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CERTAIN INTEGRALS INVOLVING ${}_2F_1$, KAMPÉ DE FÉRIET FUNCTION AND SRIVASTAVA POLYNOMIALS

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Abstract. A remarkably large number of integrals whose integrands are associated, in particular, with a variety of special functions, for example, the hypergeometric and generalized hypergeometric functions have been recorded. Here we aim at presenting certain (presumably) new and (potentially) useful integral formulas whose integrands are involved in a product of ${}_2F_1$, Srivastava polynomials, and Kampé de Fériet functions. The main results are derived with the help of some known definite integrals obtained earlier by Qureshi *et al.* [4]. Some interesting special cases of our main results are also considered.

1. Introduction and preliminaries

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ denote the sets of complex numbers, real numbers and positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For our purpose, we begin by recalling the generalized Kampé de Fériet function (see [9]; see also [8, pp. 26–33]):

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{lll} (a_p): & (b_q); & (c_k); \\ (\alpha_l): & (\beta_m); & (\gamma_n); \end{array} x,\, y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r y^r}{r!s!}, \tag{1.1}$$

where p, q, k, l, m, $n \in \mathbb{N}_0$. For the detailed account of the convergence of the series in (1.1), one may refer to [8, p. 27].

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In 1972, Srivastava [7] introduced the following family of polynomials:

$$S_n^m(x) := \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k \quad (n \in \mathbb{N}_0)\,, \tag{1.2}$$

where $A_{n,k}$ $(n,\, k \in \mathbb{N}_0)$ are arbitrary (real or complex) constants. It is noted that $S_n^m(x)$ includes a number of known polynomials as its special cases, for example, Jacobi polynomials, Bessel polynomials, Laguerre polynomials, Brafman polynomials and several others (see [10, pp. 158–161]).

The following integral formulas are required in our present investigation (see [4, p. 77, Eqs. 3.1, 3.2 and 3.3]):

$$\int_0^\infty \left[\left(ax + \frac{b}{x}\right)^2 + c\right]^{-p-1} dx = \frac{\sqrt{\pi}}{2a(4ab+c)^{p+1/2}} \frac{\Gamma(p+1/2)}{\Gamma(p+1)} \tag{1.3}$$

$$(a > 0;\, b \geq 0;\, c + 4ab > 0;\, \Re(p) + 1/2 > 0)\,;$$

$$\int_0^\infty \frac{1}{x^2} \left[\left(ax + \frac{b}{x}\right)^2 + c\right]^{-p-1} dx = \frac{\sqrt{\pi}}{2b(4ab+c)^{p+1/2}} \frac{\Gamma(p+1/2)}{\Gamma(p+1)} \tag{1.4}$$

$$(a \geq 0;\, b > 0;\, c + 4ab > 0;\, \Re(p) + 1/2 > 0)\,;$$

$$\int_0^\infty \left(a + \frac{b}{x^2}\right) \left[\left(ax + \frac{b}{x}\right)^2 + c\right]^{-p-1} dx = \frac{\sqrt{\pi}}{(4ab+c)^{p+1/2}} \frac{\Gamma(p+1/2)}{\Gamma(p+1)} \tag{1.5}$$

$$(a > 0;\, b > 0;\, c + 4ab > 0;\, \Re(p) + 1/2 > 0)\,.$$

The following Orr's relation connecting products of hypergeometric series is also needed (see, *e.g.*, [5, p. 75]): If

$$(1-y)^{\alpha+\beta-\gamma}\, {}_2F_1(2\alpha, 2\beta; 2\gamma; y) := \sum_{k=0}^{\infty} A_k\, y^k,$$

then

$${}_2F_1\left(\alpha, \beta; \gamma + \frac{1}{2}; y\right) {}_2F_1\left(\gamma - \alpha, \gamma - \beta; \gamma + \frac{1}{2}; y\right) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\left(\gamma + \frac{1}{2}\right)_k} A_k\, y^k. \tag{1.6}$$

With the aid of the integral formulas (1.3)-(1.5) due to Qureshi *et al.* [4], here, we aim at presenting certain (presumably) new and (potentially) useful integral formulas whose integrands are involved in a product of ${}_2F_1$, Srivastava polynomials, and Kampé de Fériet functions. Some interesting special cases of our main results are also considered.

