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NEW CLASS OF INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION AND THE LOGARITHMIC FUNCTION

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Abstract. Motivated essentially by Brychkov's work [1], we evaluate some new integrals involving hypergeometric function and the logarithmic function (including those obtained by Brychkov[1], Choi and Rathie [3]), which are expressed explicitly in terms of Gamma, Psi and Hurwitz zeta functions suitable for numerical computations.

1. Introduction and preliminaries

The generalized hypergeometric functions with p numeratorial and q denominatorial parameters is defined by (see, *e.g.*, [9, 11, 12])

(1.1)
$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1\ldots,\beta_q\end{matrix}\ ;\ z\right] &= {}_pF_q\left[\alpha_1,\ldots,\alpha_p;\beta_1\ldots,\beta_q\ ;\ z\right]\\ &=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n\,!}, \end{aligned}$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined for any complex number α by

$$(\alpha)_n=\begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1), & \text{if } n\in\mathbb{N}=\{1,2,\ldots\}\\ 1, & \text{if } n=0.\end{cases}$$

Using the fundamental functional relation $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, $(\alpha)_n$ can be written as

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\qquad \left(n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\right),$$

where Γ is the well-known Gamma function.

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It is well-known that the series ${}_pF_q$ converges all z if $p \leq q$ and for $|z| < 1$ if $p = q + 1$, and the series diverges for all $z \neq 0$ if $p > q + 1$. The convergence of the series for the case $|z| = 1$ when $p = q + 1$ is of much interest.

The series ${}_{q+1}F_q\left[\alpha_1, \ldots, \alpha_{q+1}; \beta_1 \ldots, \beta_q \;;\; z\right]$ *with* $|z| = 1$ *converges absolutely if* $\left(\sum \beta_j - \sum \alpha_j\right) > 0$.

The series converges conditionally if $z \neq 1$ *and* $0 \geq \left(\sum \beta_j - \sum \alpha_j\right) > -1$. *The series diverges if* $\left(\sum \beta_j - \sum \alpha_j\right) \leq -1$

It is well-known that, whenever a generalized hypergeometric function reduces to Gamma function, the results are very important from the application point of view. Thus, the classical summation theorems such as those of Gauss, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz (see, *e.g.*, [9, 11]) for the series ${}_3F_2$ and others play key role in the theory of generalized hypergeometric series. Applications of the above mentioned classical summation theorems are well-known now. In our present investigation, we are interested in the following theorems (see, *e.g.*, [11], pp. 243–244) given respectively by

$$(1.2)\qquad {}_4F_3\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b,\, c\,; \\ \frac{1}{2}a,\, 1+a-b,\, 1+a-c\,; \end{matrix}\; -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}$$
$$(\Re(a-2b-2c) > -1)$$

and

$$(1.3)\qquad {}_5F_4\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b,\, c,\, d\,; \\ \frac{1}{2}a,\, 1+a-b,\, 1+a-c,\, 1+a-d\,; \end{matrix}\; 1\right]$$
$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-b-c-d)}$$
$$(\Re(a-b-c-d) > -1).$$

In recent, Choi and Rathie [3] evaluated some single integrals involving hypergeometric functions for ${}_2F_1$ and logarithmic function in terms of Psi and Hurwitz zeta functions suitable for numerical computations. Also, we can easily evaluate some finite integrals containing hypergeometric functions ${}_4F_3(-1)$, ${}_5F_4(1)$ and the logarithmic functions.

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_0^-$ denote the sets of positive integers, real numbers, complex numbers, and nonpositive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Psi (or Digamma) function $\psi(z)$ is defined by

$$(1.4)\quad \psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \text{ or } \log\Gamma(z) = \int_1^z \psi(t)\,dt \ \left(z \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

where $\Gamma(z)$ is the Eulerian Gamma function. The polygamma functions $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ are defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) = \frac{d^n}{dz^n} \psi(z) \quad \left(n \in \mathbb{N}_0;\ z \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{1.5}$$

The Hurwitz (or generalized) zeta function $\zeta(s, a)$ is defined by

$$\zeta(s, a) := \sum_{k=0}^{\infty} (k + a)^{-s} \quad \left(\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{1.6}$$

