Commun. Korean Math. Soc. **31** (2016), No. 2, pp. 379–388
http://dx.doi.org/10.4134/CKMS.2016.31.2.379

CHARACTERIZATIONS OF SPACE CURVES WITH 1-TYPE DARBOUX INSTANTANEOUS ROTATION VECTOR

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Abstract. In this study, by using Laplace and normal Laplace operators, we give some characterizations for the Darboux instantaneous rotation vector field of the curves in the Euclidean 3-space E^3. Further, we give necessary and sufficient conditions for unit speed space curves to have 1-type Darboux vectors. Moreover, we obtain some characterizations of helices according to Darboux vector.

1. Introduction

One of the most important problems of local differential geometry is to obtain the relations characterizing special curves with respect to their curvature and torsion. The well-known types of such special curves are constant slope curves or general helices which are defined by the property that the tangent vectors of curves make a constant angle with fixed directions. A necessary and sufficient condition for a curve to be a general helix in the Euclidean 3-space E^3 is that the ratio of curvature to torsion is constant [11]. So, many mathematicians have focused their studies on these special curves in different spaces such as Euclidean space and Minkowski space [3, 4, 5, 10].

Furthermore, Chen and Ishikawa [1] classified biharmonic curves, the curves for which $\Delta\vec{H} = 0$ holds in semi-Euclidean space E^n_v where Δ is the Laplacian operator and $\vec{H}$ is mean curvature vector field of a Frenet curve. Later, Kocayiğit [6] has studied the harmonic 1-type curves and weak biharmonic curves i.e., the curves for which $\Delta^\perp\vec{H} = \lambda\vec{H}$ and $\Delta^\perp\vec{H} = 0$ hold along the curve, respectively, where $\Delta^\perp$ is the normal Laplace operator. Also, Kocayiğit and Hacısalihoğlu [7, 8] have studied 1-type curves and biharmonic curves in the Euclidean 3-space E^3 and Minkowski 3-space E^3_1. They have obtained the relations between 1-type curves and circular helix and the relations between biharmonic curves and geodesics. Moreover, Kocayiğit and et al. [9] have given some characterizations for space curves in the Euclidean space E^{2n+1}.

Received July 10, 2015.
2010 *Mathematics Subject Classification.* Primary 14H45, 14H50, 53A04.
Key words and phrases. Darboux vector, Laplacian operator, helix.

In this paper, we give the differential equations of the Darboux vector $\vec{W}$ of a space curve in E^3 and find the equations characterizing the helices. Furthermore, we give some characterizations of curves for which $\Delta\vec{W} = \lambda\vec{W}$, $\Delta\vec{W} = 0$, $\Delta^{\perp}\vec{W}^{\perp} = \lambda\vec{W}^{\perp}$ and $\Delta^{\perp}\vec{W}^{\perp} = 0$ hold, where λ is a constant. According to these conditions, we give the characterizations for helices.

2. Preliminaries

We now review some basic concepts on classical differential geometry of space curves in E^3. Let $\gamma : I \to E^3$ be a unit speed curve. Then, the velocity vector field γ' satisfies $\langle\gamma', \gamma'\rangle = 1$. Let us assume that $\langle\gamma'', \gamma''\rangle \neq 0$ holds. A unit speed curve γ is called a Frenet curve if $\langle\gamma'', \gamma''\rangle \neq 0$ and every Frenet curve γ has an orthonormal Frenet frame $\left\{\vec{V}_1, \vec{V}_2, \vec{V}_3\right\}$ along γ such that $\vec{V}_1 = \gamma'(s)$ and the following Frenet-Serret formulae hold,

$$\begin{bmatrix} \nabla_{\gamma'}\vec{V}_1 \\ \nabla_{\gamma'}\vec{V}_2 \\ \nabla_{\gamma'}\vec{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \end{bmatrix}, \tag{1}$$

