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ON THE GENERALIZATIONS OF BRÜCK CONJECTURE

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Abstract. We obtain similar types of conclusions as that of Brück [1] for two differential polynomials which in turn radically improve and generalize several existing results. Moreover a number of examples have been exhibited to justify the necessity or sharpness of some conditions used in the paper. At last we pose an open problem for future research.

1. Introduction definitions and results

In the paper by meromorphic function we always mean a function which is meromorphic in the open complex plane $\mathbb{C}$.

If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have same set of a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities) and if we do not consider the multiplicities, then f, g are said to share the value a IM (ignoring multiplicities). When $a = \infty$ the zeros of $f - a$ means the poles of f. Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a; f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(a; f)$ the set of distinct a-points of $f(z)$ with multiplicities not greater than m. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{m)}(a, f) = E_{m)}(a, g)$ $(\overline{E}_{m)}(a, f) = \overline{E}_m(a, g))$ holds for $m = \infty$ we say that f, g share the value a CM (IM).

It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)) \quad (r \to \infty,\ r \notin E).$$

A meromorphic function $a(\not\equiv \infty)$ is called a small function with respect to f provided that $T(r, a) = S(r, f)$ as $(r \to \infty,\ r \notin E)$. If $a = a(z)$ is a small function we define that f and g share a IM or a CM according as $f - a$ and $g - a$ share 0 CM or 0 IM respectively.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

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Also it is known to us that the hyper order of $f(z)$, denoted by $\rho_2(f)$, is defined by

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Nevanlinna's uniqueness theorem shows that two meromorphic functions f and g share 5 values IM are identical. Rubel and Yang [15] first showed for entire functions that in the special situation where g is the derivative of f, one usually needs sharing of only two values CM for their uniqueness. 2 years later, Mues and Steinmetz [14] proved that actually in the above case one does not even need the multiplicities. There results were as follows.

Theorem A ([14]). *Let f be a non-constant entire function. If f and $f^{'}$ share two distinct values a, b IM, then $f^{'} \equiv f$.*

Subsequently, there were more generalizations with respect to higher derivatives as well.

Natural question would be to investigate the relation between an entire function and its derivative counterpart for one CM shared value. In 1996, in this direction the following famous conjecture was proposed by Brück [1]:

Conjecture. *Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and $f^{'}$ share a finite value a CM, then $\frac{f^{'}-a}{f-a} = c$, where c is a non zero constant.*

Brück himself proved the conjecture for $a = 0$. For $a \neq 0$, Brück [1] showed that under the assumption $N(r,0;f^{'}) = S(r,f)$ the conjecture was true without any growth condition when $a = 1$.

Theorem B ([1]). *Let f be a non-constant entire function. If f and $f^{'}$ share the value 1 CM and if $N(r,0;f^{'}) = S(r,f)$, then $\frac{f^{'}-1}{f-1}$ is a nonzero constant.*

Following example shows the fact that one can not simply replace the value 1 by a small function $a(z)(\not\equiv 0,\infty)$.

Example 1.1. Let $f = 1 + e^{e^z}$ and $a(z) = \frac{1}{1-e^{-z}}$.

By Lemma 2.6 of [4, p. 50] we know that a is a small function of f. Also it can be easily seen that f and $f^{'}$ share a CM and $N(r,0;f^{'}) = 0$ but $f-a \neq c\,(f^{'}-a)$ for every nonzero constant c. We note that $f - a = e^{-z}\,(f^{'} - a)$. So in this case additional suppositions are required.

However for entire function of finite order, Yang [16] removed the supposition $N(r,0;f^{'}) = 0$ and obtained the following result.

Theorem C ([16]). *Let f be a non-constant entire function of finite order and let $a(\neq 0)$ be a finite constant. If f, $f^{(k)}$ share the value a CM, then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant, where $k(\geq 1)$ is an integer.*

Theorem C may be considered as a solution to the Brück conjecture. Next we consider the following examples which show that in Theorem B one can not simultaneously replace "CM" by "IM" and "entire function" by "meromorphic function".

Example 1.2. $f(z) = 1 + \tan z$.

Clearly $f(z)-1 = \tan z$ and $f^{'}(z)-1 = \tan^2 z$ share 1 IM and $N(r, 0; f^{'}) = 0$.

Example 1.3. $f(z) = \frac{2}{1-e^{-2z}}$.

Clearly $f^{'}(z) = -\frac{4e^{-2z}}{(1-e^{-2z})^2}$. Here $f - 1 = \frac{1+e^{-2z}}{1-e^{-2z}}$ and $f' - 1 = -\frac{(1+e^{-2z})^2}{(1-e^{-2z})^2}$. Here $N(r, 0; f^{'}) = 0$ So in both the examples we see that the conclusion of Theorem B ceases to hold.

From the above discussion it is natural to ask the following question.

Question 1.1. Can the conclusion of Theorem B be obtained for a non-constant meromorphic function sharing a small function IM together with its k-th derivative counterpart?

