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STATICAL HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD OF A QUASI-CONSTANT CURVATURE

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Abstract. In this paper, we study half lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that the characteristic vector field ζ of $\bar{M}$ is tangent to M. First, we provide a new result for such a half lightlike submanifold. Next, we investigate a statical half lightlike submanifold M of $\bar{M}$ subject such that (1) the screen distribution $S(TM)$ is totally umbilical or (2) M is screen conformal.

1. Introduction

In the theory of Riemannian geometry, Chen-Yano [2] introduced the notion of a *Riemannian manifold of a quasi-constant curvature* as a Riemannian manifold $(\bar{M}, \bar{g})$ equipped with a curvature tensor $\bar{R}$ of the following form:

$$\begin{aligned}\bar{R}(X,Y)Z = {} & f_1\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ & + f_2\{\theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y \\ & + \bar{g}(Y,Z)\theta(X)\zeta - \bar{g}(X,Z)\theta(Y)\zeta\}\end{aligned} \tag{1.1}$$

for any vector fields X, Y and Z on $\bar{M}$, where f_1 and f_2 are smooth functions, ζ is a unit vector field which is called the *characteristic vector field* of $\bar{M}$, and θ is a 1-form associated with ζ by $\theta(X) = \bar{g}(X, \zeta)$. It is well known that if $f_2 = 0$, then $\bar{M}$ is reduced to a space of constant curvature.

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics. Half lightlike submanifold M is a lightlike submanifold of codimension 2 such that $rank\{Rad(TM)\} = 1$, where $Rad(TM) = TM \cap TM^{\perp}$ is the radical distribution of M. It is a special case of general r-lightlike submanifold [4] such that $r = 1$. Its geometry is more general than that of lightlike hypersurface or coisotropic submanifold which is lightlike submanifolds M of codimension

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2 such that $rank\{Rad(TM)\} = 2$. Much of its theory will be immediately generalized in a formal way to arbitrary r-lightlike submanifolds.

In this paper, we study half lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that the characteristic vector field ζ of $\bar{M}$ is tangent to M. First, we provide a new result for such a half lightlike submanifold. Next, we investigate a statical half lightlike submanifold M of such an indefinite Kaehler manifold $\bar{M}$ subject such that (1) the screen distribution $S(TM)$ is totally umbilical or (2) M is screen conformal.

2. Preliminaries

Let (M, g) be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with the tangent bundle TM, the normal bundle $TM^{\perp}$, the radical distribution $Rad(TM) = TM \cap TM^{\perp}$, a screen distribution $S(TM)$, and a coscreen distribution $S(TM^{\perp})$ such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \qquad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We follow Duggal-Jin [5] for notations and structure equations used in this article. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E. Also denote by $(2.6)_1$ the first equation of the two equations in (2.6). We use same notations for any others. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit spacelike vector field, *i.e.*, $\bar{g}(L, L) = 1$, without loss of generality. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$, of rank 3. Certainly the vector fields ξ and L belong to $\Gamma(S(TM)^{\perp})$. Hence we have the following orthogonal decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$, of rank 2. It is known [5] that, for any null section ξ of $Rad(TM)$, there exists a uniquely defined null vector field N in $S(TM^{\perp})^{\perp}$ satisfying

$$\bar{g}(\xi, N) = 1,\ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0,\ \forall\, X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the subbundle of $S(TM^{\perp})^{\perp}$ locally spanned by N. We see that $S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to the screen distribution $S(TM)$ respectively.

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{2.1}$$

$$\bar{\nabla}_X N = -A_{_N} X + \tau(X)N + \rho(X)L, \tag{2.2}$$

$$\bar{\nabla}_X L = -A_{_L} X + \phi(X)N; \tag{2.3}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.4}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.5}$$

where ∇ and ∇^* are the induced connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local screen second fundamental form* on $S(TM)$, A_N, A^*_ξ and A_L are called the *shape operators*, and τ, ρ and ϕ are 1-forms on TM.

