

## DEFORMATION RIGIDITY OF ODD LAGRANGIAN GRASSMANNIANS

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ABSTRACT. In this paper, we study the rigidity under Kähler deformation of the complex structure of odd Lagrangian Grassmannians, i.e., the Lagrangian case  $\text{Gr}_\omega(n, 2n+1)$  of odd symplectic Grassmannians. To obtain the global deformation rigidity of the odd Lagrangian Grassmannian, we use results about the automorphism group of this manifold, the Lie algebra of infinitesimal automorphisms of the affine cone of the variety of minimal rational tangents and its prolongations.

### 1. Introduction

The study on deformations of the complex structure of complex manifolds originates with Riemann. One hundred years after Riemann, Kodaira and Spencer developed the deformation theory of higher dimensional compact complex manifolds. Kodaira and Spencer showed that an infinitesimal deformation of a compact complex manifold should be represented by the *Kodaira-Spencer class*, which is an element of the first cohomology group with coefficients in the sheaf of germs of holomorphic vector fields (see [11] and [12]).

Let  $S$  be a rational homogeneous manifold  $G/P$  for a complex simple Lie group  $G$  and a parabolic subgroup  $P \subset G$ . As a consequence of the Bott-Borel-Weil theorem,  $H^1(S, T_S) = 0$ , in fact,  $H^i(S, T_S) = 0$  for  $i \geq 1$ . By Kodaira-Spencer theory, vanishing of the first cohomology group  $H^1(S, T_S)$  implies local deformation rigidity of rational homogeneous manifolds. Furthermore, Hwang and Mok proved the *global deformation rigidity* of a rational homogeneous manifold  $S$  of Picard number 1 ([4], [5], [7], [8], [9]).

**Theorem 1.1.** *Let  $\pi: \mathcal{X} \rightarrow \Delta$  be a smooth projective morphism from a complex manifold  $\mathcal{X}$  to the unit disc  $\Delta \subset \mathbb{C}$ . Suppose for any  $t \in \Delta \setminus \{0\}$ , the fiber  $\mathcal{X}_t = \pi^{-1}(t)$  is biholomorphic to a rational homogeneous manifold  $S$  of Picard*

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number 1 different from the 7-dimensional orthogonal Grassmannian  $\mathrm{Gr}_q(2, 7)$ . Then, the central fiber  $\mathcal{X}_0$  is also biholomorphic to  $S$ .

A projective manifold  $X$  is called *quasi-homogeneous* if the identity component  $\mathrm{Aut}^0(X)$  of the automorphism group of  $X$  acts on  $X$  with open dense orbit. We may think such a variety as a smooth equivariant compactification of a homogeneous manifold  $G/H$ , where  $G = \mathrm{Aut}^0(X)$  and  $H \subset G$  is the isotropy subgroup of a general point of  $X$ . It is then natural to have questions about the deformation rigidity of quasi-homogeneous manifolds. Studying the deformation of quasi-homogeneous manifolds is also motivated by the remarkable exception of Theorem 1.1. The orthogonal Grassmannian  $\mathrm{Gr}_q(2, 7)$ , the variety of all  $q$ -isotropic complex planes in a 7-dimensional complex vector space endowed with a symmetric bilinear form  $q$ , can be deformed to the  $G_2$ -horospherical manifold called  $(G_2, \alpha_2, \alpha_1)$  in [19] (Proposition 2.3 of [20]), which is quasi-homogeneous under the complex Lie group of type  $G_2$  considered as a subgroup of the projective orthogonal group  $\mathrm{PSO}(7) = \mathrm{Aut}(\mathrm{Gr}_q(2, 7))$ .

The proof of Theorem 1.1 had been carried out over several steps for about 10 years starting with the irreducible Hermitian symmetric spaces of the compact type. As a forward step to the global deformation rigidity problem of a quasi-homogeneous Fano manifold of Picard number 1, we will deal with the *odd Lagrangian Grassmannian*  $\mathrm{Gr}_\omega(n, 2n + 1)$ , which is described in Section 2. In this case, the open orbit of its automorphism group is isomorphic to the dual tautological bundle on the Lagrangian Grassmannian  $\mathrm{Gr}_\omega(n, 2n)$ , which is the irreducible Hermitian symmetric space of type III.

**Theorem 1.2.** *Let  $\pi: \mathcal{X} \rightarrow \Delta$  be a smooth projective morphism from a complex manifold  $\mathcal{X}$  to the unit disc  $\Delta \subset \mathbb{C}$ . Suppose for any  $t \in \Delta \setminus \{0\}$ , the fiber  $\mathcal{X}_t = \pi^{-1}(t)$  is biholomorphic to an odd Lagrangian Grassmannian  $\mathrm{Gr}_\omega(n, 2n + 1)$ . Then, the central fiber  $\mathcal{X}_0$  is also biholomorphic to  $\mathrm{Gr}_\omega(n, 2n + 1)$ .*

In Section 2, we will define the odd symplectic Grassmannians and review their geometric properties. In Section 3, after recalling the notion of the variety of minimal rational tangents (VMRT) on uniruled projective manifolds, the prolongations of a linear Lie algebra and the cone structures on a complex manifold, we reduce the rigidity problem under Kähler deformation to calculating the prolongations of the Lie algebra of infinitesimal automorphisms of the cone structure given by VMRT (see Proposition 3.5). Then by considering the automorphism group of an odd Lagrangian Grassmannian, the Lie algebra of infinitesimal automorphisms of the affine cone of VMRT and its prolongations, we can prove Theorem 1.2 in Section 4.