Zhang [18] extended Theorem B to meromorphic function and also studied the CM value sharing of a meromorphic function with its k-th derivative.

Meanwhile a new notion of scalings between CM and IM known as weighted sharing ([5]-[6]), appeared in the uniqueness literature.

In 2004, Lahiri-Sarkar [9] employed weighted value sharing method to improve the results of Zhang [18]. In 2005, Zhang [19] further extended the results of Lahiri-Sarkar to a small function and proved the following result for IM sharing.

Theorem D ([19])**.** *Let f be a non-constant meromorphic function and $k(\geq 1)$ be integer. Also let $a \equiv a(z)$ $(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share 0 IM. If*

$$4\overline{N}(r,\infty;f)+3N_2\left(r,0;f^{(k)}\right)+2\overline{N}\left(r,0;(f/a)^{'}\right) < (\lambda+o(1))\,T\left(r,f^{(k)}\right) \tag{1.1}$$

for $r \in I$, where $0 < \lambda < 1$, then $\frac{f^{(k)}-a}{f-a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

We now recall the following two theorems due to Liu and Yang [11] in the direction of IM sharing related to Theorem B.

Theorem E ([11])**.** *Let f be a non-constant meromorphic function. If f and $f^{'}$ share 1 IM and if*

$$\overline{N}(r,\infty;f) + \overline{N}\left(r,0;f^{'}\right) < (\lambda + o(1))\,T\left(r,f^{'}\right) \tag{1.2}$$

for $r \in I$, where $0 < \lambda < \frac{1}{4}$, then $\frac{f^{'}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C}/\{0\}$.

Theorem F ([11]). *Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share 1 IM and*

$$(3k+6)\overline{N}(r,\infty;f)+5N(r,0;f)<(\lambda+o(1))\,T\left(r,f^{(k)}\right) \tag{1.3}$$

for $r\in I$, where $0<\lambda<1$, then $\frac{f^{(k)}-1}{f-1}\equiv c$ for some constant $c\in\mathbb{C}/\{0\}$.

In 2008, improving the result of Zhang [19], Zhang and Lü [20] further investigated the analogous problem of Brück conjecture for the n-th power of a meromorphic function sharing a small function with its k-th derivative and obtained the following theorem.

Theorem G ([20]). *Let f be a non-constant meromorphic function and $k(\geq 1)$ and $n(\geq 1)$ be integers. Also let $a\equiv a(z)$ $(\not\equiv 0,\infty)$ be a meromorphic small function. Suppose that f^n-a and $f^{(k)}-a$ share 0 IM. If*

$$\begin{aligned}4\overline{N}(r,\infty;f)+\overline{N}\left(r,0;f^{(k)}\right)+2N_2\left(r,0;f^{(k)}\right)+2\overline{N}\left(r,0;(f^n/a)'\right)\\ <(\lambda+o(1))\,T\left(r,f^{(k)}\right)\end{aligned} \tag{1.4}$$

for $r\in I$, where $0<\lambda<1$, then $\frac{f^{(k)}-a}{f^n-a}=c$ for some constant $c\in\mathbb{C}/\{0\}$.

At the end of [20] the following question was raised by Zhang and Lü [20].

What will happen if f^n and $[f^{(k)}]^m$ share a small function?

In order to answer the above question, Liu [10] obtained the following result.

Theorem H ([10]). *Let f be a non-constant meromorphic function and $k(\geq 1)$, $n(\geq 1)$ and $m(\geq 2)$ be integers. Also let $a\equiv a(z)$ $(\not\equiv 0,\infty)$ be a meromorphic small function. Suppose that f^n-a and $(f^{(k)})^m-a$ share 0 IM. If*

$$\begin{aligned}\frac{4}{m}\overline{N}(r,\infty;f)+\frac{5}{m}\overline{N}\left(r,0;f^{(k)}\right)+\frac{2}{m}\overline{N}\left(r,0;(f^n/a)'\right)\\ <(\lambda+o(1))\,T\left(r,f^{(k)}\right)\end{aligned} \tag{1.5}$$

for $r\in I$, where $0<\lambda<1$, then $\frac{(f^{(k)})^m-a}{f^n-a}=c$ for some constant $c\in\mathbb{C}/\{0\}$.

Next we recall the following definition.

Definition 1.1. Let $n_{0j},n_{1j},\ldots,n_{kj}$ be non negative integers. The expression $M_j[f]=(f)^{n_{0j}}(f^{(1)})^{n_{1j}}\ldots(f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d(M_j)=\sum_{i=0}^{k}n_{ij}$ and weight $\Gamma_{M_j}=\sum_{i=0}^{k}(i+1)n_{ij}$.

The sum $P[f]=\sum_{j=1}^{t}b_jM_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P)=\max\{d(M_j):1\leq j\leq t\}$ and weight $\Gamma_P=\max\{\Gamma_{M_j}:1\leq j\leq t\}$, where $T(r,b_j)=S(r,f)$ for $j=1,2,\ldots,t$.

The numbers $\underline{d}(P)=\min\{d(M_j):1\leq j\leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\overline{d}(P)=\underline{d}(P)$.