The relation

$$\begin{aligned} \psi^{(n)}(z) &= (-1)^{n+1}\, n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}} \\ &= (-1)^{n+1}\, n!\, \zeta(n+1, z) \quad \left(n \in \mathbb{N};\ z \in \mathbb{C} \setminus \mathbb{Z}_0^-\right) \end{aligned} \tag{1.7}$$

is well-known. It is easy to see from (1.7) that

$$\begin{aligned} \frac{\partial^k}{\partial z^k} \log \left\{ \prod_{j=1}^{m} \Gamma\left(a_j + z\right) \right\} &= \sum_{j=1}^{m} \psi^{(k-1)}\left(a_j + z\right) \quad (m,\ k \in \mathbb{N}) \\ &= (-1)^k\,(k-1)! \sum_{j=1}^{m} \zeta\left(k, a_j + z\right) \quad (m \in \mathbb{N};\ k \in \mathbb{N} \setminus \{1\}), \end{aligned} \tag{1.8}$$

where $\psi^{(0)}(z) = \psi(z)$. Details about these special functions may be found in [8].

Finally, we recall an important result for the sequel due to Edwards [4]:

$$\int_0^1 \int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}\, dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\Re(\alpha) > 0;\ \Re(\beta) > 0). \tag{1.9}$$

It is worthy to note that making the change of variable $x \to \frac{z-y}{y(z-1)}$ in equation (1.9), we obtain its simpler form as

$$\int_0^1 \int_0^y z^{\alpha-1}(1-z)^{\beta-2}\, dzdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\Re(\alpha) > 0;\ \Re(\beta) > 1). \tag{1.10}$$

2. Single definite integrals involving generalized hypergeometric function

This section aims at providing several classes of finite integrals involving generalized hypergeometric functions. Let us give, in the form of two theorems, the six integral formulas that we shall establish.

Theorem 1. *The following finite integrals formulas hold true*:

$$\int_0^1 x^{b-1}\,(1-x)^{a-2b}\;{}_3F_2\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,c\,; \\ \frac{1}{2}a,\,1+a-c\,; \end{matrix}\;-x\right]dx \tag{2.1}$$
$$= \frac{\Gamma(b)\,\Gamma(1+a-2b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)},$$

provided $\Re(b)>0$, $\Re(a-2b-2c)>-1$ *and* $\Re(1+a-2b)>0$.

$$\int_0^1 x^{c-1}\,(1-x)^{\frac{1}{2}a-c-1}\;{}_3F_2\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,b\,; \\ 1+a-b,\,1+a-c\,; \end{matrix}\;-x\right]dx \tag{2.2}$$
$$= \frac{\Gamma(c)\,\Gamma\left(\frac{1}{2}a-c\right)\,\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma(1+a)\,\Gamma(1+a-b-c)},$$

provided $\Re(c)>0$, $\Re(a-2c)>0$ *and* $\Re(a-2b-2c)>-1$,

$$\int_0^1 x^{c-1}\,(1-x)^{a-b-c}\;{}_3F_2\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,b\,; \\ \frac{1}{2}a,\,1+a-c\,; \end{matrix}\;-x\right]dx \tag{2.3}$$
$$= \frac{\Gamma(c)\,\Gamma(1+a-c)}{\Gamma(1+a)},$$

provided $\Re(c)>0$ *and* $\Re(a-2b-2c)>-1$.

Note. The result (2.3) seems to be interesting since the right-hand side of (2.3) is independent of b.

Proof. In order to prove (2.1), denote its left-hand side by I. Expressing the hypergeometric function ${}_3F_2$ as a series, changing the order of integration and summation(which is easily seen to be justified), evaluating the beta integral and summing up the series gives

$$I = \frac{\Gamma(b)\Gamma(1+a-2b)}{\Gamma(1+a-b)}\,{}_4F_3\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,b,\,c\,; \\ \frac{1}{2}a,\,1+a-b,\,1+a-c\,; \end{matrix}\;-1\right] \tag{2.4}$$
$$(\Re(a-2b-2c)>-1).$$

The ${}_4F_3$ appearing on the right-hand side of (2.4) can be evaluated by using (1.2). this produces the right-hand side of (2.1). In exactly the same manner, the integrals (2.2) and (2.3) can be established. □

The summation theorem (1.3) yields:

