition.

1. Introduction

The purpose of this article is to extend the following maximum principle by Omori [10] and Yau [18] to Alexandrov spaces.

Theorem 1.1. *Let M be a complete Riemannian manifold of dimension ≥ 2 with Ricci curvature bounded below. For every C^2-smooth function $u : M \to \mathbb{R}$ that is bounded from above, there exists a sequence $\{p_k\}$ in M such that*

$$\lim_{k\to\infty} u(p_k) = \sup_M f, \quad \lim_{k\to\infty} \|\nabla u(p_k)\| = 0, \quad \limsup_{k\to\infty} \Delta u(p_k) \leq 0.$$

This theorem has various applications in differential geometry and geometric analysis (for example, see [16], [19]).

Alexandrov spaces arise as the Gromov-Hausdorff limits of n-dimensional, compact Riemannian manifolds with sectional curvature $\geq \kappa$ and diameter $\leq D$. Thus it is natural to ask whether the key geometric analysis theorems on Riemannian manifolds extend to Alexandrov spaces.

The notion of curvature lower bound for Alexandrov spaces generalizes the notion of lower bound of sectional curvature. Kuwae and Shioya introduced *the infinitesimal Bishop-Gromov condition BG* (κ, n), which generalizes the notion of lower bound of Ricci curvature (see [7]). On the other hand, Zhang and Zhu introduced a stronger notion called *the condition RC*, which is based on the second variation formula of arc length (see [20]).

Laplacian comparison of the distance function is one of the key ingredients in the proof of the Omori-Yau maximum principle. On Alexnadrov spaces, Laplacian comparison of the distance was proved by Petrunin [13], von Renesse

Received February 27, 2015; Revised May 22, 2015.
2010 *Mathematics Subject Classification.* Primary 53C21, 53C23.
Key words and phrases. Omori-Yau maximum principle, Alexandrov space, Bishop-Gromov condition.

[17], Kuwae and Shioya [6], [8] and Zhang and Zhu [20]. The first two results were obtained under the curvature lower bound condition and the last two results were obtained under the generalized Ricci lower bound condition. Our result is based on [6].

An Alexandrov space has a C^1-structure on its subset of regular points. It is extended to DC^1-structure on Perelman regular sets, which enables one to perform the second order differential calculus for DC-functions (see Definition 2.3). Following [5] and [6] we consider DC-Laplacian $\Delta^{DC}u$, which is the distributional Laplacian $\operatorname{div}(\nabla u)$, for DC-function u. Alexandrov spaces considered in this article allow only mild singularities, the so-called *Perelman regular points*. Since such Alexandrov spaces are DC-manifolds (see Definition 2.4), we can reduce Laplacian comparison of DC-functions to gradient comparison of them through the Gauss-Green formula.

We introduce the conditions of being *regularly exhausting* and of *volume regularity* to handle a behavior of concentric geodesic spheres and the Jacobian determinant of the exponential map (see Definitions 3.2 and 3.5). We impose the following additional assumptions to apply the Laplacian comparison theorem on the distance functions (see [6]): Let (g_{ij}) be a Riemannian metric on the subset of regular points of Alexandrov space X, which is compatible with the metric of X. Denote k-dimensional Hausdorff measure by h_k. *There exist a compact set $K \subset X$, a point $x_0 \in X$, and a positive number δ such that each geodesic sphere $S_r(x)$ satisfies $h_{n-1}(\mathrm{Cut}(x_0) \cap S_r(x)) = 0$ and $|D_k g_{ij}|(S_r(x)) = 0$ for $i, j, k = 1, \ldots, n$ and for $0 < r < \delta$ whenever a geodesic ball $B_\delta(x) \subset X \setminus K$* (See Definition 2.6 and Theorem 2.10 for the detail). Set $Lip(u)(p) = \lim_{r \to 0+} \sup_{x \neq y \in B_r(p)} \frac{|u(x)-u(y)|}{|x-y|}$. Our main theorem is the following Omori-Yau maximum principle:

Theorem 1.2. *Let X be an noncompact Alexandrov space of dimension $n \geq 2$, with empty Perelman singular set. Suppose that X satisfies the condition BG (κ, n). If, in addition, X is α-volume regular ($\alpha \geq 1$) and regularly exhausting, then for every DC^1-function $u : X \to \mathbb{R}$ bounded from above such that it attains its supremum nowhere in X and $Lip(u)(p) \to 0$ as p moves away from a fixed point and for any $\epsilon > 0$, there exist a point $p_\epsilon \in X$ and a positive number R_ϵ such that*

$$u(p_\epsilon) \geq \sup_X u - \epsilon, \qquad \frac{1}{Vol(B_{R_\epsilon}(p_\epsilon))} \int_{B_{R_\epsilon}(p_\epsilon)} \Delta^{DC} u \leq \epsilon.$$

Remark 1.3. In general, the gradient estimate of the original Omori-Yau maximum principle does not hold for bounded DC-functions even on complete smooth Riemannian manifolds with lower Ricci bound. The following example shows why we need the asymptotic vanishing condition for Lipschitz constants of bounded DC-functions in the main theorem. This condition replaces the gradient estimate for bounded C^2-smooth functions on Riemannian manifolds in the classic Omori-Yau maximum principle. Let $\phi(t) = 1 - e^{-t}$ and define a function $U : \mathbb{R}^+ \to \mathbb{R}^+$,

$$U(t) = \begin{cases} \phi(2n+1)(t-2n) & \text{if } 2n \le t < 2n+1 \\ -\phi(2n+1)(t-(2n+2)) & \text{if } 2n+1 \le t < 2n+2 \end{cases}$$

for $n = 0, 1, \ldots$. Then the function $u(x) = U(|x|)$ for $x \in \mathbb{R}^2$ is a DC-function bounded above such that it does not attain supremum anywhere. In particular, it holds that $\|\nabla u(x)\| = \phi(2n+1) \ge \phi(1) > 0$ if $2n < |x| < 2n+1$ or $2n+1 < |x| < 2n+2$ for each n.

Example 1.4. The following polyhedral surface S with infinite genus satisfies the conditions of Theorem 1.2. Consider a cube $Q \subset \mathbb{R}^3$ of which vertices are $(\pm 2, \pm 2, 0)$, $(\pm 2, \pm 2, 2)$. Consider a smoothing of Q as follows. Replace the eight corners of Q with flat surfaces. For instance, at the corner $(2, 2, 2)$, take a plane passing through $(2, 1.9, 2), (1.9, 2, 2), (2, 2, 1.9)$ and cut out the smaller part of Q by this plane. Then these three points become the new corners there. Denote the resulting smoothed cube by Q_s. Since each new corner point is locally circular conic with angle $> 3\pi/2$, every singular point of Q_s is Perelman regular (see Definition 2.5).