From now and in the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and the second fundamental forms B and D are symmetric. The above local second fundamental forms are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.6}$$

$$C(X, PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{2.7}$$

$$D(X, Y) = g(A_L X, Y) - \phi(X)\eta(Y), \qquad \bar{g}(A_L X, N) = \rho(X), \tag{2.8}$$

where η is a 1-form given by $\eta(X) = \bar{g}(X, N)$. From $(2.6)_1$ and $(2.8)_1$, we get

$$B(X, \xi) = 0, \qquad D(X, \xi) = -\phi(X). \tag{2.9}$$

A^*_ξ and A_N are $S(TM)$-valued, and A^*_ξ is self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{2.10}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y). \tag{2.11}$$

But the induced connection ∇^* on $S(TM)$ is a metric connection.

Definition. A half lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *statical* [11, 12] if $\bar{\nabla}_X L \in \Gamma(S(TM))$ for any $X \in \Gamma(TM)$.

From (2.3) and $(2.8)_2$, we show that the above definition is equivalent to the conditions: $\phi = 0$ and $\rho = 0$. The condition $\phi = 0$ is equivalent to the conception: M is *irrotational*, *i.e.*, $\bar{\nabla}_X \xi \in \Gamma(TM)$ [14]. The condition $\rho = 0$ is equivalent to the conception: M is *solenoidal*, *i.e.*, $A_L X \in \Gamma(S(TM))$ [13].

We need the following Gauss-Codazzi equations (for a full set of these equations see [5]). Denote by $\bar{R}$, R and R^* the curvature tensors of $\bar{\nabla}$, ∇ and ∇^* respectively. Using the local Gauss-Weingarten formulas, we have

$$\begin{aligned}\bar{R}(X, Y)Z = {} & R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ & + D(X, Z)A_L Y - D(Y, Z)A_L X \\ & + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ & + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \\ & + \phi(X)D(Y, Z) - \phi(Y)D(X, Z)\}N \\ & + \{(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) + \rho(X)B(Y, Z) \\ & - \rho(Y)B(X, Z)\}L,\end{aligned} \tag{2.12}$$

$$\begin{aligned}(2.13)\qquad \bar{R}(X,Y)N = &-\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y]\\ &+ \tau(X)A_N Y - \tau(Y)A_N X + \rho(X)A_L Y - \rho(Y)A_L X\\ &+ \{B(Y, A_N X) - B(X, A_N Y) + 2d\tau(X,Y)\\ &+ \phi(X)\rho(Y) - \phi(Y)\rho(X)\}N\\ &+ \{D(Y, A_N X) - D(X, A_N Y) + 2d\rho(X,Y)\\ &+ \rho(X)\tau(Y) - \rho(Y)\tau(X)\}L,\end{aligned}$$

$$\begin{aligned}(2.14)\qquad R(X,Y)PZ = &\, R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A_\xi X\\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ)\\ &- \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ)\}\xi,\end{aligned}$$

$$\begin{aligned}(2.15)\qquad R(X,Y)\xi = &-\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y]\\ &- \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X\\ &+ \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X,Y)\}\xi.\end{aligned}$$

In the case $R = 0$, we say that M is *flat*. We set dim $\bar{M} = n + 3$.

The *Ricci tensor* $\bar{Ric}$ of $\bar{M}$ is defined by

$$\bar{Ric}(X,Y) = trace\{Z \to \bar{R}(X,Z)Y\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(T\bar{M}).$$

Denote by $R^{(0,\,2)}$ the induced tensor of type (0, 2) on M such that

$$(2.16)\qquad R^{(0,\,2)}(X,\, Y) = trace\{Z \to R(X,\, Z)Y\}\,, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

Due to [6], using (2.6)∼(2.8) and the Gauss equation (2.12), we get

$$\begin{aligned}(2.17)\qquad R^{(0,\,2)}(X,Y) = &\,\bar{Ric}(X,Y) + B(X,Y)tr\,A_N + D(X,Y)tr\,A_L\\ &- g(A_N X,\, A^*_\xi Y) - g(A_L X,\, A_L Y) + \rho(X)\phi(Y)\\ &- \bar{g}(\bar{R}(\xi,Y)X,\, N) - \bar{g}(\bar{R}(L,Y)X,\, L).\end{aligned}$$