## 2. Odd symplectic Grassmannians

Let  $V$  be a complex vector space endowed with a skew-symmetric bilinear form  $\omega$  of maximal rank. We denote the variety of all  $k$ -dimensional isotropic subspaces of  $V$  by  $\mathrm{Gr}_\omega(k, V) = \{W \subset V : \dim W = k, \omega|_W \equiv 0\}$ . When  $\dim V$

is even, say,  $2n$ , the form  $\omega$  is a nondegenerate symplectic form and this variety  $\text{Gr}_\omega(k, 2n)$  is the usual symplectic Grassmannian, which is homogeneous under the action of the symplectic group  $\text{Sp}(2n)$ . However, when  $\dim V$  is odd, say,  $2n + 1$ , the skew-form  $\omega$  has the one-dimensional kernel  $\text{Ker } \omega$ . The variety  $\text{Gr}_\omega(k, 2n + 1)$ , called the *odd symplectic Grassmannian*, is not homogeneous and has two orbits under the action of its automorphism group if  $2 \leq k \leq n$  (cf. [14] and Proposition 1.12 of [19]). If  $k = 1$ , then the isotropic condition holds trivially so that  $\text{Gr}_\omega(1, V)$  is just the linear space  $\mathbb{P}^{\dim V - 1}$ . Next, for  $k = n + 1$  the odd symplectic Grassmannian  $\text{Gr}_\omega(n + 1, 2n + 1)$  is isomorphic to the symplectic Grassmannian  $\text{Gr}_\omega(n, 2n)$  because any  $(n + 1)$ -dimensional isotropic subspace must contain the one-dimensional kernel of  $\omega$ .

In what follows, we will assume that  $2 \leq k \leq n$  when considering the odd symplectic Grassmannians. The odd symplectic Grassmannians have many nice geometric properties as follows.

**Proposition 2.1.** *Let  $S$  be an odd symplectic Grassmannian  $\text{Gr}_\omega(k, 2n + 1)$  for  $2 \leq k \leq n$ .*

- (1)  *$S$  is a smooth Fano manifold of Picard number 1.*
- (2) *The automorphism group  $\text{Aut}(S)$  of  $S$  is isomorphic to the semi-direct product  $(\text{Sp}(2n) \times \mathbb{C}^*) / \{\pm 1\} \ltimes \mathbb{C}^{2n}$ .*
- (3)  *$S$  has two orbits under its automorphism group  $\text{Aut}(S)$ . Furthermore, the closed orbit  $\{W \in \text{Gr}_\omega(k, 2n + 1) : \text{Ker } \omega \subset W\}$  is isomorphic to the symplectic Grassmannian  $\text{Gr}_\omega(k - 1, 2n)$  and the open orbit  $\{W \in \text{Gr}_\omega(k, 2n + 1) : \text{Ker } \omega \not\subset W\}$  is isomorphic to the dual tautological bundle on  $\text{Gr}_\omega(k, 2n)$ .*
- (4)  *$S$  does not admit local deformation of its complex structure, that is,  $H^1(S, T_S) = 0$ .*

*Proof.* (1) Because the skew-form  $\omega$  is generic, the smoothness of  $S$  follows from Bertini’s theorem (Proposition 4.1 of [14]). The Picard group of  $S$  is the free abelian group generated by the class of the closure of the codimension 1 cell (Subsection 4.13 of [14]).

(2) See Section 5 of [14] or Theorem 1.11 of [19]. Note that the semi-direct product  $(\text{Sp}(2n) \times \mathbb{C}^*) \ltimes \mathbb{C}^{2n}$  is equal to the *odd symplectic group*  $\text{Sp}(2n + 1)$  defined in [14] and  $\text{Aut}(\text{Gr}_\omega(k, 2n + 1)) = \text{PSp}(2n + 1)$  for  $2 \leq k \leq n$ .

(3) Since the odd symplectic group  $\text{Sp}(2n + 1)$  is the stabilizer in  $\text{GL}(2n + 1)$  of the odd symplectic form  $\omega$ , both orbits are stable under  $\text{Sp}(2n + 1)$ . Considering the decomposition  $\mathbb{C}^{2n + 1} = \text{Ker } \omega \oplus \mathbb{C}^{2n}$ , any  $W \in \text{Gr}_\omega(k, 2n + 1)$  containing  $\text{Ker } \omega$  corresponds to a point of  $\text{Gr}_\omega(k - 1, 2n)$ . And the projection coming from the above decomposition gives a map from the open orbit onto  $\text{Gr}_\omega(k, 2n)$  of which the fiber at a point  $E \in \text{Gr}_\omega(k, 2n)$  is  $E^*$  (see Proposition 4.3 of [14]).

(4) From Theorem 0.5 of [20], we know that  $H^i(S, T_S) = 0$  for any  $i \geq 1$ .  $\square$

From Proposition 2.1(3), an odd symplectic Grassmannian  $\text{Gr}_\omega(k, 2n + 1)$  has three orbits under the semisimple part  $\text{Sp}(2n)$  of its automorphism group. In particular, the  $\text{Sp}(2n)$ -closed orbit lying in the open orbit is isomorphic to

a symplectic Grassmannian  $\text{Gr}_\omega(k, 2n)$ , which becomes the Lagrangian Grassmannian  $\text{Gr}_\omega(n, 2n)$  when  $k = n$ . This Lagrangian case  $\text{Gr}_\omega(n, 2n + 1)$  of odd symplectic Grassmannians will be called the *odd Lagrangian Grassmannian*.

### 3. Prolongations of infinitesimal automorphisms of cone structure defined by VMRT

In this section, we review some basic facts for the variety of minimal rational tangents (VMRT) on uniruled projective manifolds, the prolongations of a linear Lie algebra and the cone structures on a complex manifold. Then we can reduce the rigidity problem under Kähler deformation to calculating the prolongations of the Lie algebra of infinitesimal automorphisms of the cone structure given by VMRT.