The polyhedral torus T is obtained from Q_s as follows. We construct an inner part of the torus as follows. Let $z = f(x)$ be a smooth convex decreasing function over $(1/2, 3/2)$ satisfying $f(1/2) = 1$, $f(3/2) = 0$ and $\frac{d^n x}{dz^n}|_{x=1/2} = 0$, $f^{(n)}(3/2) = 0$ up to $n = 3$. Consider the curve consisting of the graph of $z = f(x)$ and its reflection about $z = 1$. Consider the surface of revolution of this curve about z-axis. Cut out disks of radius $3/2$ from the top and the bottom of Q_s, whose center is respectively $(0, 0, 2)$ and $(0, 0, 0)$. Glue the surface just obtained along the two circles of the top and the bottom of the punctured Q_s, then we obtain the singular torus T. Then the convex part of T is a surface of curvature ≥ 0 and the concave part of T has Gaussian curvature bounded by a negative constant up to the boundaries.

The connector of two copies of T is constructed as follows. Let $s = g(t)$ be a smooth convex decreasing function over $(1/3, 2/3)$ satisfying $g(1/3) = 1/3$, $g(2/3) = 0$ and $\frac{d^n t}{ds^n}|_{t=1/3} = 0$, $g^{(n)}(2/3) = 0$ up to $n = 3$. The surface of revolution of $s = f(t)$ about s-axis is taken as a connector. The connector has the boundaries of the circles of radius $1/3$ and $2/3$. To glue two copies of T, make a circular hole on a side face of each copy of T and put the connector along the larger boundary. Then glue two connectors along the smaller boundaries of the connectors. Then a neighborhood of the connectors in $T \sharp T$ is C^3-smooth Riemannian surface with lower Gaussian curvature bound.

The surface S is obtained by gluing infinite number of copies of T in one direction. Since S is an Alexandrov surface of curvature bounded below by negative constant, it satisfies condition BG. Two technical conditions for Laplacian comparison Theorem 2.10 are satisfied for S. For the first condition, take a point x_0 on the copy of T which has only one connector. Then $Cut(x_0)$ consists of big circles along the inner torus parts of the copies of T, geodesics which are along the connectors and vertical to the boundaries of connectors, and singular points of S. Thus the intersection of $Cut(x_0)$ with each small geodesic circle has Hausdorff codimension 2. For the second condition, notice that the compatible Riemannian metric (g_{ij}) has an approximate limit at each regular point of S and the subset of singular points in each small geodesic circle in S has Hausdorff codimension 2 (See Definition 3.63 of [1]). By Lemma 3.76 of [1], $|D_k g_{ij}|$ vanishes on each small geodesic circle in S for $i,j,k = 1,2$. Two geometric regularity conditions are also satisfied for S. Since circular cone type singular points satisfy 1-volume regular condition, S is 1-volume regular (see Definition 3.2 and Example 3.4). It is clear that each circular cone type singular point is regularly exhausting, since the subspace of directions realizing length r geodesics from the conic singular point does not depend on r (see Definition 3.5 and (3.3)). Thus S is regularly exhausting.

As an application of Theorem 1.2, we prove the following Liouville type theorem:

Theorem 1.5. *Let X be a noncompact Alexandrov space satisfying the assumption of Theorem 1.2. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, increasing function with $F(0) = 0$. Suppose that u is a non-negative, bounded DC^1-solution of $\Delta^{DC} u = F(u)h_n$. If u attains its supremum nowhere in X, and $Lip(u)(p) \to 0$ as p moves away from a fixed point, then $u \equiv 0$.*

2. Preliminaries

In this section, we introduce the basic definitions and properties of Alexandrov spaces. We refer to [3], [4], [11].

A complete locally compact metric space $(X, |\cdot|)$ is called a *geodesic space* if for any two points $p, q \in X$ there exists a geodesic pq. A *triangle* $\triangle pqr$ in X means a set of three points $p, q, r \in X$, and of three geodesics pq, qr, rp. A *κ-plane* means a two dimensional complete simply-connected Riemannian manifold of curvature κ. We say that X satisfies *Alexandrov convexity* if for each $x \in X$ there exist a neighborhood U_x of x and a real number κ, the following hold: for any $\triangle pqr$ in U_x, there exists a triangle $\triangle \tilde{p}\tilde{q}\tilde{r}$ in κ-plane satisfying $|pq| = |\tilde{p}\tilde{q}|$, $|qr| = |\tilde{q}\tilde{r}|$, $|rp| = |\tilde{r}\tilde{p}|$ such that for any $y \in pq$, $z \in pr$, $\tilde{y} \in \tilde{p}\tilde{q}$, $\tilde{z} \in \tilde{p}\tilde{r}$ with $|py| = |\tilde{p}\tilde{y}|$, $|pz| = |\tilde{p}\tilde{z}|$, we have $|yz| \geq |\tilde{y}\tilde{z}|$.

Definition 2.1. A metric space X is called an *Alexandrov space* if it is a complete locally compact geodesic space such that it satisfies Alexandrov convexity and has a finite Hausdorff dimension.

For $p, q, r \in X$, the angle $\angle_\kappa pqr$ is defined to be the angle at the vertex $\tilde{q}$ of the comparison triangle $\triangle \tilde{p}\tilde{q}\tilde{r}$ in κ-plane. Let γ and σ be geodesics with the origin p. The angle between γ and σ is defined to be $\alpha(\gamma, \sigma) = \lim_{s,t\to 0+} \angle_\kappa \gamma(s) p\, \sigma(t)$. The metric space Σ'_p is the set of equivalence classes of geodesics with the origin p endowed with a metric in which the distance is the angle between the geodesics starting at p. The metric completion of Σ'_p is called the *space of direction at* p, which is denoted by Σ_p.

Definition 2.2. The tangent cone T_p is the Euclidean cone over the space of direction Σ_p. Its element is denoted by $v = t\gamma$ where γ is a geodesic direction and $t \in [0, \infty]$. Let o_p denote the vertex of T_p. For $v = t\gamma$, and $w = s\sigma \in T_p$, the metric of the cone is defined by $|vw| = t^2 + s^2 - 2ts \cos\alpha(\gamma, \sigma)$.