where ∇ is the Levi-Civita connection given by $\nabla_{\gamma'} = \frac{d}{ds}$ and s is arclength parameter of the curve γ. The functions κ and τ are called the curvature and torsion, respectively. The vector fields $\vec{V}_1, \vec{V}_2, \vec{V}_3$ are called unit tangent vector field, principle normal vector field and binormal vector field of γ, respectively. The Frenet formulae can be interpreted kinematically as follows: If a moving point traverses the curve in such a way that s is the time parameter, then the moving frame $\left\{\vec{V}_1, \vec{V}_2, \vec{V}_3\right\}$ moves according to equations (1). This motion contains, apart from an instantaneous translation, and instantaneous rotation with angular velocity vector given by the Darboux vector

$$\vec{W} = \tau\vec{V}_1 + \kappa\vec{V}_3.$$

The direction of the Darboux vector is that of instantaneous axis of rotation, and its length $\left\|\vec{W}\right\| = \sqrt{\kappa^2 + \tau^2}$ is the scalar angular velocity. Then, Frenet formulae (1) can be given as follows,

$$\nabla_{\gamma'}\vec{V}_i = \vec{W} \times \vec{V}_i, \quad (1 \leq i \leq 3)$$

where $\times$ shows the vector product in E^3.

Moreover, a curve can be defined by some properties according to its curvature and torsion. Some well-known definitions of such curves can be given as follows.

Definition 2.1 ([5, 6])**.** Let $\gamma : I \to E^3$ be a unit speed curve in E^3. Then we can give the following definitions:

i) The curve $\gamma : I \to E^3$ is a geodesic, if the curvature κ and the torsion τ are zero.

ii) The curve $\gamma : I \to E^3$ is a general helix, if the curvature κ and the torsion τ aren't constants, but $\frac{\kappa}{\tau}$ is constant along the curve.

iii) The curve $\gamma : I \to E^3$ is a circle, if the curvature κ is a non-zero constant and the torsion τ is zero along the curve.

iv) The curve $\gamma : I \to E^3$ is a circular helix, if the curvature κ and the torsion τ are non-zero constants along the curve.

v) If $\frac{\kappa}{\tau} = 0$, then the curve is a line and if $\frac{\kappa}{\tau} = \infty$, then the curve is a plane curve. These special cases are the examples of degenerated helices.

The Laplace operator of γ is defined by

$$\Delta = -\nabla^2_{\gamma'} = -\nabla_{\gamma'}\nabla_{\gamma'}, \tag{2}$$

and the normal connection of γ is defined by

$$\begin{aligned} &\nabla^{\perp}_{\gamma'} = \chi(\gamma(I)) \times \chi(\gamma(I))^{\perp} \to \chi(\gamma(I))^{\perp} \\ &\nabla^{\perp}_{\gamma'}\vec{\xi} = \nabla_{\gamma'}\vec{\xi} - \left\langle \nabla_{\gamma'}\vec{\xi}, \vec{V}_1 \right\rangle \vec{V}_1, \qquad \left(\forall \vec{\xi} \in \chi(\gamma(I))^{\perp}\right) \end{aligned} \tag{3}$$

where $\nabla^{\perp}_{\gamma'}\vec{\xi}$ is the normal component of $\nabla_{\gamma'}\vec{\xi}$ or normal covariant derivative of $\vec{\xi}$ with respect to γ', $\chi(\gamma(I)) = sp\left\{\vec{V}_1(s)\right\}$ and $\chi(\gamma(I))^{\perp} = sp\left\{\vec{V}_2(s),\ \vec{V}_3(s)\right\}$ is the normal bundle of the curve γ. Furthermore, the normal Laplace operator of γ is defined by

$$\Delta^{\perp} = -\nabla^{\perp\,(2)}_{\gamma'} = -\nabla^{\perp}_{\gamma'}\nabla^{\perp}_{\gamma'} \tag{4}$$

(See [1, 2]).

3. Characterizations of space curves according to Darboux vector

In this section, we give the differential equations which characterize the curves in E^3 according to the Darboux vector $\vec{W}$ and normal Darboux vector $\vec{W}^{\perp}$.