Using (2.13) and the first Bianchi's identity, we obtain

$$R^{(0,\,2)}(X,\, Y) - R^{(0,\,2)}(Y,\, X) = 2d\tau(X,\, Y).$$

This shows that $R^{(0,\,2)}$ is not symmetric. A tensor field $R^{(0,\,2)}$ of M is called its *induced Ricci tensor* and denoted by Ric if it is symmetric. In this case, M is called *Ricci flat* if $Ric = 0$. M is called an *Einstein manifold* if there exists a smooth function κ such that

$$(2.18)\qquad Ric = \kappa g.$$

Let $\nabla^\ell_X N = \pi(\bar{\nabla}_X N)$, where π is the projection morphism of $T\bar{M}$ on $ltr(TM)$. Then ∇^ℓ is a linear connection on $ltr(TM)$. We say that ∇^ℓ is a *lightlike transversal connection*. Define a curvature tensor R^ℓ on $ltr(TM)$ by

$$R^\ell(X,Y)N = \nabla^\ell_X\nabla^\ell_Y N - \nabla^\ell_Y\nabla^\ell_X N - \nabla^\ell_{[X,Y]}N.$$

If R^ℓ vanishes identically, then the lightlike transversal connection ∇^ℓ is said to be *flat*. This definition comes from the definition of *flat normal connection* [1] in the theory of classical geometry of non-degenerate submanifolds. We quote the following result (see [9, 10]).

Theorem 2.1. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. The following assertions are equivalent*:

(1) *The lightlike transversal connection of M is flat, i.e., $R^\ell = 0$.*
(2) *The 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*
(3) *The Ricci type tensor $R^{(0,2)}$ is an induced Ricci tensor of M.*

Note 1. $d\tau$ is independent to the choice of the section $\xi \in \Gamma(TM^\perp)$. Indeed, suppose τ and $\bar{\tau}$ are 1-forms with respect to the sections ξ and $\bar{\xi}$, respectively. By directed calculation, it follows that $d\tau = d\bar{\tau}$ [5]. In case $d\tau = 0$, by the cohomology theory, there exists a smooth function f such that $\tau = df$. Consequently we get $\tau(X) = X(f)$. If we take $\bar{\xi} = \lambda\xi$, it follows that $\tau(X) = \bar{\tau}(X) + X(\text{In}\,\lambda)$. Setting $\lambda = \exp(f)$ in this equation, we get $\bar{\tau} = 0$. Thus if $d\tau = 0$, we can take a 1-form τ such that $\tau = 0$. We call the pair $\{\xi, N\}$ whose corresponding 1-form τ vanishes the *canonical null pair* of M.

3. Indefinite Kaehler manifolds

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a real even dimensional indefinite Kaehler manifold, where $\bar{g}$ is a semi-Riemannian metric of index $q = 2v$, $0 < v < \frac{1}{2}(\dim \bar{M})$, and J is an almost complex structure on $\bar{M}$ such that, for all $X, Y \in \Gamma(T\bar{M})$,

$$J^2 = -I, \qquad \bar{g}(JX, JY) = \bar{g}(X, Y), \qquad (\bar{\nabla}_X J)Y = 0. \tag{3.1}$$

Let (M, g) be a half lightlike submanifold of an indefinite Kaeler manifold $\bar{M}$. Due to [7, 8], we choose a screen distribution $S(TM)$ such that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are vector subbundles of $S(TM)$. In this case, the screen distribution $S(TM)$ is expressed as follow:

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^\perp)) \oplus_{orth} H_o,$$

where H_o is a non-degenerate almost complex distribution with respect to J, *i.e.*, $J(H_o) = H_o$. Denote $H' = J(ltr(TM)) \oplus_{orth} J(S(TM^\perp))$. Then

$$TM = H \oplus H', \tag{3.2}$$

where H is a 2-lightlike almost complex distribution on M such that

$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

Consider two lightlike and one spacelike vector fields $\{U, V\}$ and W such that

$$U = -JN, \quad V = -J\xi, \quad W = -JL. \tag{3.3}$$

Denote by S the projection morphism of TM on H. By (3.2), for any vector field X on M, JX is expressed as follow

