

SOME INEQUALITIES AND ABSOLUTE MONOTONICITY FOR MODIFIED BESSEL FUNCTIONS OF THE FIRST KIND

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ABSTRACT. By employing a refined version of the Pólya type integral inequality and other techniques, the authors establish some inequalities and absolute monotonicity for modified Bessel functions of the first kind with nonnegative integer order.

1. Main results

It is well known that modified Bessel functions of the first kind $I_{\pm\nu}(z)$ are solutions of the differential equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 + \nu^2)w = 0.$$

They are holomorphic functions of z throughout the z -plane cut along the negative real axis, and are entire functions of ν for fixed $z \neq 0$. When $\nu = \pm n$, $I_\nu(z)$ are entire functions of z . In [1, p. 375, 9.6.7], it is listed that

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k+\nu}, \quad z \in \mathbb{C}, \quad \nu \in \mathbb{R} \setminus \{-1, -2, \dots\},$$

where

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

is the classical gamma function, see [1, p. 255, 6.1.2].

On [12, p. 63], the following three double inequalities are derived:

$$(1.1) \quad \frac{1 - z/2}{1 + z/2} < \Gamma(\nu + 1) \left(\frac{2}{z}\right)^\nu \frac{I_\nu(z)}{e^z} \\ < \frac{1}{2\nu + 3} + \frac{2(\nu + 1)}{2\nu + 3} \left[1 + \frac{(2\nu + 3)z}{2(\nu + 1)}\right]^{-1}, \quad z > 0, \quad \nu \geq -\frac{1}{2};$$

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$$(1.2) \quad \left[1 - \frac{z}{2(\nu+1)}\right] \left[1 + \frac{(2\nu+1)z}{2(\nu+1)}\right]^{-1} < \Gamma(\nu+1) \left(\frac{2}{z}\right)^\nu \frac{I_\nu(z)}{e^z}$$

$$< \frac{1-2\nu}{2\nu+3} + \frac{2(2\nu+1)}{2\nu+3} \left[1 + \frac{(2\nu+3)z}{2(2\nu+1)}\right]^{-1}, \quad z > 0, \quad -\frac{1}{2} \leq \nu \leq \frac{1}{2};$$

$$(1.3) \quad \frac{1}{e^z} < \Gamma(\nu+1) \left(\frac{2}{z}\right)^\nu \frac{I_\nu(z)}{e^z} < \frac{1}{2} \left(1 + \frac{1}{e^{2z}}\right), \quad z > 0, \quad \nu > -\frac{1}{2}.$$

The left-hand inequality in (1.1) is very weak unless z is quite small.

The equation (3.20) in [9, p. 226] reads that

$$(1.4) \quad \Gamma(\nu+1) \left(\frac{2}{y}\right)^\nu I_\nu(y) > \left(1 + \frac{y^2}{j_{\nu,1}^2}\right)^{j_{\nu,1}^2/4(\nu+1)}, \quad y > 0, \quad \nu > -1$$

which is more stringent than the left hand side inequality in (1.3), where $j_{\nu,1}$ is the first zero of the Bessel function of the first kind

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k! \Gamma(\nu+k+1)}, \quad \nu \in \mathbb{R}, \quad z \in \mathbb{C}.$$

In [18, Theorem 7.1] and [21, Theorem 1.3], it was derived that

$$(1.5) \quad \alpha I_1(x) > \frac{(x/2)^3}{1 - e^{-(x/2)^2}} \quad \text{and} \quad I_1(x) \geq \frac{\frac{1}{\beta} \left(\frac{x}{2}\right)^3}{1 - \exp\left[-\frac{1}{\beta} \left(\frac{x}{2}\right)^2\right]}$$

on $(0, \infty)$ if and only if $\alpha \geq 1$ and $\beta \geq 1$. More strongly, it was discovered in [18, Theorem 5.1] that

(1) when $\beta \geq 1$, the function

$$F_\beta(u) = \frac{u}{1 - e^{-u}} \frac{\sqrt{\beta u}}{I_1(2\sqrt{\beta u})}$$

is decreasing on $(0, \infty)$;

(2) when $0 < \beta < 1$, it is unimodal (that is, it has a unique maximum) and $\frac{1}{F_\beta(u)}$ is convex on $(0, \infty)$.

When $\alpha = 1$ or $\beta = 1$ and $x > 5.14\dots$, the inequality in (1.5) is better than (1.4).