$P[f]$ is called a Linear Differential Polynomial generated by f if $\overline{d}(P) = 1$. Otherwise $P[f]$ is called Non-linear Differential Polynomial. We also denote by $\mu = \max\{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = \max\{n_{1j} + 2n_{2j} + \cdots + kn_{kj} : 1 \leq j \leq t\}$.

As $(f^{(k)})^m$ is simply a special differential monomial in f, it will be interesting to investigate whether Theorems D-H can be extended up to differential polynomial generated by f. In this direction recently Li and Yang [12] improved Theorem D in the following manner.

Theorem I ([12]). *Let f be a non-constant meromorphic function $P[f]$ be a differential polynomial generated by f. Also let $a \equiv a(z)$ $(\not\equiv 0, \infty)$ be a small meromorphic function. Suppose that $f - a$ and $P[f] - a$ share 0 IM and $(t-1)\overline{d}(P) \leq \sum_{j=1}^{t} d(M_j)$. If*

$$(1.6)\; 4\overline{N}(r, \infty; f) + 3N_2(r, 0; P[f]) + 2\overline{N}\left(r, 0; (f/a)'\right) < (\lambda + o(1))\, T(r, P[f])$$

for $r \in I$, where $0 < \lambda < 1$, then $\frac{P[f]-a}{f-a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

So we see that Theorem I always holds for a monomial without any condition on its degree. But for general differential polynomial one can not eliminate the supposition $(t-1)\overline{d}(P) \leq \sum_{j=1}^{t} d(M_j)$ in the above theorem. So whether in Theorem I, the condition over the degree can be removed, sharing notion can further be relaxed, (1.6) can further be weakened, are all open problems.

We also observe that the afterward research on Brück and its generalization, one setting among the sharing functions has been restricted to only various powers of f not involving any other variants such as derivatives of f, where as the generalization have been made on the second setting. This observation must motivate oneself to find the answer of the following question.

Question 1.2. Can Brück type conclusion be obtained when two different differential polynomials share a small functions IM or even under relaxed sharing notions?

The main intention of the paper is to obtain the possible answer of the above question in such a way that it improves, unifies and generalizes all the Theorems D-H. Following theorem is the main result of the paper. Henceforth by b_j, $j = 1, 2, \ldots, t$ and c_i, $i = 1, 2, \ldots, l$ we denote small functions in f and we also suppose that $P[f] = \sum_{j=1}^{t} b_j M_j[f]$ and $Q[f] = \sum_{i=1}^{l} c_i M_i[f]$ be two differential polynomial generated by f.

Theorem 1.1. *Let f be a non-constant meromorphic function, $m(\geq 1)$ be a positive integer or infinity and $a \equiv a(z)$ $(\not\equiv 0, \infty)$ be a small meromorphic function. Suppose that $P[f]$ and $Q[f]$ be two differential polynomial generated by f such that $Q[f]$ contains at least one derivative. Suppose further that $\overline{E}_{m)}(a, P[f]) = \overline{E}_{m)}(a, Q[f])$. If*

$$(1.7) \qquad 4\overline{N}(r, \infty; f) + N_2(r, 0; Q[f]) + 2\overline{N}(r, 0; Q[f]) + \overline{N}\left(r, 0; (P[f]/a)'\right)$$

$$+\overline{N}\left(r,0;(P[f]/a)^{'}\mid(P[f]/a)\neq 0\right)<(\lambda+o(1))\,T\,(r,Q[f])$$

for $r\in I$, *where* $0<\lambda<1$, *then either* a) $\frac{Q[f]-a}{P[f]-a}=c$ *for some constant* $c\in\mathbb{C}/\{0\}$ *or* b) $P[f]Q[f]-aQ[f](1+d)\equiv -da^2$ *for a non-zero constant* $d\in\mathbb{C}$.

In particular, if i) $P[f]=b_1f^n+b_2f^{n-1}+b_3f^{n-2}+\cdots+b_{t-1}f$ *or if* ii) $\underline{d}(Q)>2\overline{d}(P)-\underline{d}(P)$ *and each monomial of* $Q[f]$ *contains a term involving a power of* f, *then the conclusion* (b) *does not hold.*

Remark 1.1. Clearly in Theorem 1.1 when $m=\infty$ we have $P[f]-a$ and $Q[f]-a$ share 0 IM where $P[f]=b_1f^n+b_2f^{n-1}+b_3f^{n-2}+\cdots+b_{t-1}f$ and we obtain the improved, extended and generalized version of Theorem I in the direction of Question 1.1.

Following five examples show that (1.7) is not necessary when (i) and (ii) of Theorem 1.1 occurs.

Example 1.4. Let $f(z)=\frac{e^z}{e^z+1}$. $P[f]=f^2$, $Q[f]=f-f^{'}$. Then clearly $P[f]$ and $Q[f]$ share 1 CM and $\frac{Q[f]-1}{P[f]-1}=1$, but (1.7) is not satisfied.