A point $p \in X$ is said to be *regular* if T_p is isometric to Euclidean space and *singular* otherwise. Denote by $Reg(X)$ (resp. $Sing(X)$) the set of regular (resp. singular) points of X. The Hausdorff dimension of $Sing(X)$ is $\leq n - 1$.

For any $p \in X$, a point $x \in X$ is said to be a *cut point of* p if any minimal segment py does not contain x in its interior. Denote by C_p the set of all cut points of p. Then $h_n(C_p) = 0$ for each $p \in X$. Define a map $\log_p : X \setminus C_p \to T_p$ by $\log_p(x) := |px| v_{px}$ for $x \in X \setminus C_p$, where $v_{px} \in \Sigma_p$ is the direction of geodesic px. Set $D_p = \log_p(X \setminus C_p)$. Consider the inverse map $\exp_p : D_p \to X$ of $\log_p$, which is called the *exponential map at* p. Then $\exp_p$ is Lipschitz on $D_p \cap B_r(o_p)$ for any $r > 0$ by Alexandrov convexity.

A C^1-structure can be established on $Reg(X)$ in the following sense: there is an atlas (U_α, ϕ_α) on $Reg(X)$, such that if $U_\alpha \cap U_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is C^1 on $\phi_\alpha(U_\alpha \cap U_\beta \cap Reg(X))$.

There exists a natural C^0-Riemannian metric g on $Reg(X)$ such that its induced distance is compatible with the metric of X and the volume form induced from g also coincides with n-dimensional Hausdorff measure.

Definition 2.3. A Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is called DC if it is locally representable as the difference of two semi-concave functions.

Let $U \subset \mathbb{R}^n$ be any open set. A Lipschitz map $F = (f_1, \ldots, f_m) : U \to \mathbb{R}^m$ is called DC if each f_i is DC. If for an open set $V \subset \mathbb{R}^m$, $F : U \to V$ and $G : V \to \mathbb{R}^m$ are DC-maps, then $G \circ F$ is DC.

Definition 2.4. Let Y be a paracompact Hausdorff topological space with dimension n. A family $\{(U_\alpha, \phi_\alpha)\}$ of local charts of Y is called DC-atlas on Y if $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is DC-map whenever $U_\alpha \cap U_\beta \neq \emptyset$. If a maximal n-dimensional DC atlas on Y is determined, Y is called DC-manifold. Additionally, if each DC-atlas is C^1-atlas on $Y \setminus Sing(Y)$, then it is called DC^1-chart.

Definition 2.5 ([12])**.** Let X be an Alexandrov space. A point $p \in X$ is called *Perelman regular* if Σ_p contains $n + 1$ directions making obtuse angles with each other.

The set of all Perelamn regular points is an open and convex subset of X, and includes $Reg(X)$. Furthermore, it is DC^1-manifold.

Definition 2.6. An L^1-function $f : U(\subset \mathbb{R}^n) \to \mathbb{R}$ is called BV (bounded variation)-function if its distributional derivatives $D_i f$, $i = 1, \ldots, n$, are all finite Radon measures. The total variation measure of $D_i f$ is denoted by $|D_i f|$ and defined as follows: for each measurable set E,

$$|D_i f|(E) = \sup\{\textstyle\sum_{h=1}^{\infty} |D_i(E_h)| : E_h \text{ measurable and pairwise disjoint,} \\ E = \cup_{h=1}^{\infty} E_h\}.$$

It is known that if f is DC-function on $U \subset \mathbb{R}^n$, then $D_i f$, $i = 1, \ldots, n$, are a.e. determined as BV-functions. Thus $D_i D_j f$, $i, j = 1, \ldots, n$, are determined as signed Radon measures. It is known that canonical Riemannian metric $g = (g_{ij})$ of the Alexandrov space X is BV. Take local coordinates $(x_1, \ldots, x_n)$ on the Alexandrov space X. A vector field $Z = \sum_i Z_i \frac{\partial}{\partial x_i}$ is said to be BV if each Z_i is BV-function.

Definition 2.7. For a locally BV-vector field Z on $\Omega \subset X$, the distributional divergence of Z is defined by

$$\mathrm{div} Z := \sum_i \frac{\partial}{\partial x_i}(\sqrt{\det(g_{ij})} Z^i)\, dx_1 \wedge \cdots \wedge dx_n$$

where $(x_1, \ldots, x_n)$ are local coordinates on Ω and $Z = \sum_i Z^i \frac{\partial}{\partial x_i}$. Let $\nabla u = \sum_i g^{ij} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_i}$ be a gradient vector field of DC-function u. Then the DC-Laplacian $\Delta^{DC} u$ is defined by

$$\Delta^{DC} u := \mathrm{div} \nabla u.$$

For any DC-local chart (U, φ) of X, a function $f : U \to \mathbb{R}$ is DC^1 if and only if $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is DC and C^1 on $\varphi(U \cap Reg(X))$. A subset N of DC^1-manifold X is called a *DC^1-hypersurface of X* if for each $x \in N$ there exists a DC^1-local chart (U, φ) around x, an open set $W \subset \mathbb{R}^{n-1}$, and a DC^1-function h on W, such that $\varphi(N \cap U)$ is a graph of h.

Kuwae, Machigashira, and Shioya [5] showed that Gauss-Green formula holds for domains whose boundaries are DC-hypersurfaces:

Proposition 2.8 (Gauss-Green formula)**.** *Let $D \subset X$ be an orientable compact subset bounded by a DC^1-hypersurface. Then, for any DC^1-function u on $\bar{D}$,*

$$\int_D \Delta^{DC} u = \int_{\partial D} \langle \nabla u, \nu \rangle \omega_{\partial D}$$

where ν is the outward normal vector on ∂D, $\langle \cdot, \cdot \rangle$ denotes the inner product induced from the metric g, and $\omega_{\partial D}$ is a volume form induced from the canonical Riemannian metric g that coincides with h_{n-1}.

For $\kappa \in \mathbb{R}$, set

$$s_\kappa(r) = \begin{cases} \sin(\sqrt{\kappa})/\sqrt{\kappa} & \kappa > 0 \\ r & \kappa = 0 \\ \sinh(\sqrt{|\kappa|})/\sqrt{|\kappa|} & \kappa < 0. \end{cases}$$

Define a map $\Phi_{p,t} : W_{p,t} \to X$ as follows: $x \in W_{p,t}$ if and only if there exists $y \in X$ such that $x \in py$ and $|px| : |py| = t : 1$. For each $x \in W_{p,t}$, such a point y is unique and we set $\Phi_{p,t}(x) := y$.