Theorem 3.1. *Let γ be a unit speed curve in E^3 with Frenet frame $\left\{\vec{V}_1, \vec{V}_2, \vec{V}_3\right\}$, curvature κ, torsion τ and Darboux vector $\vec{W}$. Then $\vec{W}$ satisfies the following differential equation*

$$\lambda_4\nabla^3_{\gamma'}\vec{W} + \lambda_3\nabla^2_{\gamma'}\vec{W} + \lambda_2\nabla_{\gamma'}\vec{W} + \lambda_1\vec{W} = 0, \tag{5}$$

where

$$\begin{aligned} &\lambda_4 = f^2, \\ &\lambda_3 = -2fg, \\ &\lambda_2 = 2g^2 + \tau f(\tau f + \kappa'') - \kappa f(\tau''' - \kappa f), \\ &\lambda_1 = -\left[2g(\kappa'\tau'' - \kappa''\tau') + \tau' f(\tau f + \kappa'') - \kappa' f(\tau''' - \kappa f)\right], \end{aligned}$$

and $f = \kappa\tau' - \kappa'\tau$ and $g = \kappa\tau'' - \kappa''\tau = f'$.

Proof. Let γ be a unit speed curve with Frenet frame $\left\{\vec{V}_1, \vec{V}_2, \vec{V}_3\right\}$ and Darboux vector

$$\vec{W} = \tau\vec{V}_1 + \kappa\vec{V}_3, \tag{6}$$

where κ and τ are curvature and torsion of the curve, respectively. By differentiating $\vec{W}$ three times with respect to s, we find the followings,

$$\nabla_{\gamma'}\vec{W} = \tau'\vec{V}_1 + \kappa'\vec{V}_3, \tag{7}$$

$$\nabla^2_{\gamma'}\vec{W} = \tau''\vec{V}_1 + (\kappa\tau' - \kappa'\tau)\vec{V}_2 + \kappa''\vec{V}_3, \tag{8}$$

$$\begin{aligned}\nabla^3_{\gamma'}\vec{W} = {} & (\tau''' + \kappa\kappa'\tau - \kappa^2\tau')\vec{V}_1 + (\kappa\tau'' + (\kappa\tau' - \kappa'\tau)' - \kappa''\tau)\,\vec{V}_2 \\ & + (\kappa\tau\tau' - \kappa'\tau^2 + \kappa''')\vec{V}_3.\end{aligned} \tag{9}$$

From (6) and (7) we have

$$\vec{V}_1 = -\left(\frac{\kappa}{\kappa'\tau - \kappa\tau'}\right)\nabla_{\gamma'}\vec{W} + \left(\frac{\kappa'}{\kappa'\tau - \kappa\tau'}\right)\vec{W}, \tag{10}$$

$$\vec{V}_3 = \left(\frac{\tau}{\kappa'\tau - \kappa\tau'}\right)\nabla_{\gamma'}\vec{W} - \left(\frac{\tau'}{\kappa'\tau - \kappa\tau'}\right)\vec{W}. \tag{11}$$

By substituting (10) and (11) in (8) we get

$$\vec{V}_2 = \left(\frac{-1}{\kappa'\tau - \kappa\tau'}\right)\nabla^2_{\gamma'}\vec{W} + \left(\frac{\kappa''\tau - \kappa\tau''}{(\kappa'\tau - \kappa\tau')^2}\right)\nabla_{\gamma'}\vec{W} + \left(\frac{\kappa'\tau'' - \kappa''\tau'}{(\kappa'\tau - \kappa\tau')^2}\right)\vec{W}. \tag{12}$$