Theorem 2. *The following finite integrals formulas hold true*:

$$\int_0^1 x^{b-1}\,(1-x)^{a-2b}\,{}_4F_3\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,c,\,d\,; \\ \frac{1}{2}a,\,1+a-b,\,1+a-c\,; \end{matrix}\; x\right] dx \tag{2.5}$$
$$= \frac{\Gamma(b)\,\Gamma(1+a-2b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-b-c-d)}{\Gamma(1+a)\,\Gamma(1+a-b-c)\,\Gamma(1+a-b-d)\,\Gamma(1+a-c-d)},$$

provided $\Re(b) > 0$, $\Re(1+a-2b) > 0$ *and* $\Re(1+a-b-c-d) > 0$,

$$\int_0^1 x^{c-1}\,(1-x)^{\frac{1}{2}a-c-1}\,{}_4F_3\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,b,\,d\,; \\ 1+a-b,\,1+a-c,\,1+a-d\,; \end{matrix}\; x\right] dx \tag{2.6}$$
$$= \frac{\Gamma(c)\,\Gamma\left(\frac{1}{2}a-c\right)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,(1+a-d)\,\Gamma(1+a-b-c-d)}{\Gamma\left(\frac{1}{2}a\right)\Gamma(1+a)\,\Gamma(1+a-b-c)\,\Gamma(1+a-b-d)\,\Gamma(1+a-c-d)},$$

provided $\Re(c) > 0$, $\Re(a-2c) > 0$ *and* $\Re(1+a-b-c-d) > 0$,

$$\int_0^1 x^{d-1}\,(1-x)^{a-2d}\,{}_4F_3\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,b,\,c\,; \\ \frac{1}{2}a,\,1+a-b,\,1+a-c\,; \end{matrix}\; x\right] dx \tag{2.7}$$
$$= \frac{\Gamma(d)\,\Gamma(1+a-2d)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-b-c-d)}{\Gamma(1+a)\,\Gamma(1+a-b-c)\,\Gamma(1+a-b-d)\,\Gamma(1+a-c-d)},$$

provided $\Re(d) > 0$, $\Re(1+a-2d) > 0$ *and* $\Re(1+a-b-c-d) > 0$.

3. New class of finite integrals involving generalized hypergeometric function and logarithmic function

In this section, we shall establish a new class of integrals containing generalized hypergeometric function and logarithmic function using the summation theorems (1.2), (1.3) and equation (1.7).

Theorem 3. *The following finite integral formula holds true*:

$$\int_0^1 x^{b-1}\,(1-x)^{a-2b}\,\log^n\left(\frac{x}{(1-x)^2}\right)\,{}_3F_2\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,c\,; \\ \frac{1}{2}a,\,1+a-c\,; \end{matrix}\; -x\right] dx \tag{3.1}$$
$$= \frac{\Gamma(1+a-c)}{\Gamma(1+a)}\,\frac{\partial^n A}{\partial b^n},$$

provided $n \in \mathbb{N}$; $\Re(b) > 0$ *and* $\Re(a-2b-2c) > -1$, *where the expression* $\frac{\partial^n A}{\partial b^n}$ *is given as follows*

$$A := \frac{\Gamma(b)\,\Gamma(1+a-2b)}{\Gamma(1+a-b-c)}, \tag{3.2}$$

$$\frac{\partial^n A}{\partial b^n} = A\left[\sum_{r_1=0}^{n-1}\left\{\binom{n-1}{r_1}\frac{\partial^{n-r_1-1}B}{\partial b^{n-r_1-1}}\sum_{r_2=0}^{r_1-1}\left[\binom{r_1-1}{r_2}\frac{\partial^{r_1-r_2-1}B}{\partial b^{r_1-r_2-1}}\cdots \sum_{r_n=0}^{r_{n-1}-1}\left\{\binom{r_{n-1}-1}{r_n}\frac{\partial^{r_{n-1}-r_n-1}B}{\partial b^{r_{n-1}-r_n-1}}\right\}\right]\right\}\right], \tag{3.3}$$

$$B := \frac{\partial \log A}{\partial b} = \psi(b) - 2\psi\,(1+a-2b) + \psi\,(1+a-b-c)\,, \tag{3.4}$$

$$\frac{\partial^{n-r-1}B}{\partial b^{n-r-1}} = (n-r-1)!\,\big\{(-1)^{n-r}\zeta(n-r,\,b) + 2^{n-r}\zeta\,(n-r,\,1+a-2b) - \zeta\,(n-r,\,1+a-b-c)\big\} \quad (n-r-1\in\mathbb{N}). \tag{3.5}$$