#### 3.1. Variety of minimal rational tangents

A *Fano manifold* is a compact complex manifold  $X$  with the positive anti-canonical bundle  $K_X^{-1}$ . By Mori's theory, Fano manifolds are *uniruled*, i.e., covered by rational curves. Rational curves play a crucial role in the study of Fano manifolds (cf. [15] and [16]). In the 1990's Hwang and Mok introduced the notion of the *variety of minimal rational tangents* on uniruled projective manifolds (see [3] and [6]). For the study of Fano manifolds, more generally uniruled manifolds, a basic tool is the deformation theory of rational curves. The study on the deformation of minimal rational curves leads to their associated varieties of minimal rational tangents, which is defined as a subvariety of the projectivized tangent bundle  $\mathbb{P}(T_X)$  consisting of tangent directions of minimal rational curves immersed in a uniruled projective manifold  $X$ .

Let  $X$  be a projective manifold of dimension  $n$ . By a *parameterized rational curve* we mean a nonconstant holomorphic map  $f: \mathbb{P}^1 \rightarrow X$  from the projective line  $\mathbb{P}^1$  into  $X$ . We say that a (parameterized) rational curve  $f: \mathbb{P}^1 \rightarrow X$  is *free* if the pullback  $f^*T_X$  of the tangent bundle is nonnegative in the sense that  $f^*T_X$  splits into a direct sum  $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$  of line bundles of degree  $a_i \geq 0$  for all  $i = 1, \dots, n$ . For a polarized uniruled projective manifold  $(X, L)$  with an ample line bundle  $L$ , a *minimal rational curve* on  $X$  is a free rational curve of minimal degree among all free rational curves on  $X$ .

Let  $\mathcal{J}$  be a connected component of the space of minimal rational curves and let  $\mathcal{K} := \mathcal{J}/\text{Aut}(\mathbb{P}^1)$  be the quotient space of unparameterized minimal rational curves. We call  $\mathcal{K}$  a *minimal rational component*. For a point  $x \in X$ , consider the subvariety  $\mathcal{K}_x$  of  $\mathcal{K}$  consisting of minimal rational curves belonging to  $\mathcal{K}$  marked at  $x$ . Define the *tangent (rational) map*  $\tau_x: \mathcal{K}_x \dashrightarrow \mathbb{P}(T_x X)$  by  $\tau_x([f(\mathbb{P}^1)]) = [df(T_o\mathbb{P}^1)]$  sending a member of  $\mathcal{K}_x$  smooth at  $x$  to its tangent direction at  $x$ , where  $f: \mathbb{P}^1 \rightarrow X$  is a minimal rational curve with  $f(o) = x$ . For a general point  $x \in X$ , by Theorem 3.4 of [10], this tangent map induces a morphism  $\tau_x: \mathcal{K}_x \rightarrow \mathbb{P}(T_x X)$ , which is finite over its image.

**Definition.** Let  $X$  be a polarized uniruled projective manifold with a minimal rational component  $\mathcal{K}$ . For a general point  $x \in X$ , the image  $\mathcal{C}_x(X) := \tau_x(\mathcal{K}_x) \subset \mathbb{P}(T_x X)$  is called the *variety of minimal rational tangents* (to be abbreviated as VMRT) of  $X$  at  $x$ . The union of  $\mathcal{C}_x$  over general points  $x \in X$  gives the fibered space  $\mathcal{C} \subset \mathbb{P}(T_X) \rightarrow X$  of varieties of minimal rational tangents associated to  $\mathcal{K}$ .

We have the following description of the VMRT of (even or odd) symplectic Grassmannians. Although this is identical to Lemma 3.6 of [2], we give details of the proof for the convenience of the reader since the proof there is omitted. For the case of (even) symplectic Grassmannians, the following lemma was originally proven in Proposition 3.2.1 of [9] or Corollary 5.5 of [13].

**Lemma 3.1.** *Let  $V$  be a complex vector space endowed with a skew-symmetric bilinear form  $\omega$  of maximal rank. The VMRT at a general point of the (even or odd) symplectic Grassmannian  $\text{Gr}_\omega(k, V)$  is isomorphic to a natural embedding*

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^{k-1}}(-1)^m \oplus \mathcal{O}_{\mathbb{P}^{k-1}}(-2)) \subset \mathbb{P}^{mk + \frac{1}{2}k(k+1) - 1}$$

for  $k \geq 2$  and  $m := \dim V - 2k$ .

*Proof.* Let  $S = \text{Gr}_\omega(k, V)$  and fix a  $k$ -dimensional isotropic subspace  $E \subset V$ . When  $\dim V$  is odd, we choose  $E$  not containing the one-dimensional kernel of  $\omega$  which lies in the open orbit by Proposition 2.1(3). Since we can view  $S$  as a subvariety of the Grassmannian  $\text{Gr}(k, V)$  of  $k$ -dimensional subspaces in  $V$  and the tangent space of  $\text{Gr}(k, V)$  at a point  $[E]$  is naturally isomorphic to  $\text{Hom}(E, V/E) = E^* \otimes V/E$ , we have

$$T_{[E]}S = \{h \in \text{Hom}(E, V/E) : \omega(h(e_1), e_2) + \omega(e_1, h(e_2)) = 0, \forall e_1, e_2 \in E\}.$$