If we write (10), (11) and (12) in (9) we get,

$$\begin{aligned}& f^2\nabla^3_{\gamma'}\vec{W} - f(g + f')\nabla^2_{\gamma'}\vec{W} + [g(g + f') + \tau f(\tau f + \kappa''') - \kappa f(\tau''' - \kappa f)]\,\nabla_{\gamma'}\vec{W} \\ & - [(g + f')(\kappa'\tau'' - \kappa''\tau') + \tau' f(\tau f + \kappa''') - \kappa' f(\tau''' - \kappa f)]\,\vec{W} = 0,\end{aligned}$$

where $f = \kappa\tau' - \kappa'\tau$ and $g = \kappa\tau'' - \kappa''\tau$. Since $f' = g$, the last equality becomes

$$\begin{aligned}& f^2\nabla^3_{\gamma'}\vec{W} - 2fg\nabla^2_{\gamma'}\vec{W} + \left[2g^2 + \tau f(\tau f + \kappa''') - \kappa f(\tau''' - \kappa f)\right]\nabla_{\gamma'}\vec{W} \\ & - [2g(\kappa'\tau'' - \kappa''\tau') + \tau' f(\tau f + \kappa''') - \kappa' f(\tau''' - \kappa f)]\,\vec{W} = 0.\end{aligned} \tag{13}$$

By writing

$$\begin{aligned}\lambda_4 &= f^2, \\ \lambda_3 &= -2fg, \\ \lambda_2 &= 2g^2 + \tau f(\tau f + \kappa''') - \kappa f(\tau''' - \kappa f), \\ \lambda_1 &= -\left[2g(\kappa'\tau'' - \kappa''\tau') + \tau' f(\tau f + \kappa''') - \kappa' f(\tau''' - \kappa f)\right],\end{aligned}$$

from (13) we get

$$\lambda_4\nabla^3_{\gamma'}\vec{W} + \lambda_3\nabla^2_{\gamma'}\vec{W} + \lambda_2\nabla_{\gamma'}\vec{W} + \lambda_1\vec{W} = 0,$$

which is desired equation. □

Assume now that γ is not a plane curve. So, we can define a 2-dimensional subbundle, say ϑ, of the normal bundle of γ into E^3 as

$$\vartheta = Sp\left\{\vec{V}_2(s),\ \vec{V}_3(s)\right\},$$

where $\vec{V}_2(s)$ and $\vec{V}_3(s)$ are Frenet vectors. Equations (3) and (4) also give how the normal connection $\nabla^{\perp}_{\gamma'}$ of γ into E^3 behaves on ϑ and we have

$$\vec{W}^{\perp} = \kappa\vec{V}_3, \tag{14}$$

$$\begin{cases} \nabla^{\perp}_{\gamma'}\vec{V}_2 = \tau\vec{V}_3, \\ \nabla^{\perp}_{\gamma'}\vec{V}_3 = -\tau\vec{V}_2, \end{cases} \tag{15}$$

where $\vec{W}^{\perp}$ is a normal Darboux instantaneous rotation vector. Then we give the followings.

Theorem 3.2. *Let γ be a unit speed curve in E^3 with Frenet frame $\left\{\vec{V}_1, \vec{V}_2, \vec{V}_3\right\}$, curvature κ, torsion τ and normal Darboux vector $\vec{W}^{\perp}$. The differential equation characterizing γ according to $\vec{W}^{\perp}$ is given by*

$$\lambda_3\left(\nabla^{\perp}_{\gamma'}\right)^2\vec{W}^{\perp} + \lambda_2\nabla^{\perp}_{\gamma'}\vec{W}^{\perp} + \lambda_1\vec{W}^{\perp} = 0 \tag{16}$$

where

$$\begin{aligned} \lambda_3 &= \kappa^2\tau, \\ \lambda_2 &= -\kappa(\kappa'\tau + (\kappa\tau)'), \\ \lambda_1 &= \kappa'(\kappa'\tau + (\kappa\tau)') - \kappa\tau(\kappa'' - \kappa\tau^2). \end{aligned}$$