2. Main results

We begin by stating three main integral formulas asserted by Theorems 1–3. To do this, for simplicity, in what follows, let X be denoted by

$$X := \left(ax + \frac{b}{x}\right)^2 + c. \tag{2.1}$$

Theorem 1. *Let p, q, k, l, m, $n \in \mathbb{N}_0$, and $M \in \mathbb{N}$. Let a, b, c, α, β, $\gamma \in \mathbb{R}$ with $a > 0$, $b \geq 0$, $c + 4ab > 0$, and $-\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$. Let λ, $\mu \in \mathbb{C}$ with $\Re(\lambda) + 1/2 > 0$. Let the coefficients A_R and $A_{N,L}$ (L, N, $R \in \mathbb{N}_0$) be arbitrary (real or complex) constants. Then the following integral formula holds true:*

(2.2)

$$\begin{aligned}
&\int_0^\infty X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X)\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)\\
&\times S_N^M[yX^\mu]\,F^{p:q;k}_{l:m;n}\left[\begin{array}{c}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{array}\ \frac{x}{X},\frac{y}{X}\right]dx\\
=&\frac{\sqrt{\pi}}{2a(4ab+c)^{\lambda+1/2}}\sum_{R=0}^{\infty}\sum_{L=0}^{[N/M]}A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\frac{(-N)_{ML}}{L!}A_{N,L}[y(4ab+c)^\mu]^L\\
&\times\frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)}F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{c}(a_p);\lambda-R-\mu L+1/2:(b_q);(c_k);\\(\alpha_l);\lambda-R-\mu L+1:(\beta_m);(\gamma_n);\end{array}\ \frac{x}{4ab+c},\frac{y}{4ab+c}\right],
\end{aligned}$$

where X is given as in (2.1).

Theorem 2. *Let p, q, k, l, m, $n \in \mathbb{N}_0$, and $M \in \mathbb{N}$. Let a, b, c, α, β, $\gamma \in \mathbb{R}$ with $a > 0$, $b \geq 0$, $c + 4ab > 0$, and $-\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$. Let λ, $\mu \in \mathbb{C}$ with $\Re(\lambda) + 1/2 > 0$. Let the coefficients A_R and $A_{N,L}$ (L, N, $R \in \mathbb{N}_0$) be arbitrary (real or complex) constants. Then the following integral formula holds true:*

(2.3)

$$\begin{aligned}
&\int_0^\infty \frac{1}{x^2}X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X)\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)\\
&\times S_N^M[yX^\mu]\,F^{p:q;k}_{l:m;n}\left[\begin{array}{c}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{array}\ \frac{x}{X},\frac{y}{X}\right]dx\\
=&\frac{\sqrt{\pi}}{2b(4ab+c)^{\lambda+1/2}}\sum_{R=0}^{\infty}\sum_{L=0}^{[N/M]}A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\frac{(-N)_{ML}}{L!}A_{N,L}[y(4ab+c)^\mu]^L\\
&\times\frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)}F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{c}(a_p);\lambda-R-\mu L+1/2:(b_q);(c_k);\\(\alpha_l);\lambda-R-\mu L+1:(\beta_m);(\gamma_n);\end{array}\ \frac{x}{4ab+c},\frac{y}{4ab+c}\right],
\end{aligned}$$

where X is given as in (2.1).

