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THE LOCAL STAR CONDITION FOR GENERIC TRANSITIVE DIFFEOMORPHISMS

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Abstract. Let $f : M \to M$ be a diffeomorphism on a closed C^∞ $d(\geq 2)$ dimensional manifold M. For C^1-generic f, if a diffeomorphism f satisfies the local star condition on a transitive set, then it is hyperbolic.

1. Introduction

Let M be a closed C^∞ $d(\geq 2)$ dimensional manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \mathrm{Diff}(M)$ and let Λ be a closed f-invariant set. We say that Λ is *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. Denote by $P(f)$ the set of all periodic points of f. We say that f satisfies *Axiom A* if the non-wandering set $\Omega(f)$ is hyperbolic and it is the closure of $P(f)$. A diffeomorphism f is Ω-*stable* if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$ there is a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that $h \circ f = g \circ h$, where $\Omega(g)$ is the non-wandering set of g. For $f \in \mathrm{Diff}(M)$, we say that f satisfies the *star diffeomorphism* (or f satisfies the *star condition*) if there is a C^1-neighborhood $\mathcal{U}$ of f such that all periodic points of $g \in \mathcal{U}$ are hyperbolic. Denote by $\mathcal{F}(M)$ the set of all star diffeomorphisms. Mañé suggested in [5] the problem: Does $f \in \mathcal{F}(M)$ imply that f satisfies Axiom A? And he proved that if $f \in \mathcal{F}(M)$ and the non-wandering set $\Omega(f)$ is locally maximal, then f is Axiom A, when $\dim M = 2$. Later, for the any dimension case, by Aoki [2] and Hayashi [3], we know that if $f \in \mathcal{F}(M)$, then f satisfies both Axiom A and no-cycle condition, that is, Ω-stable. For that, in this article, we consider the local star condition which is a local version of $\mathcal{F}(M)$.

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We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$. Here U is called a locally maximal neighborhood of Λ. Let Λ be a closed f-invariant set. We say that f satisfies the *local star condition* on Λ (or, $f|_\Lambda$ satisfies the *local star condition*) if there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, every $p \in P(g) \cap \Lambda_g(U)$ is hyperbolic, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$, and U is a locally maximal neighborhood of Λ. For the local star diffeomorphism, we consider the problem: *For a closed f-invariant set Λ, if $f \in \mathcal{F}(\Lambda)$, then is it hyperbolic?*

In this paper, we give a partial answer of the problem. Actually, we consider the transitive set of the local star diffeomorphism for the generic view point.

We say that a subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\mathrm{Diff}(M)$; in this case $\mathcal{G}$ is dense in $\mathrm{Diff}(M)$. A property "P" is said to be (C^1)-*generic* if "P" holds for all diffeomorphisms which belong to some residual subset of $\mathrm{Diff}(M)$. Then we have the following.

Theorem 1.1. *There is a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}$, if a diffeomorphism f satisfies the local star condition on a transitive set Λ, then it is hyperbolic.*

2. Proof of Theorem 1.1

Denote by $Orb(p)$ the periodic f-orbit of $p \in P(f)$. Let $p \in P(f)$ be a hyperbolic periodic point with period $\pi(p) > 0$. It is well known that if p is a hyperbolic periodic point of f with period k, then the sets

$$W^s(p) = \{x \in M : f^{k\pi(p)}(x) \to p \text{ as } n \to \infty\} \text{ and}$$
$$W^u(p) = \{x \in M : f^{-k\pi(p)}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. A point $x \in W^s(p) \cap W^u(p)$ is called a *transversal homoclinic point* of f if the intersection is transversal at x; i.e., $x \in W^s(p) \pitchfork W^u(p)$. The closure of the set of transversal homoclinic points of f associated to p is called the *transversal homoclinic class* of f associated to p, and it is denoted by $H_f(p) = \overline{W^s(p) \cap W^u(p)}$. It is clear that $H_f(p)$ is compact, invariant and transitive. The following lemma was founded in [1].

Lemma 2.1 ([1, Theorem 4.10]). *There is a residual set $\mathcal{G}_0$ such that for any $f \in \mathcal{G}_0$, if the set Λ is locally maximal and transitive, then $\Lambda = H_f(p)$ for some hyperbolic periodic point p.*

For any two periodic points $p, q \in P(f) \cap \Lambda (= P(f|_\Lambda))$, we say that p, q have the *barycenter property* if for any $\epsilon > 0$ there exists an integer $N = N(\epsilon, p, q) > 0$ such that for any two integers n_1, n_2 there is a point $x \in \Lambda$ such that

$$d(f^i(x), f^i(p)) < \epsilon,\ -n_1 \le i \le 0, \text{ and } d(f^{i+N}(x), f^i(q)) < \epsilon,\ 0 \le i \le n_2.$$

Λ satisfies the *barycenter property* if the barycenter property holds for any two periodic points $p, q \in P(f|_\Lambda)$. The notion of the barycenter property was introduced by [1].