Proof. Differentiating n times both sides of (2.1) with respect to b yields

$$\int_0^1 x^{b-1}\,(1-x)^{a-2b}\,\log^n\left(\frac{x}{(1-x)^2}\right)\,{}_3F_2\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,c\,; \\ \frac{1}{2}a,\,1+a-c\,; \end{matrix}\; -x\right]dx = \frac{\Gamma\,(1+a-c)}{\Gamma\,(1+a)}\,\frac{\partial^n A}{\partial b^n}, \tag{3.6}$$

A is the same as given in (3.2).

Now, it is easy to see that

$$\frac{\partial A}{\partial b} = A\,\frac{\partial \log A}{\partial b} = A{\cdot}B = \frac{\Gamma(b)\,\Gamma(1+a-2b)}{\Gamma(1+a-b-c)}\Big\{\psi(b) - 2\psi(1+a-2b) + \psi(1+a-b-c)\Big\}, \tag{3.7}$$

where B is the same as given in (3.4) by using (1.7). From (3.7), we have

$$\frac{\partial^n A}{\partial b^n} = \frac{\partial^{n-1}}{\partial b^{n-1}}\left(\frac{\partial A}{\partial b}\right) = \frac{\partial^{n-1}}{\partial b^{n-1}}\,(A{\cdot}B)\,, \tag{3.8}$$

which upon using Leibniz theorem becomes

$$\frac{\partial^n A}{\partial b^n} = \left[\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r A}{\partial b^r}\,\frac{\partial^{n-r-1}B}{\partial b^{n-r-1}}\right], \tag{3.9}$$

where $\frac{\partial^{n-r-1}B}{\partial b^{n-r-1}}$ is the same as given in (3.5). Finally, substituting the values of $\frac{\partial^n A}{\partial b^n}$ from (3.8) into (3.6) leads us to the asserted result (3.1). □

Next, differentiate both sides of (2.2) and (2.3) respectively, n times, with respect to c, we obtain the following theorems.

Theorem 4. *The following finite integral formula holds true*:

(3.10)
$$\int_0^1 x^{c-1}(1-x)^{\frac{1}{2}a-c-1}\log^n\left(\frac{x}{1-x}\right){}_3F_2\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b\,; \\ 1+a-b,\, 1+a-c\,; \end{matrix}\, -x\right]dx$$
$$= \frac{\Gamma(1+a-b)}{\Gamma\left(\frac{1}{2}a\right)\Gamma(1+a)}\,\frac{\partial^n A}{\partial c^n},$$

provided $n\in\mathbb{N}$; $\Re(c)>0$ *and* $\Re(a-2b-2c)>-1$, *where the expression for* $\frac{\partial^n A}{\partial c^n}$ *can be obtained from* (3.3) *by replacing* b *to* c *with*

$$A := \frac{\Gamma(c)\,\Gamma\left(\frac{1}{2}a-c\right)\Gamma(1+a-c)}{\Gamma(1+a-b-c)}, \tag{3.11}$$

$$\frac{\partial A}{\partial c} = A\cdot B, \tag{3.12}$$

$$B := \frac{\partial \log A}{\partial c} = \psi(c)-\psi\left(\frac{1}{2}a-c\right)-\psi(1+a-c)+\psi(1+a-b-c), \tag{3.13}$$

$$\frac{\partial^{n-r-1}B}{\partial c^{n-r-1}} = (n-r-1)!\left\{(-1)^{n-r}\zeta(n-r,\,c)+\zeta\left(n-r,\,\frac{1}{2}a-c\right)\right. \\ +\zeta(n-r,\,1+a-c) \\ \left.-\zeta(n-r,\,1+a-b-c)\right\}\ (n-r-1\in\mathbb{N}). \tag{3.14}$$