Example 1.5. Let $f(z)=\frac{1}{e^z+1}$. $P[f]=f^2-f^3$, $Q[f]=-ff^{'}$. Then clearly $P[f]$ and $Q[f]$ share 1 CM and $\frac{Q[f]-1}{P[f]-1}=1$, but (1.7) is not satisfied.

Example 1.6. Let $f(z)=\frac{e^z}{e^z+1}$. $P[f]=f-f^{'}$, $Q[f]=f^2-3ff^{'3}+f^3f^{'2}-ff^{'}f^{'''}+ff^{'}f^{''}$. Then clearly $P[f]$ and $Q[f]$ share 1 CM and $\frac{Q[f]-1}{P[f]-1}=1$, but (1.7) is not satisfied.

Example 1.7. Let $f(z)=\frac{1}{e^z+1}$. $P[f]=(f^{'})^2-ff^{''}$, $Q[f]=2ff^{'2}-f^2f^{''}$. Then clearly $P[f]$ and $Q[f]$ share 1 CM and $\frac{Q[f]-1}{P[f]-1}=1$, but (1.7) is not satisfied. Here we note that $3=\underline{d}(Q)>2\overline{d}(P)-\underline{d}(P)=2$.

Example 1.8. Let $f(z)=\frac{1}{e^z+1}$. $P[f]=f^{'2}$, $Q[f]=ff^{''}-f^2f^{'}$. Then clearly $P[f]=Q[f]=\frac{e^{2z}}{(e^z+1)^4}$ share $\frac{1}{z}$ CM and $\frac{Q[f]-\frac{1}{z}}{P[f]-\frac{1}{z}}=1$, but (1.7) is not satisfied.

We now give the next five examples the first two of which show that both the conditions stated in (ii) are essential in order to obtain conclusion (a) in Theorem 1.1 for homogeneous differential polynomials $P[f]$ where as the rest three substantiate the same for non homogeneous differential polynomials.

Example 1.9. Let $f(z)=\sin z$. $P[f]=f^{''2}-f^{'2}+2if^{''}f^{'''}$, $Q[f]=f^2-2iff^{'}-f^{'''2}$. Then clearly $P[f]=-e^{-2iz}$ and $Q[f]=-e^{2iz}$ share 1 CM. Here $T(r,Q)=\frac{2r}{\pi}+O(1)$, (1.7) is satisfied, but $\frac{Q[f]-1}{P[f]-1}=e^{2iz}$, rather $P[f]Q[f]=1$.

Example 1.10. Let $f(z) = \sin z$. $P[f] = 3f^2 + f^{'2} - 2iff^{'}$, $Q[f] = f^{'2} - 2iff^{'} - f^2$. Then clearly $P[f] = 2 - e^{2iz}$ and $Q[f] = e^{-2iz}$ share 1 CM. Here (1.7) is satisfied, but $\frac{Q[f]-1}{P[f]-1} = e^{-2iz}$, rather $P[f]Q[f] - 2Q[f] + 1 = 0$.

Example 1.11. Let $f(z) = \cos z$. $P[f] = f^3 + 3if^{'}f^{'''2} + 3f^{'2}f^{''} - 3if^{'} - if^{'''3}$, $Q[f] = 3f^{''} - 4f^{''3} + 3if^2f^{'} + if^{'''3}$. Then clearly $P[f] = e^{3iz}$ and $Q[f] = e^{-3iz}$ share 1 CM. Here (1.7) is satisfied, but $\frac{Q[f]-1}{P[f]-1} = -e^{-3iz}$ rather $P[f]Q[f] = 1$. We also note that here $\overline{d}(P) \neq \underline{d}(P)$, $1 = \underline{d}(Q) \not> 2\overline{d}(P) - \underline{d}(P) = 5$.

Example 1.12. Let $f(z) = \cos z$. $P[f] = -2ff^{''} + f^{'2} - f^{'}f^{'''} - f^{''} + if^{'''}$, $Q[f] = -f + if^{'''}$. Then clearly $P[f] = e^{iz} + 2$ and $Q[f] = -e^{-iz}$ and so they share 1 CM. Here (1.7) is satisfied, but $\frac{Q[f]-1}{P[f]-1} = -e^{-iz}$, rather $P[f]Q[f] - 2Q[f] + 1 = 0$. We also note that here $\overline{d}(P) \neq \underline{d}(P)$, $1 = \underline{d}(Q) \not> 2\overline{d}(P) - \underline{d}(P) = 3$.

Example 1.13. Let $f(z) = \cos z$. $P[f] = -f - if^{'} + (1+i)f^{'2} + (1+i)f^{''2}$, $Q[f] = if - f^{'''}$. Then clearly $P[f] = 1 + i - e^{-iz}$ and $Q[f] = ie^{iz}$ share both i and 1 CM. Here (1.7) is satisfied and $P[f]Q[f] - (1+i)Q[f] + i = 0$. When we consider i as the shared value then $\frac{Q[f]-i}{P[f]-i} = ie^{iz}$, on the other hand when we consider 1 as the shared value then $\frac{Q[f]-1}{P[f]-1} = e^{iz}$. We also note that here $\overline{d}(P) \neq \underline{d}(P)$, $1 = \underline{d}(Q) \not> 2\overline{d}(P) - \underline{d}(P) = 3$.