Proof. Let γ be a unit speed curve with Frenet frame $\left\{\vec{V}_1, \vec{V}_2, \vec{V}_3\right\}$ and the normal Darboux vector

$$\vec{W}^{\perp} = \kappa\vec{V}_3, \tag{17}$$

where κ and τ are curvature and torsion of the curve, respectively. By differentiating $\vec{W}^{\perp}$ two times with respect to s, we find the followings,

$$\nabla^{\perp}_{\gamma'}\vec{W}^{\perp} = -\kappa\tau\vec{V}_2 + \kappa'\vec{V}_3, \tag{18}$$

$$\left(\nabla^{\perp}_{\gamma'}\right)^2\vec{W}^{\perp} = -(\kappa'\tau + (\kappa\tau)')\vec{V}_2 + (\kappa'' - \kappa\tau^2)\vec{V}_3. \tag{19}$$

From (17) and (18), we have

$$\vec{V}_2 = \frac{1}{\kappa\tau}\left(\frac{\kappa'}{\kappa}\vec{W}^{\perp} - \nabla^{\perp}_{\gamma'}\vec{W}^{\perp}\right). \tag{20}$$

By substituting (17) and (20) in (19), we get

$$\left(\nabla^{\perp}_{\gamma'}\right)^2\vec{W}^{\perp} = \frac{\kappa'\tau + (\kappa\tau)'}{\kappa\tau}\nabla^{\perp}_{\gamma'}\vec{W}^{\perp} + \left(\frac{\kappa\tau(\kappa'' - \kappa\tau^2) - \kappa'(\kappa'\tau + (\kappa\tau)')}{\kappa^2\tau}\right)\vec{W}^{\perp}. \tag{21}$$

Equality (21) gives us

$$\begin{aligned}\kappa^2\tau\left(\nabla^{\perp}_{\gamma'}\right)^2\vec{W}^{\perp}-\kappa(\kappa'\tau+(\kappa\tau)')\nabla^{\perp}_{\gamma'}\vec{W}^{\perp}\\ +\left(\kappa'(\kappa'\tau+(\kappa\tau)')-\kappa\tau(\kappa''-\kappa\tau^2)\right)\vec{W}^{\perp}=0.\end{aligned} \tag{22}$$

By writing

$$\begin{aligned}\lambda_3&=\kappa^2\tau,\\ \lambda_2&=-\kappa(\kappa'\tau+(\kappa\tau)'),\\ \lambda_1&=\kappa'(\kappa'\tau+(\kappa\tau)')-\kappa\tau(\kappa''-\kappa\tau^2),\end{aligned}$$

in (22) we get

$$\lambda_3\left(\nabla^{\perp}_{\gamma'}\right)^2\vec{W}^{\perp}+\lambda_2\nabla^{\perp}_{\gamma'}\vec{W}^{\perp}+\lambda_1\vec{W}^{\perp}=0,$$

which is desired equation. □

If the curve γ is a circular helix, i.e., both κ and τ are non-zero constants along the curve, then from (16) we have the following corollary.

Corollary 3.3. *Let γ be a unit speed curve in E^3 with normal Darboux vector $\vec{W}^{\perp}$. If the curve γ is a circular helix, then the differential equation characterizing the curve according to the normal Darboux vector $\vec{W}^{\perp}$is given by*

$$\left(\nabla^{\perp}_{\gamma'}\right)^2\vec{W}^{\perp}+\tau^2\vec{W}^{\perp}=0. \tag{23}$$

4. Some characterizations of Darboux vector with Laplacian operator Δ

In this section, by considering the Laplace-Beltrami operator Δ, we give some characterizations of space curves according to Darboux instantaneous rotation vector field

$$\vec{W}=\tau\vec{V}_1+\kappa\vec{V}_3, \tag{24}$$

where κ and τ are the curvature and the torsion of a curve γ, respectively. First, we give the following definition.

Definition 4.1. A regular curve $\gamma : I \to E^3$ with Darboux vector $\vec{W}$ is said to have harmonic Darboux vector $\vec{W}$ if $\Delta\vec{W}=0$ holds and is said to have 1-type Darboux vector $\vec{W}$ if $\Delta\vec{W}=\lambda\vec{W}$ holds, where $\lambda\in\mathbb{R}$ is a constant.