Definition 2.9 (Infinitesimal Bishop-Gromov condition, [6]). An Alexandrov space X is said to satisfy condition BG (κ, n) if for each $p \in X$ the following holds:

$$d((\Phi_{p,t})_* \mathcal{H}^n)(x) \geq \frac{t s_\kappa(t|px|)^{n-1}}{s_\kappa(|px|)^{n-1}} d\mathcal{H}^n(x)$$

for any $x \in X$ and $t \in (0,1]$ such that $|px| < \pi/\sqrt{\kappa}$ if $\kappa > 0$, where $(\Phi_{p,t})_* \mathcal{H}^n$ is the push-forward of Hausdorff measure by $\Phi_{p,t}$.

For an n-dimensional complete Riemannian manifold, BG (κ, n) holds if and only if the Ricci curvature satisfies Ric $\geq$ (n − 1)κ (see [9]). Any Alexandrov space of curvature $\geq \kappa$ satisfies BG (κ, n) (see [7]).

The following Laplacian comparison theorem for the distance functions holds under condition BG (κ, n). Let x_0 be a fixed point of X. Set $r_0(x) = |x_0 x|$ and $\cot_\kappa(r) = s'_\kappa(r)/s_\kappa(r)$. For a BV-Riemannian metric (g_{ij}) on $Reg(X)$, recall that $D_k g_{ij}$ means distributional derivative and $|D_k g_{ij}|$ means the total variation measure of $D_k g_{ij}$.

Theorem 2.10 (Kuwae-Shioya, [6]). *Let X be an Alexandrov space of dimension $n \geq 2$. If X satisfies condition BG(κ, n), then one has*

$$\int_E \Delta^{DC} r_0 \leq (n-1) \sup_{x \in E} \cot_\kappa \circ r_0(x) h_n(E)$$

for any compact region $E \subset X^* \setminus \{x_0\}$ with Lipschitz boundary satisfying $h_{n-1}(\mathrm{Cut}(X_0) \cap \partial E) = 0$, $|D_k g_{ij}|(\partial E) = 0$ for all $i, j, k = 1, \dots, n$.

3. Proof of main theorem

We use the following elementary fact:

Lemma 3.1. *Let φ be a Lipschitz continuous function on $[0, a)$. Suppose that $\varphi(0) = 0$, $\varphi(t) \geq 0$ on $[0, a)$. Then there exists a sequence (t_k) such that $t_k \to 0$ and $\varphi'(t_k) \geq 0$.*

Proof. Suppose otherwise. Then there exists $r \in (0, a)$ such that $u' < 0$ on $(0, r)$ a.e. For each $t \in (0, r)$,

$$\varphi(t) - \varphi(0) = \int_0^t \varphi'(\tau) d\tau < 0$$

which implies that $\varphi(t) < 0$. It is a contradiction. □

Let $S_r(p)$ denote the geodesic sphere in X, centered at p with radius r. For ease of notation, set $A_r = h_{n-1}(S_r(p))$ for the fixed point p. Define the spherical mean of f at p as follows:

$$\bar{f}(r) := \frac{1}{A_r} \int_{S_r(p)} f \, dh_{n-1}. \tag{3.1}$$

Since an exponential map at p is Lipschitz continuous on $D_p \cap B_r(o_p) \subset T_p$ for each $r > 0$,

$$\int_{B_t(p)} f dh_n = \int_{\log_p(B_t(p) \setminus C_p)} f \circ \exp_p(z) \left[\frac{d((\exp_p)^* h_n)}{dh_n}(z) \right] dh_n(z). \tag{3.2}$$

Define, for $0 < r < t$

$$\Omega_p(r) = \{\theta \in \Sigma_p : \text{there exists unique } x \in X \text{ such that } px \sim \theta \text{ and } |px| = r\}.$$

Then

$$\log_p(B_t(p) \setminus C_p) = \log_p(\cup_{0 \le r < t} S_r(p) \setminus C_p) = (\cup_{0<r<t} \{r\} \times \Omega_p(r)) \cup \{o_p\} \tag{3.3}$$

and (3.2) is expressed as

$$\int_0^t \int_{\Omega_p(r)} f \circ \exp_p(r\theta) \left[\frac{d((\exp_p)^* h_n)}{dh_n}(r\theta) \right] r^{n-1} dr dh_{n-1}(\theta).$$

Define

$$J(r, \theta) = \left[\frac{d((\exp_p)^* h_n)}{dh_n}(r\theta) \right] r^{n-1}. \tag{3.4}$$

Then by the coarea formula (see [2], Theorem 9.4)

$$\int_{S_r(p)} f(x) dh_{n-1} = \int_{\Omega_p(r)} f \circ \exp_p(r\theta) J(r, \theta) dh_{n-1}(\theta).$$

For ease of notation, let $\tilde{f}_\theta(r) = f \circ \exp_p(r\theta)$.

Definition 3.2. An Alexandrov space X is said to be *α-volume regular at* $p \in X$ if there exist positive numbers $r > 0$ and $\alpha \ge 1$, and a Lipschitz continuous function E such that

$$\frac{d((\exp_p)^* h_n)}{dh_n}(z) = 1 + E(z)|z|^\alpha \quad \text{for } z \in \log_p(B_r(p) \setminus C_p). \tag{3.5}$$

An Alexandrov space X is said to be *α-volume regular* if it is α-volume regular at every point of X.

Remark 3.3. If X is a smooth Riemannian manifold, then

$$E(z)|z|^2 = -\frac{1}{6} Ric(z, z) + o(|z|^2).$$

Thus X satisfies 2-volume regular. Our condition is similar to that of *volume regularity* used in [17]. In some sense, our condition is stronger than that of von Renesse.

Example 3.4. The elliptic cone $EC(S)$ over a circle S of diameter $\frac{3}{4}\pi$ is the quotient space

$$EC(S) = S \times [0, \infty)/\sim,$$

where $x_1 = (\theta_1, r_1) \sim x_2 = (\theta_2, r_2) \Leftrightarrow r_1 = r_2 = 0$ with the metric $\cosh|x_1x_2| = \cosh r_1 \cosh r_2 - \sinh r_1 \sinh r_2 \cos|\theta_1\theta_2|$. The elliptic cone is 1-volume regular at the vertex point O : one has for $r\theta \in T_O$, $r \ll 1$ and $\delta \ll 1$

$$\begin{aligned}\frac{d(\exp_O)^*h_2}{dh_2}(r\theta) &\approx \frac{|\exp_O(r\theta), \exp_O(r(\theta+\delta))|}{|r\theta, r(\theta+\delta)|} \\ &= \frac{\cosh^{-1}(\cosh^2 r - \sinh^2 r \cos\delta)}{r\sqrt{2(1-\cos\delta)}} \\ &\approx \left(\frac{\sinh r}{r}\right)(\sqrt{\eta} + \sqrt{1+\eta})\end{aligned}$$

where $\eta = \sin^2(\frac{\delta}{2})\sinh^2 r$. It suffices to show that $\frac{1}{r}\left(\left(\frac{\sinh r}{r}\right)(\sqrt{\eta} + \sqrt{1+\eta}) - 1\right)$ is Lipschitz in r. Set $F_1 = \frac{\sinh r - r}{r^2}$ and $F_2 = \sqrt{\eta} + \sqrt{1+\eta} - 1$. Then the above equation is $\frac{(1+rF_1)(1+F_2)-1}{r}$. It is Lipschitz in r since F_1, F_2 and $\frac{F_2}{r}$ are Lipschitz in r.