The following two examples show that in order to obtain conclusions (a) or (b) of Theorem 1.1, (1.7) is essential.

Example 1.14. Let $f(z) = \sin z$. $P[f] = if + f^{'}$, $Q[f] = 2f^{'} - (f^2 + f^{'2})$. Then clearly $P[f] = e^{iz}$ and $Q[f] = e^{iz} + e^{-iz} - 1$ share 1 IM. Here neither of the conclusions of Theorem 1.1 is satisfied, nor (1.7) is satisfied. We note that $\frac{Q[f]-1}{P[f]-1} = \frac{(e^{iz}-1)}{e^{iz}}$ and $P[f]Q[f] - e^{iz}Q[f] = 0$.

Example 1.15. Let $f(z) = \cos z$. $P[f] = f - if^{'}$, $Q[f] = 2f - (f^{'2} + f^{''2})$. Then clearly $P[f] = e^{iz}$ and $Q[f] = e^{iz} + e^{-iz} - 1$ share 1 IM. Here neither of the conclusions of Theorem 1.1 is satisfied, nor (1.7) is satisfied. We note that $\frac{Q[f]-1}{P[f]-1} = \frac{(e^{iz}-1)}{e^{iz}}$ and $P[f]Q[f] - e^{iz}Q[f] = 0$.

Though we use the standard notations and definitions of the value distribution theory available in [4], we explain some definitions and notations which are used in the paper.

Definition 1.2 ([9])**.** Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid\geq p)$ ($\overline{N}(r, a; f \mid\geq p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r,a;f\mid\leq p)$ $(\overline{N}(r,a;f\mid\leq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.3 (6, cf. [17])**.** For $a\in\mathbb{C}\cup\{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+\cdots+\overline{N}(r,a;f\mid\geq p)$. Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

Definition 1.4. Let k be a positive integer and for $a\in\mathbb{C}-\{0\}$, $\overline{E}_{k)}(a;f)=\overline{E}_{k)}(a;g)$. Let z_0 be a zero of $f(z)-a$ of multiplicity p and a zero of $g(z)-a$ of multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p>q\geq 1$, by $\overline{N}_{f>s}(r,a;g)$ $(\overline{N}_{g>s}(r,a;f))$ the counting functions of those a-points of f and g for which $p>q=s(q>p=s)$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p=q=1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p=q\geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$. We denote by $\overline{N}_{f\geq k+1}(r,a;f\mid\ g\neq a)$ $(\overline{N}_{g\geq k+1}(r,a;g\mid\ f\neq a))$ the reduced counting functions of those a-points of f and g for which $p\geq k+1$ and $q=0$ $(q\geq k+1$ and $p=0)$.

Definition 1.5 ([7])**.** Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid\ g\neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

Definition 1.6 ([5, 6])**.** Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly,

$$\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)\text{ and }\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g).$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1. *Let $\overline{E}_{m)}(1;F)=\overline{E}_{m)}(1;G)$; F, G share ∞ IM and $H\not\equiv 0$. Then*

$$\begin{aligned}&N(r,\infty;H)\\ &\leq\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,\infty;F,G)\\ &\quad+\overline{N}_{F\geq m+1}(r,1;F\mid\ G\neq 1)+\overline{N}_{G\geq m+1}(r,1;G\mid\ F\neq 1)\end{aligned}$$

$$+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'}),$$

where $\overline{N}_0(r,0;F^{'})$ *is the reduced counting function of those zeros of* $F^{'}$ *which are not the zeros of* $F(F-1)$ *and* $\overline{N}_0(r,0;G^{'})$ *is similarly defined.*

Proof. We can easily verify that possible poles of H occur at (i) multiple zeros of F and G, (ii) poles of F and G with different multiplicities, (iii) the common zeros of $F-1$ and $G-1$ with different multiplicities, (iii) zeros of $F-1$ $(G-1)$ which are not the zeros of $G-1$ $(F-1)$, (iv) those 1-points of F (G) which are not the 1-points of G (F), (v) zeros of $F^{'}$ which are not the zeros of $F(F-1)$, (vi) zeros of $G^{'}$ which are not zeros of $G(G-1)$. Since H has simple pole the lemma follows from above. □

Lemma 2.2 ([19])**.** *Let f be a non-constant meromorphic function and k be a positive integer. Then*

$$N_p(r,0;f^{(k)})\le N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.3 ([8])**.** *If* $N(r,0;f^{(k)}\mid f\neq 0)$ *denotes the counting function of those zeros of* $f^{(k)}$ *which are not the zeros of* f, *where a zero of* $f^{(k)}$ *is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}\mid f\neq 0)\le k\overline{N}(r,\infty;f)+N(r,0;f\mid<k)+k\overline{N}(r,0;f\mid\ge k)+S(r,f).$$