$$JX = FX + u(X)N + w(X)L, \tag{3.4}$$

where u, v and w are 1-forms locally defined on M by

$$u(X) = g(X,\, V), \quad v(X) = g(X,\, U), \quad w(X) = g(X, W) \tag{3.5}$$

and F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$. Applying $\bar{\nabla}_X$ to (3.3) and using the Gauss-Weingarten formulas, we have

$$B(X, U) = C(X, V),\ C(X, W) = D(X, U),\ B(X, W) = D(X, V), \tag{3.6}$$

$$\nabla_X U = F(A_N X) + \tau(X)U + \rho(X)W, \tag{3.7}$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \phi(X)W, \tag{3.8}$$

$$\nabla_X W = F(A_L X) + \phi(X)U. \tag{3.9}$$

Theorem 3.1. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that ζ is tangent to M. Then*

$$f_1 = 0, \qquad f_2\theta(V) = f_2\theta(W) = 0, \qquad f_2\alpha = 0.$$

Proof. Comparing the tangential, lightlike transversal and co-screen components of the two equations (1.1) and (2.12), we get

$$\begin{aligned} R(X, Y)Z = {} & f_1\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\} \\ & + f_2\{\bar{g}(Y, Z)\theta(X)\zeta - \bar{g}(X, Z)\theta(Y)\zeta \\ & + \theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y\} \\ & + B(Y, Z)A_N X - B(X, Z)A_N Y \\ & + D(Y, Z)A_L X - D(X, Z)A_L Y, \end{aligned} \tag{3.10}$$

$$\begin{aligned} (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \\ + \phi(X)D(Y, Z) - \phi(Y)D(X, Z) = 0, \end{aligned} \tag{3.11}$$

$$(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) + \rho(X)B(Y, Z) - \rho(Y)B(X, Z) = 0. \tag{3.12}$$

Taking the scalar product with N to (2.14), we have

$$\begin{aligned} g(R(X, Y)PZ,\, N) = {} & (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ & - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ). \end{aligned}$$

Substituting (3.10) into this equation and using $(2.7)_2$ and $(2.8)_2$, we obtain

$$\begin{aligned} & (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ & - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ) \\ & - \rho(X)D(Y, PZ) + \rho(Y)D(X, PZ) \\ = {} & f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ & + f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ) \\ & + \alpha f_2\{\theta(X)g(Y, PZ) - \theta(Y)g(X, PZ)\}. \end{aligned} \tag{3.13}$$

Applying ∇_X to $(3.6)_1 : B(Y, U) = C(Y, V)$, we have

$$(\nabla_X B)(Y, U) = (\nabla_X C)(Y, V) + g(A_N Y, \nabla_X V) - g(A^*_\xi Y, \nabla_X U).$$

Using (3.1), (3.4) and (3.6)∼(3.8), the last equation is reduced to

$$\begin{aligned}&(\nabla_X B)(Y,U)\\&= (\nabla_X C)(Y,V) - 2\tau(X)C(Y,V) - \phi(X)D(Y,U) - \rho(X)D(Y,V)\\&\quad - g(A^*_\xi X, F(A_N Y)) - g(A^*_\xi Y, F(A_N X)).\end{aligned}$$

Substituting this equation into (3.11) such that $Z = U$ and using (3.7), we get

$$\begin{aligned}&(\nabla_X C)(Y,V) - (\nabla_Y C)(X,V) - \tau(X)C(Y,V) + \tau(Y)C(X,V)\\&- \rho(X)D(Y,V) + \rho(Y)D(X,V) = 0.\end{aligned}$$

Comparing this equation with (3.13) such that $PZ = V$, we obtain

$$\begin{aligned}&f_1\{\eta(X)u(Y) - \eta(Y)u(X)\} + f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(V)\\&+ f_2\alpha\{\theta(X)u(Y) - \theta(Y)u(X)\} = 0.\end{aligned} \tag{3.14}$$

Replacing Y by ξ to this equation and using the fact that $\theta(\xi) = 0$, we have

$$f_1 u(X) + f_2\theta(X)\theta(V) = 0.$$

Taking $X = V$ and $X = U$ to this equation by turns, we get

$$f_2\theta(V) = 0, \qquad f_1 + f_2\theta(U)\theta(V) = 0.$$

From these two equations, we see that $f_1 = 0$. Taking $X = \zeta$ and $Y = U$ to (3.14) and using the facts that $u(\zeta) = \theta(V)$ and $f_2\theta(V) = 0$, we have $f_2\alpha = 0$.