Theorem 3. *Let p, q, k, l, m, $n \in \mathbb{N}_0$, and $M \in \mathbb{N}$. Let a, b, c, α, β, $\gamma \in \mathbb{R}$ with $a > 0$, $b \geq 0$, $c + 4ab > 0$, and $-\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$. Let λ, $\mu \in \mathbb{C}$ with*

$\Re(\lambda)+1/2>0$. *Let the coefficients* A_R *and* $A_{N,L}$ $(L,\ N,\ R\in\mathbb{N}_0)$ *be arbitrary* (*real or complex*) *constants. Then the following integral formula holds true*:

(2.4)

$$\begin{aligned}&\int_0^\infty\left(a+\frac{b}{x^2}\right)X^{-\lambda-1}{}_2F_1\left(\alpha,\beta;\gamma+1/2;X\right){}_2F_1\left(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X\right)\\&\times S_N^M\left[yX^\mu\right]F^{p:q;k}_{l:m;n}\left[\begin{array}{c}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{array}\ \frac{x}{X},\frac{y}{X}\right]dx\\=&\frac{\sqrt{\pi}}{(4ab+c)^{\lambda+1/2}}\sum_{R=0}^\infty\sum_{L=0}^{[N/M]}A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\frac{(-N)_{ML}}{L!}A_{N,L}\left[y(4ab+c)^\mu\right]^L\\&\times\frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)}F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{c}(a_p);\lambda-R-\mu L+1/2:(b_q);(c_k);\\(\alpha_l);\lambda-R-\mu L+1:(\beta_m);(\gamma_n);\end{array}\ \frac{x}{4ab+c},\frac{y}{4ab+c}\right],\end{aligned}$$

where X *is given as in* (2.1).

Proof. We prove only Theorem 1. To do this, let $\mathcal{L}$ be the left-hand side of Eq. (2.2). Using (1.6), we get

$$\begin{aligned}\mathcal{L}=&\int_0^\infty X^{-\lambda-1}S_N^M\left[yX^\mu\right]F^{p:q;k}_{l:m;n}\left[\begin{array}{c}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{array}\ \frac{x}{X},\frac{y}{X}\right]\\&\times\left\{\sum_{R=0}^\infty A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\right\}dx.\end{aligned}\tag{2.5}$$

Then expressing Srivastava polynomials and Kampé de Fériet function in (2.5) in series with the help of (1.2) and (1.1), respectively, and interchanging the order of integration and summations, which may be guaranteed by the stated conditions, we find

(2.6)

$$\begin{aligned}\mathcal{L}=&\sum_{R=0}^\infty\sum_{L=0}^{[N/M]}A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\frac{(-N)_{ML}}{L!}A_{N,L}y^L\\&\times\sum_{r,s=0}^\infty\frac{\prod_{j=1}^p(a_j)_{r+s}\prod_{j=1}^q(b_j)_r\prod_{j=1}^k(c_j)_s}{\prod_{j=1}^l(\alpha_j)_{r+s}\prod_{j=1}^m(\beta_j)_r\prod_{j=1}^n(\gamma_j)_s}\frac{x^ry^s}{r!s!}\int_0^\infty X^{-\lambda+R+\mu L-r-s-1}dx.\end{aligned}$$

Applying (1.3) to (2.6), we obtain

$$\mathcal{L}=\sum_{R=0}^\infty\sum_{L=0}^{[N/M]}A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\frac{(-N)_{ML}}{L!}A_{N,L}y^L\tag{2.7}$$

$$\times \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \frac{x^r y^s}{r!s!}$$
$$\times \frac{\sqrt{\pi}}{2a\,(4ab+c)^{\lambda-R-\mu L+r+s+1/2}} \frac{\Gamma\left(\lambda-R-\mu L+r+s+1/2\right)}{\Gamma\left(\lambda-r-\mu L+r+s+1\right)}.$$