Lemma 2.4 ([13])**.** *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n}a_kf^k}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n\neq 0$ and $b_m\neq 0$. Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.5 ([2])**.** *Let f be a meromorphic function and $P[f]$ be a differential polynomial. Then*

$$m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)\le(\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)+S(r,f).$$

Lemma 2.6. *Let f be a meromorphic function and $P[f]$ be a differential polynomial. Then we have*

$$N\left(r,\infty;\frac{P[f]}{f^{\overline{d}(P)}}\right)\le(\Gamma_P-\overline{d}(P))\,\overline{N}(r,\infty;f)+(\overline{d}(P)-\underline{d}(P))\,N(r,0;f\mid\ge k+1)$$
$$+\mu\overline{N}(r,0;f\mid\ge k+1)+\overline{d}(P)N(r,0;f\mid\le k)+S(r,f).$$

Proof. Let z_0 be a pole of f of order r, such that $b_j(z_0) \neq 0, \infty : 1 \leq j \leq t$. Then it would be a pole of $P[f]$ of order at most $r\overline{d}(P) + \Gamma_P - \overline{d}(P)$. Since z_0 is a pole of $f^{\overline{d}(P)}$ of order $r\overline{d}(P)$, it follows that z_0 would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order at most $\Gamma_P - \overline{d}(P)$. Next suppose z_1 is a zero of f of order $s(> k)$, such that $b_j(z_1) \neq 0, \infty : 1 \leq j \leq t$. Clearly it would be a zero of $M_j(f)$ of order $s.n_{0j} + (s-1)n_{1j} + \cdots + (s-k)n_{kj} = s.d(M_j) - (\Gamma_{M_j} - d(M_j))$. Hence z_1 be a pole of $\frac{M_j[f]}{f^{\overline{d}(P)}}$ of order

$$s.\overline{d}(P) - s.d(M_j) + (\Gamma_{M_j} - d(M_j)) = s(\overline{d}(P) - d(M_j)) + (\Gamma_{M_j} - d(M_j)).$$

So z_1 would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order at most

$$\max\{s(\overline{d}(P) - d(M_j)) + (\Gamma_{M_j} - d(M_j)) : 1 \leq j \leq t)\} = s(\overline{d}(P) - \underline{d}(P)) + \mu.$$

If z_1 is a zero of f of order $s \leq k$, such that $b_j(z_1) \neq 0, \infty : 1 \leq j \leq t$, then it would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order $s\overline{d}(P)$. Since the poles of $\frac{P[f]}{f^{\overline{d}(P)}}$ comes from the poles or zeros of f and poles or zeros of $b_j(z)$'s only, it follows that

$$N\left(r, \infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\Gamma_P - \overline{d}(P))\, \overline{N}(r, \infty; f) + (\overline{d}(P) - \underline{d}(P))\, N(r, 0; f \mid\geq k+1)$$
$$+ \mu\, \overline{N}(r, 0; f \mid\geq k+1) + \overline{d}(P) N(r, 0; f \mid\leq k) + S(r, f). \qquad \square$$

Lemma 2.7 ([3])**.** *Let $P[f]$ be a differential polynomial. Then*

$$T(r, P[f]) \leq \Gamma_P T(r, f) + S(r, f).$$

Lemma 2.8. *Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial. Then $S(r, P[f])$ can be replaced by $S(r, f)$.*

Proof. From Lemma 2.7 it is clear that $T(r, P[f]) = O(T(r, f))$ and so the lemma follows. □

Lemma 2.9. *Let f be a non-constant meromorphic function and $P[f]$, $Q[f]$ be two differential polynomials. Then*

$$N(r, 0; P[f]) \leq \frac{\overline{d}(P) - \underline{d}(P)}{\underline{d}(Q)} m\left(r, \frac{1}{Q[f]}\right) + (\Gamma_P - \overline{d}(P))\, \overline{N}(r, \infty; f)$$
$$+ (\overline{d}(P) - \underline{d}(P))\, N(r, 0; f \mid\geq k+1) + \mu \overline{N}(r, 0; f \mid\geq k+1)$$
$$+ \overline{d}(P) N(r, 0; f \mid\leq k) + \overline{d}(P)\, N(r, 0; f) + S(r, f).$$

Proof. For a fixed value of r, let $E_1 = \{\theta \in [0, 2\pi] : \left|f(re^{i\theta})\right| \leq 1\}$ and E_2 be its complement. Since by definition

$$\sum_{i=0}^{k} n_{ij} \geq \underline{d}(Q)$$

for every $j = 1, 2, \ldots, l$, it follows that on E_1

$$\left|\frac{Q[f]}{f^{\underline{d}(Q)}}\right| \leq \sum_{j=1}^{l} |c_j(z)| \prod_{i=1}^{k} \left|\frac{f^{(i)}}{f}\right|^{n_{ij}} |f|^{\sum\limits_{i=0}^{k} n_{ij} - \underline{d}(Q)} \leq \sum_{j=1}^{l} |c_j(z)| \prod_{i=1}^{k} \left|\frac{f^{(i)}}{f}\right|^{n_{ij}}.$$