Applying ∇_X to $(3.6)_2$: $D(Y,U) = C(Y,W)$, and using (2.7), (2.8) and (3.7), we have

$$\begin{aligned}(\nabla_X D)(Y,U) &= (\nabla_X C)(Y,W) + g(A_N Y, \nabla_X W)\\&\quad - g(A_L Y, \nabla_X U) + \phi(Y)C(X,U).\end{aligned}$$

Using $(2.8)_2$, (3.1), (3.4), (3.6), (3.7) and (3.9), we have

$$\begin{aligned}&(\nabla_X D)(Y,U)\\&= (\nabla_X C)(Y,W) - \tau(X)C(Y,W) - \rho(X)D(Y,W) - \rho(X)B(Y,U)\\&\quad + \phi(X)C(Y,U) + \phi(Y)C(X,U) - g(A_L X, F(A_N Y)) - g(A_L Y, F(A_N X)).\end{aligned}$$

Substituting this equation into (3.11) such that $Z = U$ and using (3.7), we get

$$\begin{aligned}&(\nabla_X C)(Y,W) - (\nabla_Y C)(X,W) - \tau(X)C(Y,W) + \tau(Y)C(X,W)\\&- \rho(X)D(Y,W) + \rho(Y)D(X,W) = 0.\end{aligned}$$

Comparing this equation with (3.13) such that $PZ = W$, we obtain

$$f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(W) = 0.$$

Taking $Y = \zeta$ and $Y = \xi$ to this equation, we get $f_2\theta(W) = 0$. □

4. Totally umbilical screen distribution

If $\bar{M}$ is an indefinite Kaehler manifold of quasi-constant curvature, using (1.1) and the fact that $f_1 = 0$, we see that $\bar{g}(\bar{R}(\xi, X)Y, N) = f_2\theta(X)\theta(Y)$, $\bar{g}(\bar{R}(\xi, X)Y, N) = f_2\theta(X)\theta(Y)$ and $\bar{Ric}(X, Y) = f_2\{g(X, Y) + (n+1)\theta(X)\theta(Y)\}$. Thus (2.17) is reduced to

$$R^{(0,2)}(X, Y) = f_2\{g(X, Y) + (n-1)\theta(X)\theta(Y)\} + \rho(X)\phi(Y) + B(X, Y)tr\, A_N + D(X, Y)tr\, A_L - g(A_N X,\, A^*_\xi Y) - g(A_L X, A_L Y). \tag{4.1}$$

Definition. A screen distribution $S(TM)$ is called *totally umbilical* [4, 5] in M if there exists a smooth function γ such that $A_N X = \gamma PX$, or equivalently,

$$C(X, PY) = \gamma g(X, Y). \tag{4.2}$$

In case $\gamma = 0$, we say that $S(TM)$ is *totally geodesic* in M.

Note 2. If M is irrotational and $S(TM)$ is totally umbilical, then (4.1) reduces

$$R^{(0,2)}(X, Y) = f_2\{g(X, Y) + (n-1)\theta(X)\theta(Y)\} + B(X, Y)tr\, A_N + D(X, Y)tr\, A_L - \gamma g(X,\, A^*_\xi Y) - g(A_L X, A_L Y). \tag{4.3}$$

As A^*_ξ is self-adjoint, it follows that $R^{(0,2)}$ is symmetric, *i.e.*, $R^{(0,2)}$ is the induced Ricci tensor Ric of M. Therefore, $d\tau = 0$ and the transversal connection is flat by Theorem 2.1. As $d\tau = 0$, we can take $\tau = 0$ by Note 1.