In [17, Theorem 1.2], it was obtained that

(1) when $1 \leq k \leq 5$, the inequality

$$(1.6) \quad \frac{I_k(2\sqrt{u})}{u^{k/2}} \geq \frac{d^{k-1}}{d u^{k-1}} \left(\frac{u}{1 - e^{-u}} \right)$$

is valid on $(0, \infty)$;

(2) when $\beta \geq 1$, the function

$$G_\beta(u) = \frac{\beta u}{I_2(2\sqrt{\beta u})} \frac{d}{d u} \left(\frac{u}{1 - e^{-u}} \right)$$

is decreasing on $(0, \infty)$; when $0 < \beta < 1$, it is unimodal and $\frac{1}{G_\beta(u)}$ is convex on $(0, \infty)$.

The inequality (1.6) for $k = 1$ includes the ones in (1.5) for $\alpha = \beta = 1$. For more information on recent results of Bessel functions, please refer to [3, 4] and closely related references therein.

Recall from [13, 22, 23] that a function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I such that

$$0 \leq (-1)^k f^{(k)}(x) < \infty$$

for $x \in I$ and $k \geq 0$. Recall also from [13, 22, 23] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and

$$0 \leq f^{(k-1)}(t) < \infty$$

for $t \in I$ and $k \in \mathbb{N}$, where \mathbb{N} denotes the set of all positive integers. It is easy to see that a function $f(x)$ is completely monotonic in (a, b) if and only if $f(-x)$ is absolutely monotonic in $(-b, -a)$. See [23, p. 145, Definition 2c]. Theorem 12a in [23, p. 160] reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic in $0 \leq x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is bounded and non-decreasing and the integral converges for $0 \leq x < \infty$. Theorem 12c in [23, p. 162] states that a necessary and sufficient condition that $f(x)$ should be absolutely monotonic in $-\infty < x < 0$ is that

$$f(x) = \int_0^\infty e^{xt} d\alpha(t),$$

where $\alpha(t)$ is non-decreasing and the integral converges for $-\infty < x < 0$. For more information on these kinds of functions, please refer to [13, Chapter XIII], [23, Chapter IV], [6, 19, 22] and closely related references therein.

The main aim of this paper is, by employing a refined version of the Pólya type integral inequality and other techniques, to establish some lower and upper bounds and absolute monotonicity for modified Bessel functions of the first kind $I_n(t)$ with nonnegative integer order $n \geq 0$.

The main results may be stated as the following theorems.

Theorem 1.1. *The double inequalities*

$$(1.7) \quad |I_0(t) - \cosh t| \leq \left[1 - \frac{2}{\pi} \frac{\sinh t}{\sqrt{\frac{\sqrt{1+4t^2}-1}{2}} \exp \frac{\sqrt{1+4t^2}-1}{2}} \right] \sinh t, \quad t > 0$$

and

$$(1.8) \quad |I_0(t) - \cosh t| < \frac{\pi}{4} t \sinh t, \quad t \neq 0$$

are valid.

Theorem 1.2. *The function*

$$I_0(t) - \frac{\sinh t}{t}$$

is absolutely monotonic on $(0, \infty)$ and complete monotonic on $(-\infty, 0)$. Hence, we have

$$I_0^{(\ell)}(t) \geq \left(\frac{\sinh t}{t}\right)^{(\ell)}, \quad t > 0$$

and

$$(-1)^\ell I_0^{(\ell)}(t) \geq (-1)^\ell \left(\frac{\sinh t}{t}\right)^{(\ell)}, \quad t < 0$$

for $\ell \geq 0$. Consequently,

(1) *when $t > 0$,*

$$(1.9) \quad I_0(t) > \frac{\sinh t}{t} \quad \text{and} \quad I_1(t) > \frac{t \cosh t - \sinh t}{t^2}.$$

(2) *when $t < 0$, the first inequality in (1.9) keeps the same direction, but the second inequality in (1.9) reverses.*

2. A lemma

In order to prove Theorem 1.1, we need the Pólya type integral inequality below.

Lemma 2.1 ([2, Theorem 2] and [14, Proposition 2]). *If $f(x)$ is continuous and not identically a constant on $[a, b]$, and if $f(x)$ is differentiable and $m \leq f'(x) \leq M$ on (a, b) , then*

$$(2.1) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq - \frac{[M - S_0(a, b)][m - S_0(a, b)]}{2(M - m)}(b - a),$$

where

$$S_0(a, b) = \frac{f(b) - f(a)}{b - a}.$$

For more detailed information on the inequality (2.1), please refer to [16, pp. 25–31, Section 5] and closely related references therein.