Lemma 2.2. *There is a residual set $\mathcal{G}_1$ such that for any $f \in \mathcal{G}_1$, if the set Λ is locally maximal and transitive, then Λ satisfies the barycenter property.*

Proof. Let $f \in \mathcal{G}_1 = \mathcal{G}_0$ and let the set Λ be locally maximal and transitive. By Lemma 2.1, Λ is the homoclinic class $H_f(p)$, for some hyperbolic periodic point p. Then by [1, Proposition 4.8], Λ satisfies the barycenter property. □

Lemma 2.3. *Let Λ be a transitive with locally maximal. If Λ satisfies the barycenter property, then for any $p, q \in P(f) \cap \Lambda$,*

$$W^s(p) \cap W^u(q) \neq \emptyset \text{ and } W^u(p) \cap W^s(q) \neq \emptyset.$$

Proof. Since Λ is transitive, by Pugh's closing lemma, there exist the sequences $Orb_{g_n}(p_n)$ such that $Orb_{g_n}(p_n) \to \Lambda$ and $g_n \to f$ as $n \to \infty$. Since Λ is locally maximal, for sufficiently large n there is a periodic point p such that $p \in \Lambda$. Thus we can consider the periodic points which belong to Λ. Since Λ satisfies the barycenter property by [9, Lemma 2.2], $W^s(p) \cap W^u(q) \neq \emptyset$. Other case is similar. □

We say that f is *Kupka-Smale* if every periodic points are hyperbolic, and if for any $p, q \in P(f)$, the stable manifold $W^s(p)$ and the unstable manifold $W^u(q)$ intersect transversally. Denote by $\mathcal{KS}$ the set of all Kupka-Smale diffeomorphisms. It is well-known that $\mathcal{KS}$ is a residual set of $\mathrm{Diff}(M)$. The dimension of the stable manifold $W^s(p)$ is called *index of* p, and we denote it by $\mathrm{index}(p)$.

Lemma 2.4. *There is a residual set $\mathcal{G}_2 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_2$ if a transitive set Λ is locally maximal, then for any $p, q \in P(f) \cap \Lambda$,*

$$\mathrm{index}(p) = \mathrm{index}(q).$$

Proof. Let $f \in \mathcal{G}_2 = \mathcal{G}_1 \cap \mathcal{KS}$. Since the set Λ is transitive and locally maximal, by Lemma 2.2, one can see that Λ satisfies the barycenter property. Since f is Kupka-Smale, for any hyperbolic periodic points $p, q \in \Lambda$ we know that either $W^s(p) \cap W^u(q) = \emptyset$ or $W^s(p) \pitchfork W^u(q) \neq \emptyset$. Since Λ satisfies the barycenter property, by Lemma 2.3, for any periodic points $p, q \in \Lambda$ we have $W^s(p) \cap W^u(q) \neq \emptyset$. Therefore, we can see that $W^s(p) \pitchfork W^u(q) \neq \emptyset$. Thus $\mathrm{index}(p) = \mathrm{index}(q)$. □

Lemma 2.5. *There is a residual set $\mathcal{G}_3 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_3$, a locally maximal transitive set Λ is the closure of the set $P(f) \cap \Lambda$.*

Proof. Let $f \in \mathcal{G}_3$. Since Λ is transitive, by Pugh's closing lemma, there is a periodic orbit sequence $Orb(p_n)$ of f such that $Orb(p_n)$ converges to Λ. Since Λ is locally maximal, we have $\Lambda = \overline{P(f) \cap \Lambda}$. □

Denote by $\mathcal{F}(\Lambda)$ the set of all local star diffeomorphisms. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

Proposition 2.6. *There is a residual set $\mathcal{G}_4 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_4$, if $f \in \mathcal{F}(\Lambda)$, then there exist a C^1-neighborhood $\mathcal{U}(f)$ of f, constants $C > 0, 0 < \lambda < 1$ and $m \in \mathbb{Z}^+$ such that*

(a) *for each $g \in \mathcal{U}(f)$, if q is a periodic point of g in $\Lambda_g(U)$ with period $\pi(q,g) \geq m$, then*

$$\prod_{i=0}^{k-1} ||Dg^m|_{E^s(g^{im}(q))}|| < C\lambda^k \quad \textit{and} \quad \prod_{i=0}^{k-1} ||Dg^{-m}|_{E^u(g^{-im}(q))}|| < C\lambda^k,$$

where $k = [\pi(q,g)/m]$.