Theorem 4.1. *Let M be a tatical half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that ζ is tangent to M. If $S(TM)$ is totally umbilical, then we have the following results*:

(1) *$S(TM)$ is totally geodesic and parallel distribution,*
(2) *M is locally a product manifold $\mathcal{C}_\xi \times M^*$, where $\mathcal{C}_\xi$ is a null geodesic tangent to $TM^\perp$, and M^* is a leaf of $S(TM)$,*
(3) *$f_1 = f_2 = 0$, i.e., $\bar{M}$ is flat, and the curvature tensor R is given by*

$$R(X, Y)Z = D(Y, Z)A_L X - D(X, Z)A_L Y.$$

(4) *Moreover, if M is an Einstein manifold, then M is Ricci flat.*

Proof. As M is statical, the 1-forms ϕ and ρ are satisfied $\phi = \rho = 0$. Applying ∇_X to $C(Y, PZ) = \gamma g(Y, PZ)$ and using (2.11), we have

$$(\nabla_X C)(Y, PZ) = (X\gamma)g(Y, PZ) + \gamma B(X, PZ)\eta(Y).$$

Substituting this and (4.2) into (3.13) with $f_1 = f_2\alpha = \rho = 0$, we obtain

$$(X\gamma)g(Y, Z) - (Y\gamma)g(X, Z) + \gamma\{B(X, Z)\eta(Y) - B(Y, Z)\eta(X)\} = f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z). \tag{4.4}$$

Taking $Y = U$, $Z = V$ and $Y = V$, $Z = U$ to (4.4) by turns and using $(3.9)_1$, (4.2) and the facts that $f_2\theta(V) = 0$ and $\eta(V) = 0$, we obtain

$$X\gamma = (U\gamma)u(X), \qquad X\gamma = (V\gamma)v(X).$$

From these equations, we get $X\gamma = 0$. Thus γ is a constant. (4.4) reduces

$$\{\gamma B(X,Z) + f_2\theta(X)\theta(Z)\}\eta(Y) = \{\gamma B(Y,Z) + f_2\theta(Y)\theta(Z)\}\eta(X).$$

Taking $Y = \xi$ to this equation and using $(2.9)_1$, we have

$$\gamma B(X,Y) = -f_2\theta(X)\theta(Y). \tag{4.5}$$

Taking $Y = U$ to this equation and using (3.3), (3.5) and (4.2), we have

$$\gamma^2 u(X) = -f_2\theta(X)\theta(U). \tag{4.6}$$

Assume that $f_2 \neq 0$. Taking $X = \zeta$ to (4.6), we have

$$\gamma^2\theta(V) = -f_2\theta(U).$$

As $f_2 \neq 0$, if we product with f_2 to the last equation and use the fact that $f_2\theta(V) = 0$, then we obtain $f_2\theta(U) = 0$. Taking $X = U$ to (4.6) and using the fact that $f_2\theta(U) = 0$, we get $\gamma = 0$. As $\gamma = 0$, taking $X = Y = \zeta$ to (4.5), we have $f_2 = 0$. It is a contradiction. Therefore, $f_2 = 0$. As $f_2 = 0$, from (4.6), we obtain $\gamma = 0$.

(1) As $\gamma = C = 0$, $S(TM)$ is totally geodesic and, from (2.4) we see that $S(TM)$ is a parallel distribution.

(2) As $S(TM)$ is a parallel distribution, $Rad(TM)$ is also an auto-parallel distribution due to (2.5) and (2.10). As $TM = Rad(TM) \oplus S(TM)$, by the decomposition theorem of de Rham [3], M is locally a product manifold $\mathcal{C}_\xi \times M^*$, where $\mathcal{C}_\xi$ is a null geodesic tangent to $Rad(TM)$ and M^* is a leaf of $S(TM)$.

(3) As $f_1 = f_2 = 0$, $\bar{M}$ is flat. As $f_1 = f_2 = A_{_N} = 0$, from (3.10), R is given by

$$R(X,Y)Z = D(Y,Z)A_{_L}X - D(X,Z)A_{_L}Y.$$

(4) As $C = 0$, using (2.6), (2.8) and $(3.6)_{1,\,2}$, we have

$$B(X,U) = 0, \qquad D(X,U) = 0, \qquad A^*_\xi U = 0, \qquad A_{_L}X = 0.$$

Substituting (2.18) into (4.1) such that $f_2 = 0$ and $Y = U$ and then, using the last equations, we obtain $\kappa = 0$. Therefore, M is Ricci flat. □