Theorem 4.2. *Let γ be a unit speed curve in E^3 with Darboux vector $\vec{W}$. Then for the constant λ, $\Delta\vec{W}=\lambda\vec{W}$ holds along the curve γ, i.e., γ has 1-type Darboux vector if and only if the curvature κ and the torsion τ of the curve γ satisfy the followings,*

$$\tau''=-\lambda\tau,\quad \kappa\tau'-\kappa'\tau=0,\ \ \kappa''=-\lambda\kappa. \tag{25}$$

Proof. Let γ be a unit speed curve in E^3 with Darboux vector $\vec{W}$ and let $\Delta\vec{W} = \lambda\vec{W}$ holds along the curve γ. By using Frenet formula given in (1) and Laplacian operator Δ given in (2), from (24) we get

$$\Delta\vec{W} = -\tau''\vec{V}_1 - (\kappa\tau' - \kappa'\tau)\vec{V}_2 - \kappa''\vec{V}_3. \tag{26}$$

By (26) and using the equality $\Delta\vec{W} = \lambda\vec{W}$, we have

$$-\tau''\vec{V}_1 - (\kappa\tau' - \kappa'\tau)\vec{V}_2 - \kappa''\vec{V}_3 = \lambda\tau\vec{V}_1 + \lambda\kappa\vec{V}_3 \tag{27}$$

which gives that

$$\tau'' = -\lambda\tau, \quad \kappa\tau' - \kappa'\tau = 0, \ \ \kappa'' = -\lambda\kappa.$$

Conversely, if the equations (25) hold for the constant λ, we can write

$$-\tau''\vec{V}_1 + (\kappa\tau' - \kappa'\tau)\vec{V}_2 - \kappa''\vec{V}_3 = \lambda\tau\vec{V}_1 + \lambda\kappa\vec{V}_3$$

which shows that $\Delta\vec{W} = \lambda\vec{W}$ holds. □

Theorem 4.3. *Let γ be a unit speed curve in E^3 with Darboux vector $\vec{W}$. Then, $\Delta\vec{W} = \lambda\vec{W}$ holds along the curve γ if and only if γ is a general helix, with curvature $\kappa = d_1\cos\left(\sqrt{\lambda}s\right) + d_2\sin\left(\sqrt{\lambda}s\right) = d\tau(s)$ and torsion $\tau = d_3\cos\left(\sqrt{\lambda}s\right) + d_4\sin\left(\sqrt{\lambda}s\right)$ where $d,\ d_1,\ d_2,\ d_3,\ d_4$ are constants.*

Proof. Let γ be a unit speed curve with Darboux vector $\vec{W}$ and assume that $\Delta\vec{W} = \lambda\vec{W}$ holds along γ. Then, from Theorem 4.2, we have $\kappa\tau' - \kappa'\tau = 0$ which means that $\frac{\kappa}{\tau}$ is constant, i.e., γ is a general helix. Furthermore, since $\frac{\kappa}{\tau}$ is constant, from the first and third equations in (25) we have

$$\tau = d_3\cos\left(\sqrt{\lambda}s\right) + d_4\sin\left(\sqrt{\lambda}s\right), \kappa = d_1\cos\left(\sqrt{\lambda}s\right) + d_2\sin\left(\sqrt{\lambda}s\right) = d\tau(s),$$

respectively, where $d,\ d_1,\ d_2,\ d_3,\ d_4$ are non-zero constants.

Conversely, if γ is a general helix with curvature $\kappa = d_1\cos\left(\sqrt{\lambda}s\right) + d_2\sin\left(\sqrt{\lambda}s\right) = d\tau(s)$ and torsion $\tau = d_3\cos\left(\sqrt{\lambda}s\right) + d_4\sin\left(\sqrt{\lambda}s\right)$, we have

$$\tau'' = -\lambda\tau, \quad \kappa\tau' - \kappa'\tau = 0, \quad \kappa'' = -\lambda\kappa.$$

Then, from Theorem 4.2 we see that $\Delta\vec{W} = \lambda\vec{W}$ holds along the curve γ where λ is constant. □

From Theorem 4.3, we have the following corollary.