Putting  $E^\perp := \{v \in V : \omega(v, e) = 0, \forall e \in E\}$ ,  $E^\perp$  is a subspace of dimension  $\dim V - k$  containing  $E$  because  $E$  is an isotropic subspace (not containing  $\text{Ker } \omega$  when  $\dim V$  is odd). The isomorphism  $V/E^\perp \cong E^*$  is induced by the skew-symmetric form  $\omega$ . Then, we have the natural projection  $\psi: E^* \otimes (V/E) \rightarrow E^* \otimes E^*$  defined by the composition  $V/E \rightarrow V/E^\perp$  with the above isomorphism  $V/E^\perp \cong E^*$ . Hence, the map  $\psi: E^* \otimes (V/E) \rightarrow E^* \otimes E^*$  is given by  $e^* \otimes v \mapsto e^* \otimes v^\flat$ , where  $v^\flat \in E^*$  for  $v \in V$  is defined by  $v^\flat(e) := \omega(v, e)$ . Now, for any  $e_1, e_2 \in E$  and  $h = e^* \otimes v \in T_{[E]}S$ ,

$$\begin{aligned} \psi(e^* \otimes v)(e_1 \otimes e_2) &= (e^* \otimes v^\flat)(e_1 \otimes e_2) = e^*(e_1)v^\flat(e_2) \\ &= e^*(e_1)\omega(v, e_2) = \omega(e^*(e_1)v, e_2) \\ &= \omega(h(e_1), e_2) = \omega(h(e_2), e_1) \quad (\because h \in T_{[E]}S) \\ &= \omega(e^*(e_2)v, e_1) = \omega(v, e_1)e^*(e_2) \\ &= v^\flat(e_1)e^*(e_2) = (v^\flat \otimes e^*)(e_1 \otimes e_2). \end{aligned}$$

Thus, the tangent space  $T_{[E]}S$  is given by the inverse image  $\psi^{-1}(\text{Sym}^2 E^*)$  of  $\text{Sym}^2 E^* \subset E^* \otimes E^*$ , where  $\text{Sym}^2 E^*$  means the second symmetric power of  $E^*$ .

Since  $v^\flat = 0$  for all  $v \in E^\perp$ , there is a natural subspace  $D_{[E]}$  of the tangent space  $\psi^{-1}(\text{Sym}^2 E^*) \subset E^* \otimes (V/E)$  defined by  $D_{[E]} := E^* \otimes (E^\perp/E) = \text{Ker } \psi$ . It follows that the tangent space of  $\text{Gr}_\omega(k, V)$  at  $[E]$  can be identified with

$$T_{[E]} \text{Gr}_\omega(k, V) = (E^* \otimes (E^\perp/E)) \oplus \text{Sym}^2 E^*.$$

Minimal rational curves on  $S$  are lines of the Grassmannian  $\text{Gr}(k, V)$  lying on  $S$ . Thus the variety  $\mathcal{C}_{[E]}(S)$  of minimal rational tangents of  $S$  at a general point  $[E] \in S$  is the variety of decomposable tensors in  $T_{[E]}S$ . If a decomposable tensor  $h = e^* \otimes v$  is contained in  $T_{[E]}S$ , then we have

$$\begin{aligned} \omega(v, e') &= \omega(h(e), e') = -\omega(e, h(e')) \\ &= -\omega(e, (e^* \otimes v)e') = -\omega(e, e^*(e')v) \\ &= -\omega(e, v)e^*(e') \quad \text{for all } e' \in E, \end{aligned}$$

that is,  $\omega(v, \cdot) = v^\flat \in \mathbb{C}e^*$ . Conversely, if  $v^\flat \in \mathbb{C}e^*$ , then  $e^* \otimes v$  is contained in  $T_{[E]}S$ . Therefore, the affine cone of VMRT  $\mathcal{C}_{[E]}(S)$  is

$$\tilde{\mathcal{C}}_{[E]}(S) = \{e^* \otimes v \in E^* \otimes (V/E) : v^\flat \in \mathbb{C}e^*\} \setminus \{0\},$$

where the affine cone  $\tilde{\mathcal{A}} \subset V \setminus \{0\}$  of a complex-analytic subvariety  $\mathcal{A} \subset \mathbb{P}(V)$  means the preimage  $\pi^{-1}(\mathcal{A})$  of the canonical projection  $\pi: V \setminus \{0\} \rightarrow \mathbb{P}(V)$ . Because  $v^\flat = 0$  for  $v \in E^\perp$ , its intersection with  $D_{[E]} = \text{Ker } \psi$  is

$$\tilde{\mathcal{C}}_{[E]}(S) \cap D_{[E]} = \{e^* \otimes v \in E^* \otimes (E^\perp/E)\} \setminus \{0\}.$$

Under the projection  $e^* \otimes v \mapsto e^*$ ,  $\mathbb{P}(\{e^* \otimes v \in E^* \otimes (E^\perp/E)\}) = \mathcal{C}_{[E]}(S) \cap \mathbb{P}(D_{[E]})$  is a trivial  $\mathbb{P}^{m-1}$ -bundle on  $\mathbb{P}(E^*) = \mathbb{P}^{k-1}$ . Hence, the VMRT  $\mathcal{C}_{[E]}(S)$  is a  $\mathbb{P}^m$ -bundle on  $\mathbb{P}(E^*)$ .

Let  $F$  be the vector bundle on  $\mathbb{P}(E^*)$  such that  $\mathbb{P}(F) = \mathcal{C}_{[E]}(S)$ . Then,  $F$  has a trivial subbundle isomorphic to  $\mathbb{P}(E^*) \times (E^\perp/E)$ . From the above description, the vector bundle modulo the trivial subbundle is isomorphic to the tautological line bundle of  $\mathbb{P}(E^*)$  so that  $F \cong \mathcal{O}^m \oplus \mathcal{O}(-1)$ . The embedding  $\mathcal{C}_{[E]}(S) \subset \mathbb{P}(T_{[E]}S)$  restricts to the Segre embedding on  $\mathcal{C}_{[E]}(S) \cap \mathbb{P}(D_{[E]}) = \mathbb{P}(E^*) \times \mathbb{P}(E^\perp/E)$ , which is given by the dual tautological line bundle  $\mathcal{O}(1)$  on the projectivization  $\mathbb{P}(T_{[E]}S)$  when we view  $\mathbb{P}(E^*) \times \mathbb{P}(E^\perp/E)$  as the projectivization of the vector bundle  $\mathcal{O}(-1)^m$  on  $\mathbb{P}(E^*)$ . Thus, the variety  $\mathcal{C}_{[E]}(S)$  of minimal rational tangents at  $[E] \in S$  is isomorphic to the projectivization of the vector bundle  $\mathcal{O}(-1)^m \oplus \mathcal{O}(-2)$  on  $\mathbb{P}(E^*)$  embedded by the complete linear system associated to the dual tautological bundle of the projectivization  $\mathbb{P}(T_{[E]}S)$ . □