Theorem 5. *The following finite integral formula holds true*:

$$\int_0^1 x^{c-1}(1-x)^{a-b-c}\log^n\left(\frac{x}{1-x}\right){}_3F_2\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b\,; \\ \frac{1}{2}a,\, 1+a-c\,; \end{matrix}\, -x\right]dx \tag{3.15}$$
$$= \frac{1}{\Gamma(1+a)}\,\frac{\partial^n A}{\partial c^n},$$

provided $n\in\mathbb{N}$; $\Re(c)>0$ *and* $\Re(a-2b-2c)>-1$, *where the expression for* $\frac{\partial^n A}{\partial c^n}$ *can be obtained from* (3.3) *by replacing* b *to* c *with*

$$A := \Gamma(c)\,\Gamma(1+a-c), \tag{3.16}$$

$$\frac{\partial A}{\partial c} = A\cdot B, \tag{3.17}$$

$$B := \frac{\partial \log A}{\partial c} = \psi(c)-\psi(1+a-c), \tag{3.18}$$

$$\frac{\partial^{n-r-1}B}{\partial c^{n-r-1}} = (n-r-1)!\left\{(-1)^{n-r}\zeta(n-r,\,c)+\zeta(n-r,\,1+a-c)\right\}, \tag{3.19}$$

$(n-r-1\in\mathbb{N}).$

The equation (2.5) gives:

Formula 1.

(3.20)

$$\int_0^1 x^{b-1}\,(1-x)^{a-2b}\,\log^n\left(\frac{x}{(1-x)^2}\right)\,{}_4F_3\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,c,\,d\,; \\ \frac{1}{2}a,\,1+a-c\;1+a-d\,; \end{matrix}\; x\right]dx$$
$$=\frac{\Gamma(1+a-c)\,\Gamma(1+a-d)}{\Gamma(1+a)\,\Gamma(1+a-c-d)}\,\frac{\partial^n A}{\partial b^n},$$

where the expression for $\frac{\partial^n A}{\partial b^n}$ is the same as (3.3) with

$$A:=\frac{\Gamma(b)\,\Gamma(1+a-2b)\,\Gamma(1+a-b-c-d)}{\Gamma(1+a-b-c)\,\Gamma(1+a-b-d)}, \tag{3.21}$$

$$\frac{\partial A}{\partial b}=A{\cdot}B, \tag{3.22}$$

$$B:=\frac{\partial\,\log A}{\partial b}=\psi(b)-2\psi(1+a-2b)+\psi(1+a-b-c) \tag{3.23}$$
$$+\psi(1+a-b-d)-\psi(1+a-c-b-d)\,,$$

(3.24)

$$\frac{\partial^{n-r-1}B}{\partial b^{n-r-1}}=(n-r-1)!\,\left\{(-1)^{n-r}\zeta(n-r,\,b)+2^{n-r}\zeta(n-r,\,1+a-2b)\right.$$
$$-\zeta(n-r,\,1+a-b-c)-\zeta(n-r,\,1+a-b-d)$$
$$\left.+\zeta(n-r,\,1+a-b-c-d)\right\}\quad(n-r-1\in\mathbb{N}).$$

The equation (2.6) gives:

Formula 2.

(3.25)

$$\int_0^1 x^{c-1}\,(1-x)^{\frac{1}{2}a-c-1}\,\log^n\left(\frac{x}{1-x}\right)\,{}_4F_3\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,b,\,d\,; \\ 1+a-b,\,1+a-c\;1+a-d\,; \end{matrix}\; x\right]dx$$
$$=\frac{\Gamma(1+a-b)\,\Gamma(1+a-d)}{\Gamma\left(\frac{1}{2}a\right)\Gamma(1+a)\,\Gamma(1+a-b-d)}\,\frac{\partial^n A}{\partial c^n},$$

where the expression for $\frac{\partial^n A}{\partial c^n}$ can be obtained from (3.3) by replacing b to c with

$$A:=\frac{\Gamma(c)\,\Gamma\left(\frac{1}{2}a-c\right)\Gamma(1+a-c)\,\Gamma(1+a-b-c-d)}{\Gamma(1+a-b-c)\,\Gamma(1+a-c-d)}, \tag{3.26}$$

$$\frac{\partial A}{\partial c}=A{\cdot}B, \tag{3.27}$$

$$(3.28)\quad \begin{aligned} B := {} & \frac{\partial \log A}{\partial c} \\ & = \psi(c) - \psi\left(\frac{1}{2}a - c\right) - \psi\left(1 + a - c\right) \\ & \quad - \psi\left(1 + a - b - c - d\right) + \psi\left(1 + a - b - c\right) + \psi\left(1 + a - c - d\right), \end{aligned}$$