Definition 3.5. A singular point p in an Alexandrov space X is called *regularly exhausting* if $h_{n-1}(\Omega_p(r))$ is Lipschitz continuous in r. An Alexandrov space X is said to be *regularly exhausting* if every singular point of X is regularly exhausting and $\sup_{p\in X} Lip(h_{n-1}(\Omega_p(r))|_{r=0}$ is finite where $Lip(h_{n-1}(\Omega_p(r))|_{r=0} = \lim_{s\to 0+} Lip(h_{n-1}(\Omega_p(r))|_{\{0<r<s\}}$.

Example 3.6. (1) Consider a sector $\Sigma_F := \{(x, y) \in \mathbb{R}^2 : |y| < F(x)\}$ where F is convex, even, and nonnegative, and $F(0) = 0$. The metric space X_F is obtained by gluing $B_1(0)\setminus\Sigma_F$ along the graph of $\pm F$. Then, X_F is an Alexandrov space. Every points of the graph of $\pm F$ in X_F are singular. Consider the case in which $F(x) = 1 - \sqrt{1-x^2}$. Then $\Omega_0(r) = (\arcsin(r/2), \pi - \arcsin(r/2)) \cup (-\pi + \arcsin(r/2), -\arcsin(r/2))$ and $h_1(\Omega_0)(r) = 2(\pi - 2\arcsin(r/2))$. Clearly, $h_1(\Omega_0(r))$ is Lipschitz continuous as $r \to 0+$. Thus, the origin is regularly exhausting. Likewise, all other singular points are also regularly exhausting.

(2) Consider the case $F(x) = |x|^\gamma$, $1 < \gamma < 2$. A point $(x, F(x))$, whose distance to the origin is r, satisfies $x^2 + F(x)^2 = x^2(1 + x^{2\gamma-2}) = r^2$. Thus, if x is very small, then $x \sim r$ and $F(x) \sim F(r)$. Then $\Omega_0(r) = (\arcsin(r^{\gamma-1}), \pi - \arcsin(r^{\gamma-1})) \cup (-\pi + \arcsin(r^{\gamma-1}), -\arcsin(r^{\gamma-1}))$, and $h_1(\Omega_0(r)) = 2(\pi - \arcsin(r^{\gamma-1}))$. In this case, $h_1(\Omega_0(r))$ is not Lipschitz continuous as $r \to 0+$. The origin is not regularly exhausting.

Assuming the two conditions given in Definitions 3.2 and 3.5 to be satisfied, one can obtain the following derivative comparison of spherical means:

Lemma 3.7. *Suppose that an Alxandrov space X is α-volume regular ($\alpha \geq 1$) and is regularly exhausting. Let f and g be Lipschitz continuous functions on X. Suppose that $f(p) = g(p)$ and $f \geq g$ on a neighborhood of p. Then there* *exists a sequence* (r_k) *converging to* 0 *such that the spherical means* $\bar{f}, \bar{g}$ *satisfy* $\bar{f}'(r_k) \geq \bar{g}'(r_k)$.

Proof. Clearly $\bar{f}(0) = \bar{g}(0)$ and there exists $a > 0$ such that $\bar{f}(r) \geq \bar{g}(r)$ for $r \in [0, a)$. It suffices from Lemma 3.1 to show that $\bar{f}$ (likewise $\bar{g}$) is Lipschitz continuous. From the α-volume regularity condition with $\alpha \geq 1$, there exist $0 < a' < a$ and a Lipschitz continuous function E on T_p such that $J(r, \theta) = r^{n-1}(1 + E(r\theta)r^{\alpha})$ for $0 < r < a'$. Then one has

$$\begin{aligned}
\bar{f}(r) &= \frac{\int_{\Omega_p(r)} \tilde{f}_\theta(r) J(r,\theta) dh_{n-1}(\theta)}{\int_{\Omega_p(r)} J(r,\theta) dh_{n-1}(\theta)} \\
&= \frac{r^{n-1} \int_{\Omega_p(r)} \tilde{f}_\theta(r) dh_{n-1}(\theta) + r^{n-1+\alpha} \int_{\Omega_p(r)} \tilde{f}_\theta(r) E(r\theta) dh_{n-1}(\theta)}{r^{n-1} h_{n-1}(\Omega_p(r)) + r^{n-1+\alpha} \int_{\Omega_p(r)} E(r\theta) dh_{n-1}(\theta)} \\
&= \frac{\int_{\Omega_p(r)} \tilde{f}_\theta(r) dh_{n-1}(\theta) + r^{\alpha} \int_{\Omega_p(r)} \tilde{f}_\theta(r) E(r\theta) dh_{n-1}(\theta)}{h_{n-1}(\Omega_p(r)) + r^{\alpha} \int_{\Omega_p(r)} E(r\theta) dh_{n-1}(\theta)}.
\end{aligned}$$

Since X is regularly exhausting, $h_{n-1}(\Omega_p(r))$ is Lipschitz continuous in r. For $r > s$, one clearly has $\Omega_r(p) \subset \Omega_s(p)$ and

$$h_{n-1}(\Omega_s(p) \setminus \Omega_r(p)) = h_{n-1}(\Omega_s) - h_{n-1}(\Omega_r) \leq C_2|r - s|.$$

Thus, for $0 < s < r < a'$

$$\begin{aligned}
&\left| \int_{\Omega_r} \tilde{f}_\theta(r) dh_{n-1}(\theta) - \int_{\Omega_s} \tilde{f}_\theta(s) dh_{n-1}(\theta) \right| \\
&\leq \int_{\Omega_r} |\tilde{f}_\theta(r) - \tilde{f}_\theta(s)| dh_{n-1}(\theta) + \left| \int_{\Omega_s \setminus \Omega_r} \tilde{f}_\theta(s) dh_{n-1}(\theta) \right| \\
&\leq C_1 |\Sigma_p| |r - s| + C_2 \sup_{B_a(p)} |f| |r - s|
\end{aligned}$$

which implies that $\int_{\Omega_r(p)} \tilde{f}_\theta(r) dh_{n-1}(\theta)$ is Lipschitz continuous.