3. Proofs of Theorems 1.1 and 1.2

Now we are in a position to prove Theorems 1.1 and 1.2.

Proof of the inequality (1.7). Let $f_t(x) = e^{t \cos x}$ on $[0, \pi]$ and $t > 0$. Then

$$f'_t(x) = -t \sin x e^{t \cos x}, \quad f''_t(x) = t(t \sin^2 x - \cos x) e^{t \cos x},$$

$$f_t(0) = e^t, \quad f_t(\pi) = e^{-t}, \quad f'_t(0) = f'_t(\pi) = 0.$$

Since the equation $t \sin^2 x - \cos x = t - \cos x - t \cos^2 x = 0$ of the variable x has only one positive root $x_0 = \arccos \frac{\sqrt{1+4t^2}-1}{2t}$ and

$$f'_t(x_0) = -\sqrt{\frac{\sqrt{1+4t^2}-1}{2}} \exp \frac{\sqrt{1+4t^2}-1}{2}$$

which is the minimum of $f'_t(x)$ for $x \in [0, \pi]$.

In [1, p. 376, 9.6.16], it was listed that

$$(3.1) \quad I_0(z) = \frac{1}{\pi} \int_0^\pi e^{\pm z \cos \theta} d\theta = \frac{1}{\pi} \int_0^\pi \cosh(z \cos \theta) d\theta.$$

Combining (2.1) and (3.1), using the above properties of the function $f_t(x)$, and directly arranging give

$$\left| I_0(t) - \frac{e^t + e^{-t}}{2} \right| \leq \frac{e^t - e^{-t}}{2} \left[1 - \frac{\sqrt{2}}{\pi} \frac{e^t - e^{-t}}{\sqrt{\sqrt{1+4t^2}-1}} \exp \frac{1 - \sqrt{1+4t^2}}{2} \right]$$

which is equivalent to the double inequality (1.7). The proof of the inequality (1.7) is complete. \square

Proof of the double inequality (1.8). Let $h_t(x) = \cosh(t \cos x)$ for $x \in [0, \pi]$ and $t \in \mathbb{R}$. Then

$$h_t(0) = h_t(\pi) = \cosh t.$$

Since $f_t(x)$ is symmetric with respect to $x_0 = \frac{\pi}{2}$, its derivative $f'_t(x)$ is anti-symmetric with respect to $x_0 = \frac{\pi}{2}$. Hence, we have

$$\max_{x \in [0, \pi]} h'_t(x) = - \min_{x \in [0, \pi]} h'_t(x).$$

Applying $h_t(x)$ to the inequality (2.1) yields

$$|I_0(t) - \cosh t| \leq \frac{\pi}{4} \max_{x \in [0, \pi]} h'_t(x).$$

A simple computation gives

$$h'_t(x) = -t \sin x \sinh(t \cos x) \leq t \sinh t, \quad x \in [0, \pi].$$

Therefore, the double inequality (1.8) follows immediately. \square

Proof of Theorem 1.2. From the power series expansion

$$\frac{\sinh t}{t} = \sum_{k=0}^\infty \frac{t^{2k}}{(2k+1)!}$$

and

$$I_0(t) = \sum_{k=0}^\infty \frac{1}{(k!)^2} \left(\frac{t}{2}\right)^{2k},$$

it follows that

$$(3.2) \quad I_0(t) - \frac{\sinh t}{t} = \sum_{k=0}^\infty \left[\frac{1}{(2k+1)!} - \frac{1}{2^{2k}(k!)^2} \right] t^{2k}.$$

Since

$$(2k+1)! = (2k+1)!!(2k)!! > [(2k)!!]^2 = 2^{2k}(k!)^2,$$

the coefficients $\frac{1}{(2k+1)!} - \frac{1}{2^{2k}(k!)^2}$ in the power series (3.2) is nonnegative for all $k \geq 0$. This implies the absolute monotonicity on $(0, \infty)$ and complete monotonicity on $(-\infty, 0)$ of the function $I_0(t) - \frac{\sinh t}{t}$. As a result, by definitions of the absolute and complete monotonicity, we obtain

$$\left[I_0(t) - \frac{\sinh t}{t} \right]^{(\ell)} \geq 0, \quad t > 0$$

and

$$(-1)^\ell \left[I_0(t) - \frac{\sinh t}{t} \right]^{(\ell)}, \quad t < 0$$

for $\ell \geq 0$. The proof of Theorem 1.2 is complete. \square

4. Remarks and comparisons

Remark 4.1. The lower bound in the double inequality (1.7) is better than $\frac{\sinh t}{t}$, but the lower bound in the double inequality (1.8) is not better than $\frac{\sinh t}{t}$.