After a little simplification, we have

$$\mathcal{L} = \frac{\sqrt{\pi}}{2a\,(4ab+c)^{\lambda+1/2}} \sum_{R=0}^{\infty} \sum_{L=0}^{[N/M]} A_R \frac{(\gamma)_R}{(\gamma+1/2)_R} \frac{(-N)_{ML}}{L!} A_{N,L} \tag{2.8}$$
$$\times y^L (4ab+c)^{\mu L} \frac{\Gamma\left(\lambda-R-\mu L+1/2\right)}{\Gamma\left(\lambda-R-\mu L+1\right)}$$
$$\times \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \frac{(\lambda-R-\mu L+1/2)_{r+s}}{(\lambda-R-\mu L+1)_{r+s}}$$
$$\times \left(\frac{x}{4ab+c}\right)^r \left(\frac{y}{4ab+c}\right)^s \frac{1}{r!s!}.$$

Finally, interpreting (2.8) in terms of (1.1) is easily seen to arrive at the right-hand side of (2.2). This completes the proof of Theorem 1.

A similar argument as in the proof of Theorem 1 will establish Theorems 2 and 3. Details of their proofs are omitted. □

3. Special cases and remarks

The main results presented in Section 2 are general enough to include many interesting integral formulas, some of which are illustrated here.

• Setting $M = 2$ and $A_{N,L} = (-1)^L$ in (2.2), (2.3) and (2.4), it is seen that $S_n^2(x)$ becomes the Hermite polynomials $x^{N/2} H_N\left[\frac{1}{2\sqrt{x}}\right]$ (see [11, 12]). Then the resulting formulas are recorded in Corollary 1.

Corollary 1. *Let p, q, k, l, m, n, $N \in \mathbb{N}_0$. Let a, b, c, α, β, $\gamma \in \mathbb{R}$ with $a > 0$, $b \geq 0$, $c+4ab > 0$, and $-\frac{1}{2} < \alpha-\beta-\gamma < \frac{1}{2}$. Let λ, $\mu \in \mathbb{C}$ with $\Re(\lambda)+1/2 > 0$. Let the coefficients A_R ($R \in \mathbb{N}_0$) be arbitrary (real or complex) constants. Then the following integral formulas hold true*:

$$\int_0^{\infty} X^{-\lambda-1} {}_2F_1\left(\alpha,\beta;\gamma+1/2;X\right) {}_2F_1\left(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X\right) \tag{3.1}$$
$$\times \left(yX^{\mu}\right)^{N/2} H_N\left(\frac{1}{2\sqrt{yX^{\mu}}}\right) F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p): & (b_q); & (c_k); \\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{matrix}\ \frac{x}{X}, \frac{y}{X}\right] dx$$

$$= \frac{\sqrt{\pi}}{2a\,(4ab+c)^{\lambda+1/2}} \sum_{R=0}^{\infty} \sum_{L=0}^{[N/2]} A_R \frac{(\gamma)_R}{(\gamma+1/2)_R} \frac{(-N)_{2L}}{L!}$$
$$\times \left(-y\,(4ab+c)^{\mu}\right)^L \frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)}$$
$$\times F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{c}(a_p);\lambda-R-\mu L+1/2:(b_q);(c_k);\\ (\alpha_l);\lambda-R-\mu L+1:(\beta_m);(\gamma_n);\end{array}\ \frac{x}{4ab+c},\frac{y}{4ab+c}\right];$$

$$\int_0^{\infty} \frac{1}{x^2} X^{-\lambda-1}\,{}_2F_1(\alpha,\beta;\gamma+1/2;X)\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X) \tag{3.2}$$
$$\times (yX^{\mu})^{N/2}\, H_N\left(\frac{1}{2\sqrt{yX^{\mu}}}\right) F^{p:q;k}_{l:m;n}\left[\begin{array}{ccc}(a_p): & (b_q); & (c_k);\\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{array}\ \frac{x}{X},\frac{y}{X}\right] dx$$
$$= \frac{\sqrt{\pi}}{2b\,(4ab+c)^{\lambda+1/2}} \sum_{R=0}^{\infty} \sum_{L=0}^{[N/2]} A_R \frac{(\gamma)_R}{(\gamma+1/2)_R}$$
$$\times \frac{(-N)_{2L}}{L!} \left(-y\,(4ab+c)^{\mu}\right)^L \frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)}$$
$$\times F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{c}(a_p);\lambda-R-\mu L+1/2:(b_q);(c_k);\\ (\alpha_l);\lambda-R-\mu L+1:(\beta_m);(\gamma_n);\end{array}\ \frac{x}{4ab+c},\frac{y}{4ab+c}\right];$$