Since $E(r\theta)$ is Lipschitz continuous in r, $\int_{\Omega_p(r)} \tilde{f}_\theta(r) E(r\theta) dh_{n-1}(\theta)$ and $\int_{\Omega_p(r)} E(r\theta) dh_{n-1}(\theta)$ are likewise Lipschitz continuous. Thus, the denominator and numerator of $\bar{f}$ are Lipschitz continuous. Since $h_{n-1}(\Omega_p(r)) \to h_{n-1}(\Sigma_p)$ as $r \to 0+$, the denominator of $\bar{f}$ is bounded below by positive constant as $r \to 0+$. It implies that $\bar{f}$ are Lipschitz continuous near 0. □

We now give the key technical ingredient for the main theorem.

Proposition 3.8. *Let* X *be an Alexandrov space satisfying the assumptions of Lemma 3.7. Additionally assume that* X *has no Perelman singular points. If* f *and* g *are* DC^1*-functions such that* $f(p) = g(p)$ *and* $f \geq g$ *on* $B_a(p)$*, then there exists a sequence of positive numbers* (R_i) *converging to* 0 *such that*

$$\int_{B_{R_i}(p)} \Delta^{DC} f \geq \int_{B_{R_i}(p)} \Delta^{DC} g + \tau_p(R_i) R_i^n,$$

where a function $\tau_p(r)$ satisfies

$$|\tau_p(r)| \leq 16(Lip(f)(p) + Lip(g)(p))Lip(h_{n-1}(\Omega_p(r))|_{r=0} + O(r^{\alpha})$$

as $r \to 0+$.

Proof. Since $r_p(x) = |px|$ is a DC^1-function in x, $r_p \circ \varphi^{-1}$ is a DC^1-function on a open subset $\varphi(U)$ of $\mathbb{R}^n$ for local DC^1-chart (φ, U) around p such that $B_a(p) \subset U$ (one can take smaller a if it is needed). Since the level set $\{r_p \circ \varphi^{-1} = r\}$ has no critical point, as a consequence of Proposition 2.13 of [15], it is a DC^1-hypersurface in $\mathbb{R}^n$. Thus $S_r(p)$ is a DC^1-hypersurface in X. Applying the Gauss-Green formula (Proposition 2.8) one has

$$\begin{aligned}\int_{B_r(p)} \Delta^{DC}(f-g) &= \int_{S_r(p)} \langle \nabla(f-g), \nu\rangle dh_{n-1} \\ &= \int_{\Omega_p(r)} \left(\frac{\partial \tilde{f}_\theta}{\partial r} - \frac{\partial \tilde{g}_\theta}{\partial r}\right) J(r,\theta) dh_{n-1}(\theta),\end{aligned} \tag{3.6}$$

where ν is the outward normal vector on $S_r(p)$. Since

$$\begin{aligned}&\frac{d}{dr}\int_{\Omega_p(r)} \tilde{f}_\theta(r) J(r,\theta) dh_{n-1}(\theta) \\ =& \int_{\Omega_p(r)} \frac{\partial}{\partial r}(\tilde{f}_\theta(r) J(r,\theta)) dh_{n-1}(\theta) \\ &- \lim_{h\to 0+} \frac{1}{h}\int_{\Omega_p(r-h)\setminus\Omega_p(r)} \tilde{f}_\theta(r-h) J(r-h,\theta) dh_{n-1}(\theta)\end{aligned}$$

for f (likewise g) one has for a.e. r,

$$\begin{aligned}&\int_{\Omega_p(r)} \frac{\partial \tilde{f}_\theta}{\partial r} J(r,\theta) dh_{n-1}(\theta) \\ =& \bar{f}'(r) A_r + \bar{f}(r) A_r' - \int_{\Omega_p(r)} \tilde{f}_\theta \frac{\partial J}{\partial r} dh_{n-1}(\theta) \\ &+ \lim_{h\to 0+} \frac{1}{h}\int_{\Omega_p(r-h)\setminus\Omega_p(r)} \tilde{f}_\theta(r-h) J(r-h,\theta) dh_{n-1}(\theta).\end{aligned}$$

Thus, from (3.6),

$$\begin{aligned}&\int_{B_r(p)} \Delta^{DC} f - \int_{B_r(p)} \Delta^{DC} g \\ =& (\bar{f}'(r) - \bar{g}'(r)) A_r + \int_{\Omega_p(r)} (\tilde{f}_\theta - \tilde{g}_\theta)\, J \left(\frac{A_r'}{A_r} - \frac{\partial J/\partial r}{J}\right) dh_{n-1}(\theta) \\ &+ \lim_{h\to 0+} \frac{1}{h}\int_{\Omega_p(r-h)\setminus\Omega_p(r)} (\tilde{f}_\theta(r-h) - \tilde{g}_\theta(r-h)) J(r-h,\theta) dh_{n-1}(\theta).\end{aligned}$$

By Lemma 3.7, there exists a sequence $R_i \to 0$ such that

$$
\begin{aligned}
(3.7) \qquad & \int_{B_{R_i}(p)} \Delta^{DC} f - \int_{B_{R_i}(p)} \Delta^{DC} g \\
& \geq \int_{\Omega_p(r)} (\tilde{f}_\theta - \tilde{g}_\theta)\, J \left(\frac{A_r'}{A_r} - \frac{\partial J/\partial r}{J} \right) dh_{n-1}|_{r=R_i} \\
& \quad + \lim_{h \to 0+} \frac{1}{h} \int_{\Omega_p(r-h)\setminus\Omega_p(r)} (\tilde{f}_\theta - \tilde{g}_\theta)(r-h)\, J(r-h,\theta) dh_{n-1}|_{r=R_i}.
\end{aligned}
$$

Now, our aim is to estimate the right hand side of the inequality (3.7). First notice that

$$
A_r' = \int_{\Omega_p(r)} \frac{\partial J}{\partial r}(r,\theta) dh_{n-1}(\theta) - \lim_{h \to 0+} \frac{1}{h} \int_{\Omega_p(r-h)\setminus\Omega_p(r)} J(r-h,\theta) dh_{n-1}(\theta).
$$