$$(3.29)\quad \begin{aligned} \frac{\partial^{n-r-1} B}{\partial c^{n-r-1}} = {} & (n-r-1)! \left\{ (-1)^{n-r} \zeta(n-r,\, c) + \zeta\left(n-r,\, \frac{1}{2}a - c\right) \right. \\ & + \zeta\left(n-r,\, 1+a-c\right) + \zeta\left(n-r,\, 1+a-b-c-d\right) \\ & \left. - \zeta\left(n-r,\, 1+a-b-c\right) - \zeta\left(n-r,\, 1+a-c-d\right) \right\} \\ & (n-r-1 \in \mathbb{N}). \end{aligned}$$

The equation (2.6) also gives another formula:

Formula 3.

(3.30)

$$\int_0^1 x^{c-1}\, (1-x)^{\frac{1}{2}a-c-1} \log^n \sqrt{1-x}\; {}_4F_3\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b,\, d\,; \\ 1+a-b,\, 1+a-c\; 1+a-d\,; \end{matrix}\, x\right] dx$$

$$= \Gamma\left(c\right) \frac{\partial^n A}{\partial a^n},$$

where the expression for $\frac{\partial^n A}{\partial a^n}$ can be obtained from (3.3) by replacing b to a with

(3.31)

$$A := \frac{\Gamma\left(\frac{1}{2}a - c\right)\Gamma\left(1+a-b\right)\Gamma\left(1+a-c\right)\Gamma\left(1+a-d\right)\Gamma\left(1+a-b-c-d\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(1+a-b-c\right)\Gamma\left(1+a-b-d\right)\Gamma\left(1+a-c-d\right)},$$

$$(3.32)\quad \frac{\partial A}{\partial a} = A{\cdot}B,$$

$$(3.33)\quad \begin{aligned} B := {} & \frac{\partial \log A}{\partial a} \\ & = \frac{1}{2}\psi\left(\frac{1}{2}a - c\right) + \psi\left(1+a-b\right) + \psi\left(1+a-c\right) + \psi\left(1+a-d\right) \\ & \quad + \psi\left(1+a-b-c-d\right) - \frac{1}{2}\psi\left(\frac{1}{2}a\right) - \psi\left(1+a\right) \\ & \quad - \psi\left(1+a-b-c\right) - \psi\left(1+a-b-d\right) - \psi\left(1+a-c-d\right), \end{aligned}$$

$$(3.34)\quad \begin{aligned} \frac{\partial^{n-r-1} B}{\partial a^{n-r-1}} = {} & (-1)^{n-r}(n-r-1)! \left\{ \left(\frac{1}{2}\right)^{n-r} \zeta(n-r,\, \frac{1}{2}a - c) \right. \\ & + \zeta\left(n-r,\, 1+a-b\right) + \zeta\left(n-r,\, 1+a-c\right) \end{aligned}$$

$$
\begin{aligned}
&+\zeta\,(n-r,\,1+a-d)+\zeta\,(n-r,\,1+a-b-c-d)\\
&-\left(\frac{1}{2}\right)^{n-r}\zeta(n-r,\,\frac{1}{2}a)-\zeta\,(n-r,\,1+a)\\
&-\zeta\,(n-r,\,1+a-b-c)-\zeta\,(n-r,\,1+a-b-d)\\
&\left.-\zeta\,(n-r,\,1+a-c-d)\right\}\quad (n-r-1\in\mathbb{N}).
\end{aligned}
$$

The equation (2.7) gives:

Formula 4.

(3.35)

$$
\int_0^1 x^{d-1}\,(1-x)^{a-2d}\log^n\left(\frac{x}{(1-x)^2}\right){}_4F_3\left[\begin{matrix} a,\,1+\frac{1}{2}a,\,b,\,c\,; \\ \frac{1}{2}a,\,1+a-b\;1+a-c\,; \end{matrix}\; x\right]dx
$$

$$
=\frac{\Gamma\,(1+a-b)\,\Gamma\,(1+a-c)}{\Gamma\,(1+a)\,\Gamma\,(1+a-b-c)}\,\frac{\partial^n A}{\partial d^n},
$$