(3.3)

$$\int_0^{\infty} \left(a+\frac{b}{x^2}\right) X^{-\lambda-1}\,{}_2F_1(\alpha,\beta;\gamma+1/2;X)\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)$$
$$\times (yX^{\mu})^{N/2}\, H_N\left(\frac{1}{2\sqrt{yX^{\mu}}}\right) F^{p:q;k}_{l:m;n}\left[\begin{array}{ccc}(a_p): & (b_q); & (c_k);\\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{array}\ \frac{x}{X},\frac{y}{X}\right] dx$$
$$= \frac{\sqrt{\pi}}{(4ab+c)^{\lambda+1/2}} \sum_{R=0}^{\infty} \sum_{L=0}^{[N/2]} A_R \frac{(\gamma)_R}{(\gamma+1/2)_R}$$
$$\times \frac{(-N)_{2L}}{L!} \left(-y\,(4ab+c)^{\mu}\right)^L \frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)}$$
$$\times F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{ccc}(a_p);\lambda-R-\mu L+1/2:(b_q); & (c_k); & \\ (\alpha_l);\lambda-R-\mu L+1:(\beta_m); & (\gamma_n); & \end{array}\ \frac{x}{4ab+c},\frac{y}{4ab+c}\right],$$

where X is given as in (2.1).

• Setting $M = 2$ and $A_{N,L} = \binom{N+\nu}{N}\frac{1}{\nu+1}$ in (2.2), (2.3) and (2.4), $S_n^2(x)$ is seen to become the Laguerre polynomials $L_N^{(\nu)}(x)$ (see [11, 12]). Then the resulting formulas are recorded in Corollary 2.

Corollary 2. *Let p, q, k, l, m, n, $N \in \mathbb{N}_0$. Let a, b, c, α, β, $\gamma \in \mathbb{R}$ with $a > 0$, $b \geq 0$, $c+4ab > 0$, and $-\frac{1}{2} < \alpha-\beta-\gamma < \frac{1}{2}$. Let λ, μ, $\nu \in \mathbb{C}$ with $\Re(\lambda)+1/2 > 0$. Let the coefficients A_R ($R \in \mathbb{N}_0$) be arbitrary (real or complex) constants. Then*

the following integral formulas hold true:

$$\begin{aligned}
(3.4)\quad & \int_0^\infty X^{-\lambda-1}{}_2F_1\left(\alpha,\beta;\gamma+1/2;X\right){}_2F_1\left(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X\right) \\
& \times L_N^{(\alpha')}\left[yX^\mu\right]F^{p:q;k}_{l:m;n}\left[\begin{array}{lll}(a_p): & (b_q); & (c_k);\\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{array}\frac{x}{X},\frac{y}{X}\right]dx \\
= & \frac{\sqrt{\pi}}{2a\left(4ab+c\right)^{\lambda+1/2}}\sum_{R=0}^{\infty}\sum_{L=0}^{[N/2]}A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\frac{(-N)_{2L}}{L!} \\
& \times\binom{N+\nu}{N}\frac{1}{\nu+1}\left(y\left(4ab+c\right)^\mu\right)^L\frac{\Gamma\left(\lambda-R-\mu L+1/2\right)}{\Gamma\left(\lambda-R-\mu L+1\right)} \\
& \times F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{lll}(a_p);\lambda-R-\mu L+1/2: & (b_q); & (c_k);\\ (\alpha_l);\lambda-R-\mu L+1: & (\beta_m); & (\gamma_n);\end{array}\frac{x}{4ab+c},\frac{y}{4ab+c}\right];
\end{aligned}$$