### 3.2. Prolongations of a linear Lie algebra

Let  $G \subset \text{GL}(V)$  be a linear Lie group and  $\mathfrak{g} \subset \mathfrak{gl}(V)$  its Lie algebra. The (algebraic) prolongations  $\mathfrak{g}^{(k)}$  of a linear Lie algebra  $\mathfrak{g} \subset \mathfrak{gl}(V)$  originate from the higher order derivatives of the infinitesimal automorphisms of the flat  $G$ -structure, which is a subbundle  $V \times G$  of the frame bundle  $\mathcal{F}(V) = V \times \text{GL}(V)$ .

Taking a global coordinate system  $(x_1, \dots, x_n)$  given by a basis of  $V$ , we can interpret every vector field  $\xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$  on  $V$  as a  $V$ -valued function  $F_\xi$  on  $V$ . Then the derivative  $dF_\xi$  of  $F_\xi$  is expressed as the coefficient matrix under the identification  $V \cong T_v V$  for  $v \in V$ . Now, we consider this as a  $\mathfrak{gl}(V)$ -valued linear map on  $V$ . In addition, if the lift of a vector field on  $V$  to the frame bundle  $\mathcal{F}(V)$  is tangent to a  $G$ -subbundle, then the coefficient matrix of the derivative  $dF_\xi$  is a  $\mathfrak{g}$ -valued linear map on  $V$ . Likewise, the coefficient matrix of the second derivative of  $F_\xi$  is a  $\mathfrak{gl}(V) \otimes V^*$ -valued linear map on  $V$ , in fact, actually takes values in  $\text{Sym}^2 V^* \otimes V$  and  $V^* \otimes \mathfrak{g}$ .

**Definition.** Let  $V$  be a complex vector space and  $\mathfrak{g} \subset \mathfrak{gl}(V)$  a linear Lie algebra. For an integer  $k \geq 0$ , the space  $\mathfrak{g}^{(k)}$ , called the  $k$ -th prolongation of  $\mathfrak{g}$ , is the vector space of symmetric multi-linear homomorphisms  $A: \text{Sym}^{k+1} V \rightarrow V$  such that for any fixed vectors  $v_1, \dots, v_k \in V$ , the endomorphism

$$v \in V \mapsto A(v_1, \dots, v_k, v) \in V$$

belongs to  $\mathfrak{g}$ . That is,  $\mathfrak{g}^{(k)} = \text{Hom}(\text{Sym}^{k+1} V, V) \cap \text{Hom}(\text{Sym}^k V, \mathfrak{g})$ .

We are interested in the case where a Lie algebra  $\mathfrak{g}$  arises from geometric situations, in particular, the Lie algebra of infinitesimal linear automorphisms of an irreducible projective subvariety.

**Definition.** Let  $S \subset \mathbb{P}V$  be an irreducible projective variety. The projective automorphism group of  $S$  is  $\text{Aut}(S) = \{g \in \text{PGL}(V) : g(S) = S\}$ . Its Lie algebra is denoted by  $\mathfrak{aut}(S)$ . Denote by  $\tilde{S} \subset V$ , the affine cone of  $S$  and by  $T_\alpha \tilde{S} \subset V$ , the affine tangent space at a smooth point  $\alpha \in \tilde{S}$ . The Lie algebra of infinitesimal linear automorphisms of  $\tilde{S}$  is defined by

$$\mathfrak{aut}(\tilde{S}) = \{A \in \mathfrak{gl}(V) : A(\alpha) \in T_\alpha \tilde{S} \text{ for any smooth point } \alpha \in \tilde{S} \subset V\}.$$

Its prolongation  $\mathfrak{aut}(\tilde{S})^{(k)}$  will be called the  $k$ -th prolongation of  $S \subset \mathbb{P}V$ .

In [9], Hwang and Mok studied the prolongations  $\mathfrak{aut}(\tilde{S})^{(k)}$  of a projective variety  $S \subset \mathbb{P}V$  using the projective geometry of  $S$  and the deformation theory of rational curves on  $S$ . In particular, the vanishing of the second prolongation  $\mathfrak{aut}(\tilde{S})^{(2)}$  for an irreducible smooth nondegenerate projective variety was proven, which is Theorem 1.1.2 of [9].

**Proposition 3.2.** *Let  $S \subset \mathbb{P}V$  be an irreducible smooth nondegenerate projective variety. If  $S \neq \mathbb{P}V$ , then the second prolongation of  $S$  vanishes, that is,  $\mathfrak{aut}(\tilde{S})^{(2)} = 0$ .*

From the definition of prolongations, it is immediate that  $\mathfrak{g}^{(k)} = 0$  for some  $k \geq 0$  implies  $\mathfrak{g}^{(k+1)} = 0$ . Thus, if  $S \subsetneq \mathbb{P}V$  is an irreducible smooth nondegenerate projective variety, then  $\mathfrak{aut}(\tilde{S})^{(k)} = 0$  for  $k \geq 2$ .

**3.3. Cone structures defined by VMRT**

Cone structure  $\mathcal{C}$  on complex manifold  $M$  is a closed analytic subvariety  $\mathcal{C} \subset \mathbb{P}(T_M)$  such that the natural projection  $\pi: \mathcal{C} \rightarrow M$  is proper, flat and surjective with connected fibers. We denote the fiber  $\pi^{-1}(x)$  by  $\mathcal{C}_x$  for a point  $x \in M$ . A germ of holomorphic vector field  $\xi$  at  $x \in M$  is said to preserve the cone structure if the local 1-parameter family of biholomorphisms integrating  $\xi$  lifts to local biholomorphisms of  $\mathbb{P}(T_M)$  preserving  $\mathcal{C}$ . The Lie algebra  $\mathfrak{aut}(\mathcal{C}, x)$  of infinitesimal automorphisms of the cone structure  $\mathcal{C}$  at  $x$  is the set of all germs of holomorphic vector fields preserving the cone structure at  $x$ .