On the other hand, one has

$$
\frac{1}{J}\frac{\partial J}{\partial r} = \frac{(n-1)r^{n-2} + (n-1+\alpha) r^{n-2+\alpha} E + r^{n-1+\alpha} \frac{\partial E}{\partial r}}{r^{n-1} + r^{n-1+\alpha} E}
$$

for almost everywhere in r and θ, and

$$
\begin{aligned}
& \frac{\int_{\Omega_p(r)} \frac{\partial J}{\partial r}}{\int_{\Omega_p(r)} J} \\
= \; & \frac{(n-1) r^{n-2} h_{n-1}(\Omega_p(r)) + (n-1+\alpha) r^{n-2+\alpha} \int_{\Omega_p(r)} E + r^{n-1+\alpha} \int_{\Omega_p(r)} \frac{\partial E}{\partial r}}{r^{n-1} h_{n-1}(\Omega_p(r)) + r^{n-1+\alpha} \int_{\Omega_p(r)} E}
\end{aligned}
$$

for almost everywhere in r. Thus, using Lipschitz continuity of E in r, we see

$$
\begin{aligned}
& \frac{\int_{\Omega_p(r)} \frac{\partial J}{\partial r}}{\int_{\Omega_p(r)} J} - \frac{\frac{\partial J}{\partial r}}{J} \\
& = \alpha r^{\alpha-1} \left(\frac{1}{h_{n-1}(\Omega_p(r))} \int_{\Omega_p(r)} E(r\theta) dh_{n-1}(\theta) - E(r\theta) \right) + O(r^\alpha) \\
& = O(r^\alpha)
\end{aligned}
$$

as $r \to 0+$. Since $f-g$ is Lipschitz continuous and $f(p) = g(p)$, one has $|\tilde{f}_\theta - \tilde{g}_\theta| \leq Cr$ for $r\theta \in \cup_{0<r<a} \{r\} \times \Omega_p(r)$ as $r \to 0+$. Furthermore, by the condition on J, one has $(\tilde{f}_\theta - \tilde{g}_\theta) J = O(r^n)$. It implies that

$$
\int_{\Omega_p(r)} (\tilde{f}_\theta - \tilde{g}_\theta)\, J \left(\frac{\int_{\Omega_p(r)} \frac{\partial J}{\partial r}}{A_r} - \frac{\frac{\partial J}{\partial r}}{J} \right) dh_{n-1} = O(r^{n+\alpha}).
$$

So, for the estimate of the righthand side of (3.7), it remains to control

$$
(3.8) \quad - \left(\frac{\lim_{h\to 0+} \frac{1}{h} \int_{\Omega_p(r-h)\setminus\Omega_p(r)} J(r-h,\theta) dh_{n-1}}{\int_{\Omega_p(r)} J dh_{n-1}} \right) \int_{\Omega_p(r)} (\tilde{f}_\theta - \tilde{g}_\theta)(r)\, J dh_{n-1}
$$

$$+\lim_{h\to 0+}\frac{1}{h}\int_{\Omega_p(r-h)\setminus\Omega_p(r)}(\tilde{f}_\theta-\tilde{g}_\theta)(r-h)\,J(r-h,\theta)dh_{n-1}(\theta).$$

Here, from (3.5) we see

$$\begin{aligned}&\frac{\lim_{h\to 0+}\frac{1}{h}\int_{\Omega_p(r-h)\setminus\Omega_p(r)}J(r-h,\theta)dh_{n-1}(\theta)}{\int_{\Omega_p(r)}Jdh_{n-1}}\\ &=\frac{\frac{\partial}{\partial r}\int_{\Omega_p(r)}Jdh_{n-1}-\int_{\Omega_p(r)}\frac{\partial J}{\partial r}dh_{n-1}}{\int_{\Omega_p(r)}Jdh_{n-1}}\\ &=\frac{\frac{\partial}{\partial r}h_{n-1}(\Omega_p(r))}{h_{n-1}(\Omega_p(r))}+O(r^\alpha).\end{aligned}$$

Thus, the absolute value of (3.8) is bounded by

$$\begin{aligned}&4\sup_{x\in B_p(r)}\frac{|f(x)-g(x)|}{|x-p|}\left|\frac{\partial}{\partial r}h_{n-1}(\Omega_p(r))\right|r^n+O(r^{n+\alpha})\\ &\le 16(Lip(f)(p)+Lip(g)(p))\,Lip(h_{n-1}(\Omega_p(r))|_{r=0}\,r^n+O(r^{n+\alpha})\end{aligned}$$

as $r\to 0+$. It completes the proof. □

Now we give a proof of the main theorem.

Proof of Theorem 1.2. Let x_0 be a point in X satisfying the technical assumption for main theorem. Denote the distance function from x_0 by r_0. Take a positive number b such that $B_b(x_0)$ includes the compact set K in the technical assumption for main theorem. Define a function

$$\tilde{r}_0(x)=\begin{cases}b & \text{if } r_0(x)<b\\ r_0(x) & \text{if } r_0(x)\ge b.\end{cases}$$

Consider $\frac{1}{m}\tilde{r}_0-u$. Then there exists $p_m\in X$ such that $\frac{1}{m}\tilde{r}_0-u$ has a local minimum at p_m. Take $B_{R_m}(p_m)$ where $\frac{1}{m}\tilde{r}_0\ge u$. Set $u_m=u+\frac{1}{m}\tilde{r}_0(p_m)-u(p_m)$.

First, one can show that $\lim_{m\to\infty}u(p_m)=\sup_X u$ as follows. Let $\epsilon>0$. Then there exists p_ϵ such that $u(p_\epsilon)>\sup_X u-\epsilon/2$. Choose m sufficiently large such that $2\tilde{r}_0(p_\epsilon)<m\epsilon$. Then

$$u(p_m)\ge u(p_m)-\frac{1}{m}\tilde{r}_0(p_m)\ge u(p_\epsilon)-\frac{1}{m}\tilde{r}_0(p_\epsilon)\ge\sup_X u-\epsilon.$$

Next, we apply Proposition 3.8 to $\frac{1}{m}\tilde{r}_0$ and u_m on $B_{R_m}(p_m)$. Then, there exists a sequence $R_{m,i}(<R_m)$ which goes to 0 as $i\to\infty$ such that

$$\frac{1}{m}\int_{B_{R_{m,i}}(p_m)}\Delta^{DC}\tilde{r}_0\ge\int_{B_{R_{m,i}}(p_m)}\Delta^{DC}u+\tau_{p_m}(R_{m,i})R^n_{m,i}.$$