Corollary 4.4. *Let γ be a unit speed curve in E^3 with Darboux vector $\vec{W}$ and non zero curvatures κ, τ. Then, $\Delta\vec{W} = 0$ holds along the curve γ if and only if γ is a general helix with curvature $\kappa(s) = m_1s + m_2 = m\tau(s)$ and torsion $\tau(s) = m_3s + m_4$ where $m_1,\ m_2,\ m_3,\ m_4,\ m$ are constants.*

Proof. Assume that $\Delta\vec{W} = 0$ holds. By taking $\lambda = 0$ in Theorem 4.2, we have

$$\tau'' = 0, \;\; \kappa\tau' - \kappa'\tau = 0, \;\; \kappa'' = 0. \tag{28}$$

The second equation of (28) gives us that $\frac{\kappa}{\tau}$ is constant i.e., γ is a general helix, and from the first and third equations of (28), we obtain that $\kappa(s) = m_1 s + m_2 = m\tau(s)$ and torsion $\tau(s) = m_3 s + m_4$ where $m,\ m_1,\ m_2,\ m_3,\ m_4$ are constants.

Conversely, if γ is a general helix with curvature $\kappa(s) = m_1 s + m_2 = m\tau(s)$ and torsion $\tau(s) = m_3 s + m_4$ where $m,\ m_1,\ m_2,\ m_3,\ m_4$ are constants, we have the equalities (28). So, Theorem 4.2 shows that $\Delta\vec{W} = 0$ holds along the curve γ.

□

Theorem 4.5. *Let γ be a unit speed curve in E^3 with Darboux vector $\vec{W}$. Then, for the constants λ and μ,*

$$\Delta\vec{W} + \lambda\nabla_{\gamma'}\vec{W} + \mu\vec{W} = 0 \tag{29}$$

holds along the curve γ if and only if γ is a general helix with curvature

$$\kappa = n_1 \exp\left(\frac{\lambda + \sqrt{\lambda^2 + 4\mu}}{2}s\right) + n_2 \exp\left(\frac{\lambda - \sqrt{\lambda^2 + 4\mu}}{2}s\right) = n\tau(s)$$

and the torsion

$$\tau = n_3 \exp\left(\frac{\lambda + \sqrt{\lambda^2 + 4\mu}}{2}s\right) + n_4 \exp\left(\frac{\lambda - \sqrt{\lambda^2 + 4\mu}}{2}s\right),$$

where $n,\ n_1,\ n_2,\ n_3,\ n_4$ are constants.

Proof. Assume that (29) holds along the curve γ. Then from the equalities (6), (7) and (26) we have

$$\begin{cases} \tau'' - \lambda\tau' - \mu\tau = 0, \\ \kappa\tau' - \kappa'\tau = 0, \\ \kappa'' - \lambda\kappa' - \mu\kappa = 0. \end{cases} \tag{30}$$

The second equation of the system (30) gives that $\frac{\kappa}{\tau}$ is constant, i.e., γ is a general helix and from the first and third equations of system (30), we obtain

$$\tau = n_3 \exp\left(\frac{\lambda + \sqrt{\lambda^2 + 4\mu}}{2}s\right) + n_4 \exp\left(\frac{\lambda - \sqrt{\lambda^2 + 4\mu}}{2}s\right) \tag{31}$$

and

$$\kappa = n_1 \exp\left(\frac{\lambda + \sqrt{\lambda^2 + 4\mu}}{2}s\right) + n_2 \exp\left(\frac{\lambda - \sqrt{\lambda^2 + 4\mu}}{2}s\right) = n\tau(s), \tag{32}$$

respectively, where $n,\ n_1,\ n_2,\ n_3,\ n_4$ are constants.