Also we note that

$$\frac{1}{f^{\underline{d}(Q)}} = \frac{Q[f]}{f^{\underline{d}(Q)}} \frac{1}{Q[f]}.$$

Since on E_2, $\frac{1}{|f(z)|} < 1$, we have

$$\begin{aligned}
&\underline{d}(Q) m\left(r, \frac{1}{f}\right) \\
&= \frac{1}{2\pi} \int\limits_{E_1} \log^+ \frac{1}{|f(re^{i\theta})|^{\underline{d}(Q)}} d\theta + \frac{1}{2\pi} \int\limits_{E_2} \log^+ \frac{1}{|f(re^{i\theta})|^{\underline{d}(Q)}} d\theta \\
&\leq \frac{1}{2\pi} \sum_{j=1}^{l} \left[\int\limits_{E_1} \log^+ |c_j(z)|\, d\theta + \sum_{i=1}^{k} \int\limits_{E_1} \log^+ \left|\frac{f^{(i)}}{f}\right|^{n_{ij}} d\theta\right] \\
&\quad + \frac{1}{2\pi} \int\limits_{E_1} \log^+ \left|\frac{1}{Q[f(re^{i\theta})]}\right| d\theta \\
&\leq \frac{1}{2\pi} \int\limits_{0}^{2\pi} \log^+ \left|\frac{1}{Q[f(re^{i\theta})]}\right| d\theta + S(r, f) = m\left(r, \frac{1}{Q[f]}\right) + S(r, f).
\end{aligned}$$

So using Lemmas 2.5 and 2.6 and the first fundamental theorem we get

$$\begin{aligned}
&N(r, 0; P[f]) \\
&\leq N\left(r, \infty; \frac{f^{\overline{d}(P)}}{P[f]}\right) + \overline{d}(P) N(r, 0; f) \\
&\leq m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) + N\left(r, \infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) + \overline{d}(P) N(r, 0; f) + S(r, f) \\
&\leq (\overline{d}(P) - \underline{d}(P)) m\left(r, \frac{1}{f}\right) + (\Gamma_P - \overline{d}(P))\, \overline{N}(r, \infty; f) \\
&\quad + (\overline{d}(P) - \underline{d}(P))\, N(r, 0; f \mid\geq k+1) + \mu \overline{N}(r, 0; f \mid\geq k+1) \\
&\quad + \overline{d}(P) N(r, 0; f \mid\leq k) + \overline{d}(P)\, N(r, 0; f) + S(r, f) \\
&\leq \frac{(\overline{d}(P) - \underline{d}(P))}{\underline{d}(Q)} m\left(r, \frac{1}{Q[f]}\right) + (\Gamma_P - \overline{d}(P))\, \overline{N}(r, \infty; f) \\
&\quad + (\overline{d}(P) - \underline{d}(P))\, N(r, 0; f \mid\geq k+1) + \mu \overline{N}(r, 0; f \mid\geq k+1) \\
&\quad + \overline{d}(P) N(r, 0; f \mid\leq k) + \overline{d}(P)\, N(r, 0; f) + S(r, f).
\end{aligned}$$

□

3. Proof of the theorem

Proof of Theorem 1.1. Let $F=\frac{P[f]}{a}$ and $G=\frac{Q[f]}{a}$. Then $F-1=\frac{P[f]-a}{a}$, $G-1=\frac{Q[f]-a}{a}$. Since $\overline{E}_{m)}(a,P[f])=\overline{E}_{m)}(a,Q[f])$, it follows that $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. Let $H\not\equiv 0$.

Let z_0 be a simple zero of $F-1$. Then by a simple calculation we see that z_0 is a zero of H and hence

$$N_E^{1)}(r,1;F)=N_E^{1)}(r,1;G)\le N(r,0;H)\le N(r,\infty;H)+S(r,F). \tag{3.1}$$

Using (3.1), Lemmas 2.1 and 2.8 and noting that $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)+S(r,f)=\overline{N}(r,\infty;f)+S(r,f)$ and $\overline{N}_{F>1}(r,1;G)+\overline{N}(r,1;G\mid\ge 2)=\overline{N}_E^{(2}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;F)+\overline{N}_{G\ge m+1}(r,1;G\mid\ F\neq 1)+S(r,f)$, we get from the second fundamental theorem that

(3.2)

$$\begin{aligned}
&T(r,G)\\
\le\ &\overline{N}(r,\infty;G)+\overline{N}(r,0;G)+N_E^{1)}(r,1;G)+\overline{N}_{F>1}(r,1;G)+\overline{N}(r,1;G\mid\ge 2)\\
&-N_0(r,0;G^{'})+S(r,G)\\
\le\ &2\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}(r,0;F\mid\ge 2)+2\overline{N}_L(r,1;F)\\
&+2\overline{N}_L(r,1;G)+\overline{N}_{F\ge m+1}(r,1;F\mid\ G\neq 1)+2\overline{N}_{G\ge m+1}(r,1;G\mid\ F\neq 1)\\
&+\overline{N}_E^{(2}(r,1;G)+\overline{N}_0(r,0;F^{'})+S(r,f).
\end{aligned}$$