Since u does not attain its supremum in X, $r_0(p_m) \to +\infty$ as $m \to +\infty$. Thus for given $\epsilon > 0$, there exists m such that $(1/m + Lip(u)(p_m)) < \epsilon$. Once m is chosen, take $R_{m,i} < R_m$ such that

$$\begin{aligned}|\tau_{p_m}(R_{m,i})| &\leq 16Lip(h_{n-1}(\Omega_{p_m}(r))|_{r=0}(1/m + Lip(u)(p_m)) \\ &\leq 16C\epsilon,\end{aligned}$$

where $C = \sup_{p\in X} Lip(h_{n-1}(\Omega_p(r))|_{r=0} < +\infty$ by regular exhausting condition. Since $r_0(p_m) \geq b$ for large m, Theorem 2.10 implies that

$$\begin{aligned}\int_{B_{R_{m,i}}(p_m)} \Delta^{DC} r_0 &\leq (n-1) \sup_{x\in B_{R_{m,i}}(p_m)} \frac{s'_\kappa(r_0(x))}{s_\kappa(r_0(x))} h_n(B_{R_{m,i}}(p_m)) \\ &\leq (n-1)\, C_\kappa\, h_n(B_{R_{m,i}}(p_m)),\end{aligned}$$

where

$$C_\kappa = \begin{cases} 1/b & \text{if } \kappa = 0 \\ \sqrt{|\kappa|}\coth(\sqrt{|\kappa|}b) & \text{if } \kappa < 0 \end{cases}$$

(since X is noncompact, the case $\kappa > 0$ is excluded). Then one has

$$\frac{1}{h_n(B_{R_{m,i}}(p_m))} \int_{B_{R_{m,i}}(p_m)} \Delta^{DC} u \leq \epsilon \left((n-1)C_\kappa + 32C \frac{R^n_{m,i}}{V^n_\kappa(R_{m,i})} \right),$$

where $V^n_\kappa(R)$ is the volume of geodesic ball of radius R in the n-dimensional space form of curvature κ and $R^n_{m,i}/V^n_\kappa(R_{m,i})$ are bounded by uniform constant as $R_{m,i} \to 0+$. □

Proof of Theorem 1.5. Given $\epsilon > 0$, there exist $p_\epsilon \in X$ and $R_\epsilon > 0$ such that

$$\int_{B_{R_\epsilon}(p_\epsilon)} F(u) dh_n = \int_{B_{R_\epsilon}(p_\epsilon)} \Delta^{DC} u < \epsilon h_n(B_{R_\epsilon}(p_\epsilon)).$$

Since u is continuous, the condition $u(p_\epsilon) > \sup_X u - \epsilon$ can be replaced with the condition $\inf_{x\in B_{R_\epsilon}(p_\epsilon)} u(x) \geq \sup_X u - \epsilon$ by taking small R_ϵ. Then,

$$h_n(B_{R_\epsilon}(p_\epsilon)) F(\sup_X u - \epsilon) \leq \int_{B_{R_\epsilon}(p_\epsilon)} F(u) dh_n$$

which implies

$$F(\sup_X u - \epsilon) \leq \epsilon,$$

equivalently

$$\sup_X u \leq \epsilon + F^{-1}(\epsilon).$$

It implies that $u \equiv 0$. □

Acknowledgements. The author would like to express gratitude to Professor Jaigyoung Choe for his warm hospitality during the author's visit to Korea Institute for Advanced Study.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, New York, 2000.

[2] L. Ambrosio and B. Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math. Ann. **318** (2000), no. 3, 527–555.

[3] D. Burago, Y. Burago, and S. Ivanov, *A Course in Metric Geometry*, Graduate Studies in Mathematics, vol.33, American Mathematical Society, Providence, RI, 2001.

[4] Y. Burago, M. Gromov, and G. Perelman, *A. D. Aleksandrov spaces with curvature bounded below*, Russian Math. Surveys **47** (1992), no. 2, 1–58.

[5] K. Kuwae, Y. Machigahira, and T. Shioya, *Sobolev sapces, Laplacian, and heat kernel on Alexandrov spaces*, Math. Z. **238** (2001), no. 2, 269–316.

[6] K. Kuwae and T. Shioya, *Laplacian comparison for Alexandrov spaces*, arXiv:0709.0788, 2007.

[7] ———, *Infinitesimal Bishop-Gromov condition for Alexandrov spaces*, Probabilistic approach to geometry, 293–302, Adv. Stud. Pure Math., 57, Math. Soc. Japan, Tokyo, 2010.

[8] ———, *A topological splitting theorem for weighted Alexandrov spaces*, Tohoku Math. J. (2) **63** (2011), no. 1, 59–76.

[9] S. Ohta, *On the measure contraction property of metric measure spaces*, Comment. Math. Helv. **82** (2007), no. 4, 805–828.

[10] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–214.

[11] Y. Otsu and T. Shioya, *The Riemannian structure of Alexandrov spaces*, J. Differential Geom. **39** (1994), no. 3, 629–658.

[12] G. Perelman, *DC-structure on Alexandrov spaces*, preprint.

[13] A. Petrunin, *Alexandrov meets Lott-Villani-Sturm*, Münster J. Math. **4** (2011), 53–64.

[14] S. Pigola, M. Rigoli, and A. G. Setti, *A remark on the maximum principle and stochastic completeness*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1283–1288.

[15] J. Rataj and L. Zajìček, *Critical values and level sets of distance functions in Riemannian, Alexandrov and Minkowski spaces*, Houston J. Math. **38** (2012), no. 2, 445–467.

[16] A. Ratto, M. Rigoli, and A. G. Setti, *On the Omori-Yau maximum principle and its application to differential equations and geometry*, J. Funct. Anal. **134** (1995), no. 2, 486–510.

[17] M.-K. von Renesse, *Heat kernel comparison on Alexandrov spaces with curvature bounded below*, Potential Anal. **21** (2004), no. 2, 151–176.

[18] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

[19] ———, *A general Schwarz lemma for Kahler manifolds*, Amer. J. Math. **100** (1978), no. 1, 197–203.

[20] H. Zhang and X. Zhu, *Ricci curvature on Alexandrov spaces and rigidity theorems*, Comm. Anal. Geom. **18** (2010), no. 3, 503–553.

Hanjin Lee
Global Leadership School
Handong Global University
Pohang 791-708, Korea
and
School of Mathematics
Korea Institute for Advanced Study
Seoul 130-722, Korea
E-mail address: hxl@handong.edu