Theorem 4.2. *Let M be an Einstein statical half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that ζ is tangent to M. If $S(TM)$ is totally umbilical, then M is Ricci flat.*

Proof. As $C = 0$, from $(3.6)_2$ and the facts that $\phi = 0$ and $\rho = 0$, we obtain

$$D(X,\,U) = 0, \qquad A_{_L}U = 0. \tag{4.7}$$

As $f_2 = \gamma = A_{_N} = 0$, from (4.3), the induced Ricci tensor $R^{(0,\,2)}$ is given by

$$R^{(0,\,2)}(X,Y) = D(X,Y)tr\,A_{_L} - g(A_{_L}X, A_{_L}Y), \tag{4.8}$$

where $\ell = tr\, A_L$. As M is Einstein, substituting (2.18) into (4.8), we have

$$g(A_L X, A_L Y) - \ell g(A_L X, Y) + \kappa g(X, Y) = 0.$$

Taking $X = U$ and $Y = V$ to this equation and using (4.7), we obtain $\kappa = 0$. Therefore M is Ricci flat. □

Denote by $\mathcal{G} = J(Rad(TM)) \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o$. Then $\mathcal{G}$ is a complementary vector subbundle to $J(ltr(TM))$ in $S(TM)$ and we have

$$S(TM) = J(ltr(TM)) \oplus \mathcal{G}.$$

Theorem 4.3. *Let M be a statical half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that ζ is tangent to M. If $S(TM)$ is totally umbilical, then M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_\xi$ and $\mathcal{C}_U$ are null geodesics tangent to $Rad(TM)$ and $J(ltr(TM))$ respectively and $M^\sharp$ is a leaf of $\mathcal{G}$.*

Proof. As M is statical and $S(TM)$ is totally umbilical, we have

$$\nabla_X U = 0, \tag{4.9}$$

due to $A_N = \tau = \rho = 0$. Thus $J(ltr(TM))$ is a parallel distribution on M. From (2.5) and (2.10), $Rad(TM)$ is also a parallel distribution on M. Using (4.9), we derive

$$g(\nabla_X Y,\, U) = 0, \qquad g(\nabla_X V,\, U) = 0, \qquad g(\nabla_X W,\, U) = 0,$$

for all $X \in \Gamma(\mathcal{G})$ and $Y \in \Gamma(H_o)$. Thus $\mathcal{G}$ is also parallel. By the decomposition theorem of de Rham [3], M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_\xi$ and $\mathcal{C}_U$ are null geodesics tangent to $Rad(TM)$ and $J(ltr(TM))$ respectively and $M^\sharp$ is a leaf of $\mathcal{G}$. □

5. Screen conformal lightlike hypersurfaces

Definition. A half lightlike submanifold M is called *screen conformal* [6, 7] if there exists a non-vanishing function φ such that $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, Y). \tag{5.1}$$

If φ is a non-zero constant, then we say that M is *screen homothetic*.

Note 3. If M is irrotational and screen conformal, then (4.1) is reduced to

$$\begin{aligned} R^{(0,2)}(X, Y) = {} & f_2\{g(X, Y) + (n-1)\theta(X)\theta(Y)\} \\ & + B(X, Y) tr\, A_N + D(X, Y) tr\, A_L \\ & - \varphi g(A^*_\xi X,\, A^*_\xi Y) - g(A_L X, A_L Y). \end{aligned} \tag{5.2}$$

Thus $R^{(0,2)}$ is symmetric, $d\tau = 0$ and the transversal connection is flat by Theorem 2.1. In this section, since $d\tau = 0$, we also take $\tau = 0$ as Section 4.