The Lie algebra  $\mathfrak{aut}(\mathcal{C}, x)$  is naturally filtered by the vanishing order of vector fields at  $x$ . More precisely, letting  $\mathfrak{aut}(\mathcal{C}, x)_k$  be the subalgebra of  $\mathfrak{aut}(\mathcal{C}, x)$  consisting of vector fields which vanishes at  $x$  to order  $\geq k + 1$  for each integer  $k \geq 0$ , the Lie bracket gives the structure of filtration  $\mathfrak{aut}(\mathcal{C}, x) \supset \mathfrak{aut}(\mathcal{C}, x)_0 \supset \mathfrak{aut}(\mathcal{C}, x)_1 \supset \mathfrak{aut}(\mathcal{C}, x)_2 \supset \dots$ . Note that the graded pieces (under natural filtration) of the Lie algebra of infinitesimal symmetries of  $G$ -structure on complex manifold  $M$  are contained in the prolongations  $\mathfrak{g}^{(k)}$  of the Lie algebra  $\mathfrak{g} \subset \mathfrak{gl}(V)$  of  $G$  with  $\dim V = \dim M$  and equal to the prolongations when the  $G$ -structure on  $M$  is flat.

Let  $\xi$  be a germ of holomorphic vector field on  $M$  vanishing to order  $\geq k + 1$  at  $x$ . Then, its  $(k + 1)$ -jet  $J_x^{k+1}(\xi)$  defines an element of  $\text{Sym}^{k+1}(T_x^*M) \otimes T_xM$ . Because  $J_x^{k+1}(\zeta) = 0$  for a vector field  $\zeta$  vanishing to order  $\geq k + 2$  at  $x$ , this defines the inclusion  $\mathfrak{aut}(\mathcal{C}, x)_k / \mathfrak{aut}(\mathcal{C}, x)_{k+1} \subset \text{Hom}(\text{Sym}^{k+1}(T_xM), T_xM)$ . The following result is from Proposition 1.2.1 of [9] (also, see Section 5 of [2]).

**Proposition 3.3.** *Let  $\mathcal{C} \subset \mathbb{P}(T_M)$  be a cone structure on a complex manifold  $M$  and  $x \in M$  a point. For each  $k \geq 0$ , if the quotient space  $\mathfrak{aut}(\mathcal{C}, x)_k / \mathfrak{aut}(\mathcal{C}, x)_{k+1}$  is regarded as a subspace of  $\text{Hom}(\text{Sym}^{k+1}(T_xM), T_xM)$ , then we have the inclusion*

$$\mathfrak{aut}(\mathcal{C}, x)_k / \mathfrak{aut}(\mathcal{C}, x)_{k+1} \subset \mathfrak{aut}(\widetilde{\mathcal{C}}_x)^{(k)}.$$

From Proposition 3.3, we have the natural inequalities

$$\begin{aligned} \dim \mathfrak{aut}(\mathcal{C}, x)_0 &\leq \dim \mathfrak{aut}(\widetilde{\mathcal{C}}_x) + \dim \mathfrak{aut}(\mathcal{C}, x)_1 \\ &\leq \dim \mathfrak{aut}(\widetilde{\mathcal{C}}_x) + \dim \mathfrak{aut}(\widetilde{\mathcal{C}}_x)^{(1)} + \dim \mathfrak{aut}(\mathcal{C}, x)_2 \leq \dots \end{aligned}$$

Because the codimension of  $\mathfrak{aut}(\mathcal{C}, x)_0$  in  $\mathfrak{aut}(\mathcal{C}, x)$  is at most  $\dim M$ , we obtain the following direct consequence.

**Corollary 3.4.** *Let  $\mathcal{C} \subset \mathbb{P}(T_M)$  be a cone structure on a complex manifold  $M$  and  $x \in M$ . If  $\mathfrak{aut}(\widetilde{\mathcal{C}}_x)^{(k+1)} = 0$  for some  $k \geq 0$ , then*

$$\dim \mathfrak{aut}(\mathcal{C}, x) \leq \dim M + \dim \mathfrak{aut}(\widetilde{\mathcal{C}}_x) + \dim \mathfrak{aut}(\widetilde{\mathcal{C}}_x)^{(1)} + \dots + \dim \mathfrak{aut}(\widetilde{\mathcal{C}}_x)^{(k)}.$$

In summary of the above results, we have a criterion on the biholomorphic equivalence of locally rigid Fano manifolds with smooth nondegenerate projectively equivalent VMRTs.

**Proposition 3.5.** *Let  $\pi: \mathcal{X} \rightarrow \Delta$  be a smooth and projective morphism from a complex manifold  $\mathcal{X}$  to the unit disc  $\Delta$ . Suppose for any  $t \in \Delta \setminus \{0\}$ , the fiber  $\mathcal{X}_t = \pi^{-1}(t)$  is biholomorphic to a locally rigid Fano manifold  $S$ , and the variety  $\mathcal{C}_s(S)$  of minimal rational tangents at a general point  $s$  of  $S$  is irreducible, smooth, nondegenerate, and not linear. If the following conditions hold:*

- (a) *the VMRT at a general point of  $\mathcal{X}_0$  is projectively isomorphic to  $\mathcal{C}_s(S)$ , and*
- (b)  *$\dim \mathbf{aut}(S) = \dim S + \dim \mathbf{aut}(\widetilde{\mathcal{C}}_s) + \dim \mathbf{aut}(\widetilde{\mathcal{C}}_s)^{(1)}$  for the cone structure  $\mathcal{C}$  on  $S$  given by VMRT,*