where the expression for $\frac{\partial^n A}{\partial d^n}$ can be obtained from (3.3) by replacing b to d with

$$
A:=\frac{\Gamma(d)\,\Gamma\,(1+a-2d)\,\Gamma\,(1+a-b-c-d)}{\Gamma\,(1+a-b-d)\,\Gamma\,(1+a-c-d)}, \tag{3.36}
$$

$$
\frac{\partial A}{\partial d}=A{\cdot}B, \tag{3.37}
$$

$$
\begin{aligned}
B:=\frac{\partial\,\log A}{\partial d}&=\psi(d)-2\psi\,(1+a-2d)+\psi\,(1+a-b-d)\\
&\quad+\psi\,(1+a-c-d)-\psi\,(1+a-c-b-d)\,,
\end{aligned} \tag{3.38}
$$

(3.39)

$$
\begin{aligned}
\frac{\partial^{n-r-1}B}{\partial d^{n-r-1}}&=(n-r-1)!\,\left\{(-1)^{n-r}\zeta(n-r,\,d)+2^{n-r}\zeta\,(n-r,\,1+a-2d)\right.\\
&\quad-\zeta\,(n-r,\,1+a-b-d)-\zeta\,(n-r,\,1+a-c-d)\\
&\quad\left.+\zeta\,(n-r,\,1+a-b-c-d)\right\}\quad (n-r-1\in\mathbb{N}).
\end{aligned}
$$

4. New class of finite double integrals involving generalized hypergeometric function

In this section, we shall establish a new class of double integrals containing generalized hypergeometric function. Note that while deriving the integrals, we shall use he the well-known double integrals (1.9) due to Edwards [4].

Theorem 6. *Under the conditions given in* (2.1), *the following double integrals formulas hold true*:

$$\int_0^1 \int_0^1 y^b \,(1-x)^{b-1}\,(1-y)^{(1+a-2b)-1}\,(1-xy)^{1-(1+a-b)} \cdot {}_3F_2\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, c\,; \\ \frac{1}{2}a,\, 1+a-c\,; \end{matrix} \; -\frac{y(1-x)}{(1-xy)}\right] dx\,dy \;=\mathcal{A}_1, \tag{4.1}$$

$$\int_0^1 \int_0^1 y^{1+a-2b}\,(1-x)^{1+a-2b-1}\,(1-y)^{b-1}\,(1-xy)^{1-(1+a-b)} \cdot {}_3F_2\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, c\,; \\ \frac{1}{2}a,\, 1+a-c\,; \end{matrix} \; -\frac{1-y}{(1-xy)}\right] dx\,dy \;=\mathcal{A}_1,$$

where $\mathcal{A}_1$ *is the right-hand side of* (2.1).

Theorem 7. *Under the conditions given in* (2.2), *the following double integrals formulas hold true*:

$$\int_0^1 \int_0^1 y^c\,(1-x)^{c-1}\,(1-y)^{(\frac{1}{2}a-c-1)}\,(1-xy)^{1-\frac{1}{2}a} \cdot {}_3F_2\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b\,; \\ 1+a-b,\, 1+a-c\,; \end{matrix} \; -\frac{y(1-x)}{(1-xy)}\right] dx\,dy \;=\mathcal{A}_2, \tag{4.2}$$

$$\int_0^1 \int_0^1 y^{\frac{1}{2}a-c}\,(1-x)^{\frac{1}{2}a-c-1}\,(1-y)^{c-1}\,(1-xy)^{1-\frac{1}{2}a} \cdot {}_3F_2\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b\,; \\ 1+a-b,\, 1+a-c\,; \end{matrix} \; -\frac{1-y}{(1-xy)}\right] dx\,dy \;=\mathcal{A}_2,$$

where $\mathcal{A}_2$ *is the right-hand side of* (2.2).