Conversely, if γ is a general helix with curvature κ and torsion τ given by (32) and (31), respectively, it is easily seen that (29) holds. □

Example 4.6. Let consider the Frenet curve $\gamma : I \to E^3$ with the parametrization

$$\gamma(s) = \left(3\cos\frac{s}{5},\ 3\sin\frac{s}{5},\ \frac{4}{5}s\right).$$

Frenet vectors and curvatures of γ are obtained as follows:

$$\begin{aligned}
&\vec{V}_1(s) = \left(-\tfrac{3}{5}\sin\tfrac{s}{5},\ \tfrac{3}{5}\cos\tfrac{s}{5},\ \tfrac{4}{5}\right),\\
&\vec{V}_2(s) = \left(-\cos\tfrac{s}{5},\ -\sin\tfrac{s}{5},\ 0\right),\\
&\vec{V}_3(s) = \left(\tfrac{4}{5}\sin\tfrac{s}{5},\ -\tfrac{4}{5}\cos\tfrac{s}{5},\ \tfrac{3}{5}\right),\\
&\kappa = \tfrac{3}{25},\quad \tau = \tfrac{4}{25}.
\end{aligned}$$

Then the Darboux vector is given by $\vec{W} = \left(0, 0, \frac{1}{5}\right)$ and it is easily seen that $\Delta\vec{W} = 0$ holds along the curve γ.

5. Some characterizations of Darboux vector with normal Laplace operator $\boldsymbol{\Delta}^{\perp}$

Let us denote the normal Laplace operator of γ by $\Delta^{\perp}$ and normal Darboux instantaneous rotation vector field along γ by $\vec{W}^{\perp}$. Then, we can give the followings:

Theorem 5.1. *Let γ be a unit speed curve in E^3 with Darboux vector $\vec{W}$. Then, for the constant λ, $\Delta^{\perp}\vec{W}^{\perp} = \lambda\vec{W}^{\perp}$ holds along the curve γ if and only if*

$$\kappa'' + (\lambda - \tau^2)\kappa = 0, \quad 2\kappa'\tau + \tau'\kappa = 0, \tag{33}$$

holds.

Proof. Let γ be a unit speed curve in E^3 with normal Darboux vector $\vec{W}^{\perp}$ and assume that $\Delta^{\perp}\vec{W}^{\perp} = \lambda\vec{W}^{\perp}$ holds along the curve γ for the constant λ. From equations (14) and (15) we have

$$\Delta^{\perp}\vec{W}^{\perp} = (2\kappa'\tau + \kappa\tau')\vec{V}_2 - (\kappa'' - \kappa\tau^2)\vec{V}_3. \tag{34}$$

Using the equality $\Delta^{\perp}\vec{W}^{\perp} = \lambda\vec{W}^{\perp}$, from (34) we have

$$(2\kappa'\tau + \kappa\tau')\vec{V}_2 - (\kappa'' - \kappa\tau^2)\vec{V}_3 = \lambda\kappa\vec{V}_3, \tag{35}$$

which gives that

$$\kappa'' + (\lambda - \tau^2)\kappa = 0, \quad 2\kappa'\tau + \tau'\kappa = 0.$$

Conversely, if the equations (33) are satisfied, then it is easily seen that the equality $\Delta^{\perp}\vec{W}^{\perp} = \lambda\vec{W}^{\perp}$ holds along the curve γ for the constant λ. □

Corollary 5.2. *Let γ be a unit speed curve in E^3 with Darboux vector $\vec{W}$, non-zero curvature function κ and non-zero constant torsion τ. Then, $\Delta^{\perp}\vec{W}^{\perp} = \lambda\vec{W}^{\perp}$ holds along the curve γ if and only if γ is a circular helix with torsion $\tau^2 = \lambda$.*

Proof. Assume that $\Delta^{\perp}\vec{W}^{\perp} = \lambda\vec{W}^{\perp}$ holds along the curve γ and τ be a non-zero constant. Then from second equation of (33) we have that κ is a constant which gives that γ is a circular helix. Moreover, from the first equation of (33) it is obtained that $\tau^2 = \lambda$.

Conversely, if γ is a circular helix with torsion $\tau^2 = \lambda$, then it is easily seen that $\Delta^{\perp}\vec{W}^{\perp} = \lambda\vec{W}^{\perp}$ holds along the curve. □

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