*then the central fiber  $\mathcal{X}_0$  is also biholomorphic to  $S$ .*

*Proof.* First, we recall the standard fact that the Euler-Poincaré characteristic of the holomorphic tangent bundle  $T_X$  on a Fano manifold  $X$  is given by  $\chi(X, T_X) = h^0(X, T_X) - h^1(X, T_X)$ . In fact, the Serre duality and Kodaira-Nakano vanishing theorem imply that  $H^i(X, T_X) = H^{n-i}(X, T_X^* \otimes K_X) = 0$  for  $i \geq 2$ . Since the Euler-Poincaré characteristic is constant in a smooth family,  $\chi(\mathcal{X}_0, T_{\mathcal{X}_0}) = \chi(S, T_S)$ . Because  $S$  is a locally rigid Fano manifold,  $h^1(S, T_S) = 0$  and so  $\chi(\mathcal{X}_t, T_{\mathcal{X}_t}) = h^0(S, T_S)$  for all  $t$ . Thus, if the inequality  $h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) \leq h^0(S, T_S)$  holds, then we get  $h^1(\mathcal{X}_0, T_{\mathcal{X}_0}) = 0$ , which implies the desired rigidity result from the Kodaira-Spencer deformation theory.

Because  $\mathbf{aut}(\mathcal{X}_0)$  is isomorphic to the space  $H^0(\mathcal{X}_0, T_{\mathcal{X}_0})$  of global sections of the tangent bundle  $T_{\mathcal{X}_0}$  (Section 2.3 of [1]), we know  $h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) = \dim \mathbf{aut}(\mathcal{X}_0)$ . Also,  $\mathbf{aut}(\widetilde{\mathcal{C}}_x)^{(k)} = 0$  for  $k \geq 2$  by Proposition 3.2. Then, from the fact that any global vector field must preserve the cone structure given by VMRT and Corollary 3.4, for a general point  $x \in \mathcal{X}_0$ , we have the inequality

$$\begin{aligned} h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) &= \dim \mathbf{aut}(\mathcal{X}_0) \\ &\leq \dim \mathbf{aut}(\mathcal{C}, x) \\ &\leq \dim \mathcal{X}_0 + \dim \mathbf{aut}(\widetilde{\mathcal{C}}_x) + \dim \mathbf{aut}(\widetilde{\mathcal{C}}_x)^{(1)} \\ &= \dim S + \dim \mathbf{aut}(\widetilde{\mathcal{C}}_s) + \dim \mathbf{aut}(\widetilde{\mathcal{C}}_s)^{(1)} \\ &= \dim \mathbf{aut}(S) \\ &= h^0(S, T_S), \end{aligned}$$

where the first equality in the last line is just the assumption (b). Therefore, we have  $h^1(\mathcal{X}_0, T_{\mathcal{X}_0}) = h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) - h^0(S, T_S) = 0$ , hence the central fiber  $\mathcal{X}_0$  is also biholomorphic to the general fiber  $S$ . □

#### 4. Proof of Theorem 1.2

The proof of Theorem 1.2 consists of two parts. The first part is to show that the variety of minimal rational tangents at a general point of the central fiber  $\mathcal{X}_0$  agrees with that of the model manifold as a projective subvariety. Next, to obtain the global deformation rigidity of an odd Lagrangian Grassmannian, we

use results about the automorphism group of this manifold, the Lie algebra of infinitesimal automorphisms of the affine cone of VMRT and its prolongations.

For the projective equivalence of VMRTs, let us consider the situation more general than Theorem 1.2. Let  $W$  and  $Q$  be complex vector spaces of dimensions  $k \geq 2$  and  $m$ , respectively. By Lemma 3.1, the variety of minimal rational tangents at a general point of the odd symplectic Grassmannian  $\text{Gr}_\omega(k, 2k + m)$  is a projective subvariety  $Z \subset \mathbb{P}^{mk + \frac{1}{2}k(k+1) - 1}$  of which affine cone  $\tilde{Z} \subset (W \otimes Q) \oplus \text{Sym}^2 W$  is given by  $\{\lambda \otimes \mu + \mathbb{C}\lambda^2 : \lambda \in W, \mu \in Q\}$ .

**Proposition 4.1.** *Let  $\pi: \mathcal{X} \rightarrow \Delta$  be a smooth projective morphism from a complex manifold  $\mathcal{X}$  to the unit disc  $\Delta$ . Suppose for any  $t \in \Delta \setminus \{0\}$ , the fiber  $\mathcal{X}_t$  is biholomorphic to  $\text{Gr}_\omega(k, 2k + m)$ . Then, the VMRT at a general point  $x$  of  $\mathcal{X}_0$  is projectively isomorphic to  $Z \subset \mathbb{P}((W \otimes Q) \oplus \text{Sym}^2 W) = \mathbb{P}^{mk + \frac{1}{2}k(k+1) - 1}$ .*

*Proof.* This was proven in Proposition 3.5.2 of [9] for the case of (even) symplectic Grassmannians. The proof was presented under the assumption that  $m = \dim Q$  is even. However, the proof did not use this assumption. Thus, the proof there also works for odd symplectic Grassmannians.  $\square$

Now recall the results about the Lie algebra of infinitesimal automorphisms of the affine cone of VMRT and its prolongations. The following is from Proposition 3.8 of [2].