When $0 < t < 0.523\dots$, the double inequality (1.8) is better than (1.7); when $t > 0.523\dots$, the double inequality (1.7) is better than (1.8).

Remark 4.2. If $\nu = 0$, the inequalities in (1.1) and (1.2) become the same inequality

$$(4.1) \quad \frac{2-z}{2+z}e^z < I_0(z) < \frac{6+3z}{3(2+3z)}e^z, \quad z > 0$$

and the inequality (1.3) is reduced to

$$(4.2) \quad 1 < I_0(z) < \frac{1}{2} \left(e^z + \frac{1}{e^z} \right), \quad z > 0.$$

It is easy to see that the first inequality in (1.9) is better than the lower bounds in inequalities (4.1) and (4.2).

The inequality (1.8) is worse than the double inequalities (4.1) and (4.2).

The upper bound in (1.7) is worse than the upper bounds of the double inequalities (4.1) and (4.2).

When $x > 0.85\dots$ and $x > 1.1\dots$ respectively, the lower bound in (1.7) is better than the lower bounds in the double inequalities (4.1) and (4.2).

In conclusion, the inequalities for I_0 obtained in Theorems 1.1 and 1.2 are more or less significant.

Remark 4.3. When $\nu = 1$, from (1.1) and (1.3), it follows that

$$(4.3) \quad \frac{z(2-z)}{2(z+2)}e^z < I_1(z) < \frac{z(z+4)}{2(5z+4)}e^z$$

and

$$(4.4) \quad \frac{z}{2} < I_1(z) < \frac{z}{4} \left(e^z + \frac{1}{e^z} \right)$$

for $z > 0$. The inequality (1.4) for $\nu = 1$ is

$$(4.5) \quad I_1(y) > \frac{y}{2} \left(1 + \frac{y^2}{j_{1,1}^2} \right)^{j_{1,1}^2/8}$$

for $y > 0$, where $j_{1,1} = 3.83 \dots$ is the first zero of J_1 .

It is clear that the inequality (4.5) is better than the left-hand side inequalities in (4.3) and (4.4).

When $t > 1.2894 \dots$, the second inequality in (1.9) is better than the left hand side inequality in (4.3).

When $t > 6.14 \dots$, the second inequality in (1.9) is better than (4.5).

When $t > 5.898 \dots$, the second inequality in (1.9) is better than the corresponding one in (1.5) for $\alpha = \beta = 1$.

In a word, the inequality for I_1 in Theorem 1.2 is somewhat significant.

Remark 4.4. The inequality (2.1) was generalized in [15] as

$$\begin{aligned} & \frac{m(b^3 - a^3)}{6} + \frac{\{f(a) - f(b) + bf'(b) - af'(a) + m(a^2 - b^2)/2\}^2}{2[(a - b)m - f'(a) + f'(b)]} \\ & \leq \int_a^b f(x) \, dx - bf(b) + af(a) + \frac{b^2f'(b) - a^2f'(a)}{2} \\ & \leq \frac{M(b^3 - a^3)}{6} + \frac{\{f(a) - f(b) + bf'(b) - af'(a) + M(a^2 - b^2)/2\}^2}{2[(a - b)M - f'(a) + f'(b)]}, \end{aligned}$$

where $f(x)$ is a 2-times differentiable function satisfying $m \leq f''(x) \leq M$ on $[a, b]$. For more information on the inequality (2.1) and its generalizations, please refer to the papers [2, 5, 10, 11, 15], especially the expository and survey article [16], and plenty references therein.

Remark 4.5. The method used in this paper has been employed in [7] to evaluate the complete elliptic integrals. It can also be used to derive bounds for

$$I_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_0^\pi e^{\pm z \cos \theta} \sin^{2\nu} \theta \, d\theta, \quad \Re(\nu) > -\frac{1}{2}.$$

Remark 4.6. The first inequality in (1.9) has been applied in [20]. All inequalities for I_0 can be applied as did in [20].

Remark 4.7. This paper is a revised version of the preprint [8].

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