$$\begin{aligned}
(3.5)\quad & \int_0^\infty \frac{1}{x^2}X^{-\lambda-1}{}_2F_1\left(\alpha,\beta;\gamma+1/2;X\right){}_2F_1\left(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X\right) \\
& \times L_N^{(\alpha')}\left[yX^\mu\right]F^{p:q;k}_{l:m;n}\left[\begin{array}{lll}(a_p): & (b_q); & (c_k);\\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{array}\frac{x}{X},\frac{y}{X}\right]dx \\
= & \frac{\sqrt{\pi}}{2b\left(4ab+c\right)^{\lambda+1/2}}\sum_{L=0}^{[N/2]}A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\frac{(-N)_{2L}}{L!} \\
& \times\binom{N+\nu}{N}\frac{1}{\nu+1}\left(y\left(4ab+c\right)^\mu\right)^L\frac{\Gamma\left(\lambda-R-\mu L+1/2\right)}{\Gamma\left(\lambda-R-\mu L+1\right)} \\
& \times F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{lll}(a_p);\lambda-R-\mu L+1/2: & (b_q); & (c_k);\\ (\alpha_l);\lambda-R-\mu L+1: & (\beta_m); & (\gamma_n);\end{array}\frac{x}{4ab+c},\frac{y}{4ab+c}\right];
\end{aligned}$$

(3.6)

$$\begin{aligned}
& \int_0^\infty\left(a+\frac{b}{x^2}\right)X^{-\lambda-1}{}_2F_1\left(\alpha,\beta;\gamma+1/2;X\right){}_2F_1\left(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X\right) \\
& \times L_N^{(\alpha')}\left[yX^\mu\right]F^{p:q;k}_{l:m;n}\left[\begin{array}{lll}(a_p): & (b_q); & (c_k);\\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{array}\frac{x}{X},\frac{y}{X}\right]dx \\
= & \frac{\sqrt{\pi}}{2a\left(4ab+c\right)^{\lambda+1/2}}\sum_{R=0}^{\infty}\sum_{L=0}^{[N/2]}A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\frac{(-N)_{2L}}{L!}\binom{N+\nu}{N}\frac{1}{\nu+1} \\
& \times\binom{N+\nu}{N}\frac{1}{\nu+1}\left(y\left(4ab+c\right)^\mu\right)^L\frac{\Gamma\left(\lambda-R-\mu L+1/2\right)}{\Gamma\left(\lambda-R-\mu L+1\right)} \\
& \times F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{lll}(a_p);\lambda-R-\mu L+1/2: & (b_q); & (c_k);\\ (\alpha_l);\lambda-R-\mu L+1: & (\beta_m); & (\gamma_n);\end{array}\frac{x}{4ab+c},\frac{y}{4ab+c}\right],
\end{aligned}$$

where X is given as in (2.1).

• Setting $\alpha = \gamma$ in (2.2), (2.3) and (2.4), the part of the integrand ${}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)$ reduces to 1 and $A_R = (\beta)_R/R!$. Then the resulting formulas are recorded in Corollary 3.

Corollary 3. *Let p, q, k, l, m, $n \in \mathbb{N}_0$, and $M \in \mathbb{N}$. Let a, b, c, α, $\beta \in \mathbb{R}$ with $a > 0$, $b \geq 0$, $c+4ab > 0$, and $-\frac{1}{2} < \beta < \frac{1}{2}$. Let λ, $\mu \in \mathbb{C}$ with $\Re(\lambda)+1/2 > 0$. Let the coefficients $A_{N,L}$ (L, $N \in \mathbb{N}_0$) be arbitrary (real or complex) constants. Then the following integral formulas hold true*:

$$\begin{aligned}
(3.7)\qquad & \int_0^\infty X^{-\lambda-1}{}_2F_1(\alpha,\beta;\alpha+1/2;X)\,S_N^M[yX^\mu] \\
& \times F^{p:q;k}_{l:m;n}\left[\begin{array}{lll}(a_p): & (b_q); & (c_k);\\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{array}\frac{x}{X},\frac{y}{X}\right]dx \\
&= \frac{\sqrt{\pi}}{2a(4ab+c)^{\lambda+1/2}}\sum_{R=0}^{\infty}\sum_{L=0}^{[N/M]}\frac{(\alpha)_R(\beta)_R}{(\alpha+1/2)_R\,R!} \\
& \times \frac{(-N)_{ML}}{L!}A_{N,L}\left[y(4ab+c)^\mu\right]^L\frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)} \\
& \times F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{l}(a_p);\lambda-R-\mu L+1/2:(b_q);(c_k);\\ (\alpha_l);\lambda-R-\mu L+1:(\beta_m);(\gamma_n);\end{array}\frac{x}{4ab+c},\frac{y}{4ab+c}\right];
\end{aligned}$$

$$\begin{aligned}
(3.8)\qquad & \int_0^\infty \frac{1}{x^2}X^{-\lambda-1}{}_2F_1(\alpha,\beta;\alpha+1/2;X)\,S_N^M[yX^\mu] \\
& \times F^{p:q;k}_{l:m;n}\left[\begin{array}{lll}(a_p): & (b_q); & (c_k);\\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{array}\frac{x}{X},\frac{y}{X}\right]dx \\
&= \frac{\sqrt{\pi}}{2b(4ab+c)^{\lambda+1/2}}\sum_{R=0}^{\infty}\sum_{L=0}^{[N/M]}\frac{(\alpha)_R(\beta)_R}{(\alpha+1/2)_R\,R!} \\
& \times \frac{(-N)_{ML}}{L!}A_{N,L}\left[y(4ab+c)^\mu\right]^L\frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)} \\
& \times F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{l}(a_p);\lambda-R-\mu L+1/2:(b_q);(c_k);\\ (\alpha_l);\lambda-R-\mu L+1:(\beta_m);(\gamma_n);\end{array}\frac{x}{4ab+c},\frac{y}{4ab+c}\right];
\end{aligned}$$

$$\begin{aligned}
(3.9)\qquad & \int_0^\infty \left(a+\frac{b}{x^2}\right)X^{-\lambda-1}{}_2F_1(\alpha,\beta;\alpha+1/2;X)\,S_N^M[yX^\mu] \\
& \times F^{p:q;k}_{l:m;n}\left[\begin{array}{lll}(a_p): & (b_q); & (c_k);\\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{array}\frac{x}{X},\frac{y}{X}\right]dx \\
&= \frac{\sqrt{\pi}}{(4ab+c)^{\alpha+1/2}}\sum_{R=0}^{\infty}\sum_{L=0}^{[N/M]}\frac{(\alpha)_R(\beta)_R}{(\alpha+1/2)_R\,R!} \\
& \times \frac{(-N)_{ML}}{L!}A_{N,L}\left[y(4ab+c)^\mu\right]^L\frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)}
\end{aligned}$$

$$\times F^{p+1:q;k}_{l+1:m;n}\left[\begin{array}{c}(a_p);\lambda-R-\mu L+1/2:(b_q);(c_k);\\(\alpha_l);\lambda-R-\mu L+1:(\beta_m);(\gamma_n);\end{array}\ \frac{x}{4ab+c},\frac{y}{4ab+c}\right],$$

where X is given as in (2.1).

- Recalling the following known relation (see, *e.g.*, [8, pp. 28–32]):

$$F^{p:0;0}_{q:0;0}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p\\\beta_1,\ldots,\beta_q\end{array};x,y\right]={}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p\\\beta_1,\ldots,\beta_q\end{array};x+y\right],\tag{3.10}$$

the results (2.2), (2.3) and (2.4) reduce to certain interesting integral formulas involving ${}_pF_q$ given in Corollary 4.