Proposition 5.1. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that ζ is tangent to M. If M is irrotational and screen conformal, then the curvature function f_2 is satisfied $f_2\theta(U) = 0$. Moreover, M is statical and screen homothetic, then $f_2 = 0$.*

Proof. Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this into (3.13) such that $\tau = 0$ and using (3.11), we obtain

$$\begin{aligned}&(X\varphi)B(Y, PZ) - (Y\varphi)B(X, PZ) - \rho(X)D(Y, PZ) + \rho(Y)D(X, PZ)\\ &= f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ).\end{aligned}$$

Replacing Y by ξ to this and using (2.9) and the fact that $\theta(\xi) = 0$, we get

$$(\xi\varphi)B(X, Y) - \rho(\xi)D(X, Y) = f_2\theta(X)\theta(Y). \tag{5.3}$$

Taking $Y = V$ to (5.3) and using $(3.6)_3$ and the fact that $f_2\theta(V) = 0$, we have

$$(\xi\varphi)B(X, V) - \rho(\xi)B(X, W) = 0.$$

Replacing Y by U to (5.3) and using $(3.6)_{1,\,2}$, we have

$$(\xi\varphi)C(X, V) - \rho(\xi)C(X, W) = f_2\theta(X)\theta(U).$$

From the last two equations and (5.1), we obtain $f_2\theta(X)\theta(U) = 0$. Replacing X by ζ, we get $f_2\theta(U) = 0$. If M is statical and screen homothetic, then $\xi\varphi = 0$ and $\rho(\xi) = 0$. Therefore, taking $X = Y = \zeta$ to (5.3), we get $f_2 = 0$. □

Theorem 5.2. *Let M be an Einstein half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that ζ is tangent to M. If M is irrotational and screen conformal, then the function κ, given by* (2.18), *is satisfied $\kappa = f_2$. Moreover, M is statical and screen homothetic, then it is Ricci flat, i.e., $\kappa = 0$.*

Proof. As $\{U, V\}$ is a null basis of $J(Rad(TM)) \oplus J(ltr(TM))$, the vector fields

$$\mu = U - \varphi V, \qquad \nu = U + \varphi V$$

form an orthogonal basis of $J(Rad(TM)) \oplus J(ltr(TM))$. From (3.5) and (5.1), we obtain

$$B(X, \mu) = 0, \qquad A^*_\xi \mu = 0. \tag{5.4}$$

From (2.8), $(3.6)_{2,\,3}$ and the fact that $\phi = 0$, we also obtain

$$D(X, \mu) = 0, \qquad A_L\mu = \rho(\mu)\xi. \tag{5.5}$$

As $f_2\theta(V) = 0$ and $f_2\theta(U) = 0$, we also have

$$f_2\theta(\mu) = 0, \qquad f_2\theta(\nu) = 0. \tag{5.6}$$

Taking $X = Y = \mu$ to (5.2) and using (5.4)~(5.6), we have $\kappa = f_2$. If M is statical and screen homothetic, then $\kappa = 0$ as $f_2 = 0$. □

Let $\mathcal{H}' = Span\{\mu\}$. Then $\mathcal{H} = H_o \oplus_{orth} Span\{\nu, W\}$ is a complementary vector subbundle to $\mathcal{H}'$ in $S(TM)$ and we have the following decomposition

$$S(TM) = \mathcal{H}' \oplus_{orth} \mathcal{H}. \tag{5.7}$$

Theorem 5.3. *Let M be a statical and screen homothetic half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that ζ is tangent to M. Then M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\natural$, where $\mathcal{C}_\xi$ and $\mathcal{C}_\mu$ are null and non-null geodesics tangent to $Rad(TM)$ and $\mathcal{H}'$, respectively and $M^\natural$ is a leaf of $\mathcal{H}$.*

Proof. As M is statical and screen homothetic, using (3.7), (3.8) and the fact that F is linear operator, we have

$$\nabla_X \mu = 0. \tag{5.8}$$

This implies that $\mathcal{H}'$ is a parallel distribution on M. From (2.5) and (2.10), $Rad(TM)$ is also a parallel distribution on M. Using (5.8), we derive

$$g(\nabla_X Y, \mu) = 0, \qquad g(\nabla_X \nu, \mu) = 0, \qquad g(\nabla_X W, \mu) = 0,$$

for all $X \in \Gamma(\mathcal{H})$ and $Y \in \Gamma(H_o)$. Thus $\mathcal{H}$ is also parallel. By the decomposition theorem of de Rham [3], M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\natural$, where $\mathcal{C}_\xi$ and $\mathcal{C}_\mu$ are null and non-null geodesics tangent to $Rad(TM)$ and $\mathcal{H}'$ respectively and $M^\natural$ is a leaf of $\mathcal{H}$. □

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