(b) Λ *admits a dominated splitting* $T_\Lambda M = E \oplus F$ *with* $\dim E = \mathrm{index}(p)$.

Proof. Let $f \in \mathcal{G}_4 = \mathcal{G}_2 \cap \mathcal{G}_3$ and let $\mathcal{U}(f)$ be a neighborhood of f. We may assume that $f \in \mathcal{F}(\Lambda)$. Then by Mañé's result, we have the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\mathrm{dimM}}$ generated by $Dg(g \in \mathcal{U}_0(f) \subset \mathcal{U}(f))$ along the hyperbolic periodic points $q \in P(g) \cap \Lambda_g(U)(= \bigcap_{n\in\mathbb{Z}} g^n(U))$ is uniformly hyperbolic. Since $f \in \mathcal{G}_4$, by Lemma 2.4 and Lemma 2.5, we have (a) and (b) (see [6]). $\square$

Then following was proved by Mañé [7, Lemma I.5].

Lemma 2.7. *Let Λ be a closed invariant set of f and $E \subset T_\Lambda M$ be a continuous invariant subbundle. If there is $m > 0$ such that*

$$\int \log \|Df^m|_E\| d\mu < 0$$

for every ergodic $\mu \in \mathcal{M}_{f^m}(\Lambda)$, then E is contracting.

Let us recall Mañé's ergodic closing lemma in [6]. For any $\epsilon > 0$, let $B_\epsilon(f,x)$ be an ϵ-tubular neighborhood of f-orbit of x, that is,

$$B_\epsilon(f,x) = \{y \in M : d(f^n(x), y) < \epsilon \text{ for any } n \in \mathbb{Z}\}.$$

We say that a point $x \in M$ is *well closable* for $f \in \mathrm{Diff}(M)$, if for any $\epsilon > 0$ there is $g \in \mathrm{Diff}(M)$ with $d_1(f,g) < \epsilon$ such that $d(f^n(x), g^n(p)) < \epsilon$ for any $0 \leq n \leq \pi(p)$, where d_1 is the C^1-metric. Let Σ_f denote the set of well closable points of f. In [6], Mañé showed that for any f-invariant Borel probability measure μ on M, $\mu(\Sigma_f) = 1$. Let $\mathcal{M}$ be the space of all Borel measures μ on M with the weak* topology. Then we know that for any ergodic measure $\mu \in \mathcal{M}$

of f, μ is supported on a periodic orbit $Orb(p) = \{p, f(p), \ldots, f^{\pi(p)-1}(p)\}$ if and only if

$$\mu = \frac{1}{\pi(p)} \sum_{i=0}^{\pi(p)-1} \delta_{f^i(p)},$$

where δ_x is the atomic measure respecting x. The following lemma is proved by Lee and Wen [4].

Lemma 2.8 ([4, Lemma 2.3]). *There is a residual set $\mathcal{G}_5 \subset \mathrm{Diff}(M)$ such that every $f \in \mathcal{G}_5$ satisfies the following property: Any ergodic invariant measure μ of f is the limit of sequence of ergodic invariant measures supported by periodic orbits $Orb(p_n)$ of f in the weak* topology. Moreover, the orbits $Orb(p_n)$ converge to the support of μ in the Hausdorff topology.*

Proof of Theorem 1.1. Let $f \in \mathcal{G}_3 \cap \mathcal{G}_4 \cap \mathcal{G}_5$, and let Λ be locally maximal in U. Then by Proposition 2.6(b), Λ admits a dominated splitting $T_\Lambda M = E \oplus F$ such that $E(p) = E^s(p)$ and $F(p) = E^u(p)$ for every $p \in P(f) \cap \Lambda$. Let μ be an ergodic measure with $supp(\mu) \subset \Lambda$. Since $f \in \mathcal{G}_5$, there is a sequence of periodic orbits $Orb(p_n)$ such that $Orb(p_n)$ converges to Λ. We write $Orb(p_n) = P_n$. Then we know

$$\int \log \|Df^m|_E\| d\mu_{P_n} < 0.$$

Thus

$$\lim_{n\to\infty} \int \log \|Df^m|_E\| d\mu_{P_n} = \int \log \|Df^m|_E\| d\mu < 0.$$

Since Λ is locally maximal $P_n \in \Lambda$. Thus by Lemma 2.7, E is contracting. With the same argumentation, F is expanding. □

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