Corollary 4. *Let p, $q\in\mathbb{N}_0$, and $M\in\mathbb{N}$. Let a, b, c, α, β, $\gamma\in\mathbb{R}$ with $a>0$, $b\geq 0$, $c+4ab>0$, and $-\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2}$. Let λ, $\mu\in\mathbb{C}$ with $\Re(\lambda)+1/2>0$. Let the coefficients A_R and $A_{N,L}$ (L, N, $R\in\mathbb{N}_0$) be arbitrary (real or complex) constants. Then the following integral formula holds true*:

(3.11)

$$\begin{aligned}&\int_0^\infty X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X)\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)\\&\times S_N^M[yX^\mu]\,{}_pF_l\left[\begin{array}{c}a_1,\ldots,a_p;\\\alpha_1,\ldots,\alpha_l;\end{array}\ \frac{x+y}{X}\right]dx\\&=\frac{\sqrt{\pi}}{2a(4ab+c)^{\lambda+1/2}}\sum_{R=0}^{\infty}\sum_{L=0}^{[N/M]}A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\frac{(-N)_{ML}}{L!}A_{N,L}[y(4ab+c)^\mu]^L\\&\times\frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)}{}_{p+1}F_{l+1}\left[\begin{array}{c}a_1,\ldots,a_p,\lambda-R-\mu L+1/2;\\\alpha_1,\ldots,\alpha_l,\lambda-R-\mu L+1;\end{array}\ \frac{x+y}{4ab+c}\right];\end{aligned}$$

(3.12)

$$\begin{aligned}&\int_0^\infty \frac{1}{x^2}X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X)\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)\\&\times S_N^M[yX^\mu]\,{}_pF_l\left[\begin{array}{c}a_1,\ldots,a_p;\\\alpha_1,\ldots,\alpha_l;\end{array}\ \frac{x+y}{X}\right]dx\\&=\frac{\sqrt{\pi}}{2b(4ab+c)^{\lambda+1/2}}\sum_{R=0}^{\infty}\sum_{L=0}^{[N/M]}A_R\frac{(\gamma)_R}{(\gamma+1/2)_R}\frac{(-N)_{ML}}{L!}A_{N,L}[y(4ab+c)^\mu]^L\\&\times\frac{\Gamma(\lambda-R-\mu L+1/2)}{\Gamma(\lambda-R-\mu L+1)}{}_{p+1}F_{l+1}\left[\begin{array}{c}a_1,\ldots,a_p,\lambda-R-\mu L+1/2;\\\alpha_1,\ldots,\alpha_l,\lambda-R-\mu L+1;\end{array}\ \frac{x+y}{4ab+c}\right];\end{aligned}$$

(3.13)

$$\begin{aligned}&\int_0^\infty\left(a+\frac{b}{x^2}\right)X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X)\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)\\&\times S_N^M[yX^\mu]\,{}_pF_l\left[\begin{array}{c}a_1,\ldots,a_p;\\\alpha_1,\ldots,\alpha_l;\end{array}\ \frac{x+y}{X}\right]dx\end{aligned}$$

$$= \frac{\sqrt{\pi}}{(4ab+c)^{\lambda+1/2}} \sum_{R=0}^{\infty} \sum_{L=0}^{[N/M]} A_R \frac{(\gamma)_R}{(\gamma+1/2)_R} \frac{(-N)_{ML}}{L!} A_{N,L} \left[y(4ab+c)^{\mu}\right]^L$$
$$\times \frac{\Gamma(\lambda - R - \mu L + 1/2)}{\Gamma(\lambda - R - \mu L + 1)} {}_{p+1}F_{l+1} \left[\begin{array}{c} a_1, \ldots, a_p, \lambda - R - \mu L + 1/2; \\ \alpha_1, \ldots, \alpha_l, \lambda - R - \mu L + 1; \end{array} \frac{x+y}{4ab+c} \right],$$

where X is given as in (2.1).

We conclude this paper by emphasizing that many other interesting integral formulas can be derived as special cases of our main results (2.2), (2.3) and (2.4), which, upon choosing the parameters suitably, is also seen to yield those results in the works of Agarwal [1], Agarwal and Chand [2], and Chand [3].

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