**Lemma 4.2.** *If  $Z \subset \mathbb{P}((W \otimes Q) \oplus \text{Sym}^2 W)$  is the variety of minimal rational tangents at a general point  $[W]$  of  $\text{Gr}_\omega(k, 2k + m)$ , then*

$$\text{aut}(\tilde{Z}) = (W^* \otimes Q) \rtimes (\mathfrak{gl}(W) \oplus \mathfrak{gl}(Q)) \quad \text{and} \quad \text{aut}(\tilde{Z})^{(1)} = \text{Sym}^2 W^*.$$

*Proof of Theorem 1.2.* If  $n = 1$ , then  $\text{Gr}_\omega(1, 3)$  is just the projective plane  $\mathbb{P}^2$ , which is the case already established in Theorem 1.1. Hence, we suppose  $n \geq 2$ . Let us continue to use the previous notations. In this case,  $\dim W = n$  and  $\dim Q = 1$ . Since  $\text{Gr}_\omega(n, 2n + 1)$  is a locally rigid Fano manifold (Proposition 2.1 (1) and (4)) and the VMRT at a general point of the central fiber  $\mathcal{X}_0$  is projectively isomorphic to  $Z \subset \mathbb{P}((W \otimes Q) \oplus \text{Sym}^2 W)$  by Proposition 4.1, it suffices to check the condition (b) of Proposition 3.5.

Now, we can easily obtain the dimension computation

$$\begin{aligned} \dim \text{Gr}_\omega(n, 2n + 1) &= \dim((W \otimes Q) \oplus \text{Sym}^2 W) \\ &= n + \frac{1}{2}n(n + 1) \\ &= \frac{1}{2}n(n + 3). \end{aligned}$$

In addition, Lemma 4.2 implies that  $\dim \text{aut}(\tilde{Z}) = n + n^2 + 1$  and  $\dim \text{aut}(\tilde{Z})^{(1)} = \frac{1}{2}n(n + 1)$ . Hence, we have

$$\dim \text{Gr}_\omega(n, 2n + 1) + \dim \text{aut}(\tilde{Z}) + \dim \text{aut}(\tilde{Z})^{(1)} = 2n^2 + 3n + 1.$$

On the other hand, by Proposition 2.1(2),

$$\begin{aligned} \dim \operatorname{Aut}(\operatorname{Gr}_\omega(n, 2n + 1)) &= \dim \operatorname{Sp}(2n + 1) \\ &= \dim \operatorname{Sp}(2n) + 1 + 2n \\ &= n(2n + 1) + 2n + 1 \\ &= 2n^2 + 3n + 1. \end{aligned}$$

Therefore,  $\mathcal{X}_0$  is also biholomorphic to  $\operatorname{Gr}_\omega(n, 2n + 1)$  by Proposition 3.5.  $\square$

We have obtained the rigidity result only for the Lagrangian case  $\operatorname{Gr}_\omega(n, 2n + 1)$  among odd symplectic Grassmannians  $\operatorname{Gr}_\omega(k, 2n + 1)$  because the condition (b) of Proposition 3.5 does not hold in other cases. However, the VMRT at a general point of the central fiber is also projectively isomorphic to the VMRT  $\mathcal{C}_s(\operatorname{Gr}_\omega(k, 2n + 1))$  at a general point  $s \in \operatorname{Gr}_\omega(k, 2n + 1)$  regardless of  $k$  by Proposition 4.1. Thus, we guess that a similar result to Theorem 1.2 may hold for more broad cases of odd symplectic Grassmannians.

On the other hand, we know the global deformation rigidity of a general hyperplane section  $S$  of the 10-dimensional spinor variety  $\mathbb{S}_5$ . It is well-known that  $S$  is a 9-dimensional Fano manifold of coindex 3 and the quasi-homogeneous horospherical manifold of type  $(B_3, \alpha_1, \alpha_3)$  considered in [19]. Mukai [17] classified the Fano manifolds of coindex  $\leq 3$ . By Mukai’s result, any projective and smooth deformation of  $S$  must be a general hyperplane section of  $\mathbb{S}_5$  and so is biholomorphic to  $S$  because all smooth hyperplane sections of  $\mathbb{S}_5$  are conjugate to each other. However, we can also obtain the deformation rigidity result from the consideration on the prolongations of the cone structure defined by VMRT in the same way as the above proof.

**Corollary 4.3.** *Let  $\pi: \mathcal{X} \rightarrow \Delta$  be a smooth projective morphism from a complex manifold  $\mathcal{X}$  to the unit disc  $\Delta$ . Suppose for any  $t \in \Delta \setminus \{0\}$ , the fiber  $\mathcal{X}_t$  is biholomorphic to a general hyperplane section  $S$  of the 10-dimensional spinor variety  $\mathbb{S}_5$ . Then, the central fiber  $\mathcal{X}_0$  is also biholomorphic to  $S$ .*

*Proof.* The argument is similar to the one used for odd Lagrangian Grassmannians. Since the VMRT  $Z = \mathcal{C}_s(S)$  at a general point  $s$  of  $S$  is isomorphic to a natural embedding  $\operatorname{Gr}_\omega(2, 5) \subset \mathbb{P}(T_s S) = \mathbb{P}^8$  induced by the Plücker embedding, the condition (a) of Proposition 3.5 is satisfied by Theorem 1.2 and Zak’s theorem on tangencies (Proposition 1.3.2 of Hwang-Mok [6] and Zak [21]). We use the facts that from Propositions 3.11 and 3.12 of [2]

$$\operatorname{aut}(\tilde{Z}) = \mathbb{C} \oplus W \rtimes (\mathfrak{so}(Q) \oplus \mathbb{C}) \quad \text{and} \quad \operatorname{aut}(\tilde{Z})^{(1)} = Q^*,$$

where  $Q$  is a 5-dimensional orthogonal vector space and  $W$  is the 4-dimensional spin representation of  $\mathfrak{so}(5)$ . Because  $\operatorname{Aut}(S) = (\operatorname{SO}(7) \times \mathbb{C}^*) \rtimes \mathbb{C}^8$  from Theorem 1.11 of [19], the result follows from Proposition 3.5.  $\square$

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