Theorem 8. *Under the conditions given in* (2.3), *the following double integrals formulas hold true*:

$$\int_0^1 \int_0^1 y^c\,(1-x)^{c-1}\,(1-y)^{(1+a-b-c)-1}\,(1-xy)^{1-(1+a-b)} \cdot {}_3F_2\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b\,; \\ \frac{1}{2}a,\, 1+a-c\,; \end{matrix} \; -\frac{y(1-x)}{(1-xy)}\right] dx\,dy \;=\mathcal{A}_3, \tag{4.3}$$

$$\int_0^1 \int_0^1 y^{1+a-b-c}\,(1-x)^{1+a-b-c-1}\,(1-y)^{c-1}\,(1-xy)^{1-(1+a-b)}$$

$$\cdot\, {}_3F_2\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b\,; \\ \frac{1}{2}a,\, 1+a-b\,; \end{matrix}\; -\frac{1-y}{(1-xy)}\right] dx\,dy\; = \mathcal{A}_3,$$

where $\mathcal{A}_3$ *is the right-hand side of* (2.3)*.*

Theorem 9. *Under the conditions given in* (2.5)*, the following double integrals formulas hold true*:

$$\int_0^1\int_0^1 y^b\,(1-x)^{b-1}\,(1-y)^{(1+a-2b)-1}\,(1-xy)^{1-(1+a-b)} \tag{4.4}$$

$$\cdot\, {}_4F_3\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, c,\, d\,; \\ \frac{1}{2}a,\, 1+a-c\, 1+a-d\,; \end{matrix}\; \frac{y(1-x)}{(1-xy)}\right] dx\,dy\; = \mathcal{A}_4,$$

$$\int_0^1\int_0^1 y^{1+a-2b}\,(1-x)^{1+a-2b-1}\,(1-y)^{b-1}\,(1-xy)^{1-(1+a-b)}$$

$$\cdot\, {}_4F_3\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, c,\, d\,; \\ \frac{1}{2}a,\, 1+a-c\, 1+a-d\,; \end{matrix}\; \frac{1-y}{(1-xy)}\right] dx\,dy\; = \mathcal{A}_4,$$

where $\mathcal{A}_4$ *is the right-hand side of* (2.5)*.*

Theorem 10. *Under the conditions given in* (2.6)*, the following double integrals formulas hold true*:

$$\int_0^1\int_0^1 y^c\,(1-x)^{c-1}\,(1-y)^{(\frac{1}{2}a-c-1)}\,(1-xy)^{1-\frac{1}{2}a} \tag{4.5}$$

$$\cdot\, {}_4F_3\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b,\, d\,; \\ 1+a-b,\, 1+a-c,\, 1+a-d\,; \end{matrix}\; -\frac{y(1-x)}{(1-xy)}\right] dx\,dy\; = \mathcal{A}_5,$$

$$\int_0^1\int_0^1 y^{\frac{1}{2}a-c}\,(1-x)^{\frac{1}{2}a-c-1}\,(1-y)^{c-1}\,(1-xy)^{1-\frac{1}{2}a}$$

$$\cdot\, {}_4F_3\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b,\, d\,; \\ 1+a-b,\, 1+a-c,\, 1+a-d\,; \end{matrix}\; \frac{1-y}{(1-xy)}\right] dx\,dy\; = \mathcal{A}_5,$$

where $\mathcal{A}_5$ *is the right-hand side of* (2.6)*.*

Theorem 11. *Under the conditions given in* (2.7)*, the following double integrals formulas hold true*:

$$\int_0^1\int_0^1 y^d\,(1-x)^{d-1}\,(1-y)^{(1+a-2d)-1}\,(1-xy)^{1-(1+a-d)} \tag{4.6}$$

$$\cdot\, {}_4F_3\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b,\, c\,; \\ \frac{1}{2}a,\, 1+a-b\, 1+a-c\,; \end{matrix}\, \frac{y(1-x)}{(1-xy)}\right] dx\, dy\; = \mathcal{A}_6,$$

$$\int_0^1 \int_0^1 y^{1+a-2d}\, (1-x)^{1+a-2d-1}\, (1-y)^{d-1}\, (1-xy)^{1-(1+a-d)}$$

$$\cdot\, {}_4F_3\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b,\, c\,; \\ \frac{1}{2}a,\, 1+a-b\, 1+a-c\,; \end{matrix}\, \frac{1-y}{(1-xy)}\right] dx\, dy\; = \mathcal{A}_6,$$

where $\mathcal{A}_6$ *is the right-hand side of* (2.7).

Remark. For similar integrals, we refer [3, 5, 6].

5. Concluding remarks

Classical theorems on hypergeometric functions have been used to evaluate a variety of definite integrals involving hypergeometric functions and logarithmic functions. Alternative forms of some of the results by Brychkov [1] are also established.

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