Using Lemmas 2.2 and 2.3 we see that

(3.3)

$$\begin{aligned}
&\overline{N}(r,0;G\mid\ge 2)+2\overline{N}_{G\ge m+1}(r,1;G\mid\ F\neq 1)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;G)\\
\le\ &\overline{N}(r,0;G^{'}\mid G\neq 0)+\overline{N}(r,0;G^{'})+S(r,f)\\
\le\ &2\overline{N}(r,\infty;f)+\overline{N}(r,0;Q[f])+N_2(r,0;Q[f])+S(r,f)
\end{aligned}$$

and

(3.4)

$$\begin{aligned}
&\overline{N}(r,0;F\mid\ge 2)+\overline{N}_{F\ge m+1}(r,1;F\mid\ G\neq 1)+2\overline{N}_L(r,1;F)+\overline{N}_0(r,0;F^{'})\\
\le\ &\overline{N}(r,0;F^{'}\mid F\neq 0)+\overline{N}(r,0;F^{'})+S(r,f)\\
\le\ &\overline{N}(r,0;(P[f]/a)^{'}\mid (P[f]/a)\neq 0)+\overline{N}(r,0;(P[f]/a)^{'})+S(r,f).
\end{aligned}$$

Using (3.3) and (3.4) in (3.1) we have

$$\begin{aligned}
T\left(r,Q[f]\right)\le\ &4\overline{N}(r,\infty;f)+2\overline{N}\left(r,0;Q[f]\right)+N_2\left(r,0;Q[f]\right)+\overline{N}\left(r,0;(P[f]/a)^{'}\right)\\
&+\overline{N}\left(r,0;(P[f]/a)^{'}\mid (P[f]/a)\neq 0\right)+S(r,f).
\end{aligned}$$

This contradicts (1.7).

Case 2. Let $H \equiv 0$.

On integration we get from

$$\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \tag{3.5}$$

where C, D are constants and $C \neq 0$. From (3.5) it is clear that F and G share 1 CM. We first assume that $D \neq 0$. Then by (3.5) we get

$$\overline{N}(r,\infty;f) = S(r,f). \tag{3.6}$$

Clearly $\overline{N}(r,\infty;G) = \overline{N}(r,\infty;f) + S(r,f) = S(r,f)$.

From (3.5) we get

$$\frac{1}{F-1} = \frac{D\left(G-1+\frac{C}{D}\right)}{G-1}. \tag{3.7}$$

Clearly from (3.7) we have

$$\overline{N}\left(r,1-\frac{C}{D};G\right) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;G) = S(r,f). \tag{3.8}$$

If $\frac{C}{D} \neq 1$, by the second fundamental theorem, Lemma 2.8 and (3.8) we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\left(r,1-\frac{C}{D};G\right) + S(r,G) \\ &\leq \overline{N}(r,0;G) + S(r,f) \leq N_2(r,0;G) + S(r,f) \\ &\leq T(r,G) + S(r,f). \end{aligned}$$

So $T(r,G) = N_2(r,0;G) + S(r,f)$ that is, $T\left(r,Q[f]\right) = N_2\left(r,0;Q[f]\right) + S(r,f)$, which contradicts (1.7).

If $\frac{C}{D} = 1$ we get from (3.5)

$$\left(F-1-\frac{1}{C}\right)G \equiv -\frac{1}{C}, \tag{3.9}$$

i.e.,

$$P[f]Q[f] - aQ(1+d) \equiv -da^2$$

for a non zero constant $d = \frac{1}{C} \in \mathbb{C}$. From (3.9) it follows that

$$N(r,0;f \mid\geq k+1) \leq N(r,0;Q[f]) \leq N(r,0;G) \leq N(r,0;a) = S(r,f). \tag{3.10}$$

When $P[f] = b_1 f^n + b_2 f^{n-1} + b_3 f^{n-2} + \cdots + b_{t-1} f$, we see from (3.9) that

$$\frac{1}{f^{\overline{d}(Q)}\left(P[f]-(1+1/C)a\right)} \equiv -\frac{C}{a^2}\ \frac{Q[f]}{f^{\overline{d}(Q)}}.$$

Hence by the first fundamental theorem, (3.6), (3.10), Lemmas 2.4, 2.5 and 2.6 we get that

$$(n+\overline{d}(Q))T(r,f) \tag{3.11}$$

$$
\begin{aligned}
&= T\left(r, f^{\overline{d}(Q)}(P[f]-(1+\frac{1}{C})a)\right)+S(r,f)\\
&= T\left(r, \frac{1}{f^{\overline{d}(Q)}(P[f]-(1+\frac{1}{C})a)}\right)+S(r,f)\\
&= T\left(r, \frac{Q[f]}{f^{\overline{d}(Q)}}\right)+S(r,f)\\
&\leq m\left(r, \frac{Q[f]}{f^{\overline{d}(Q)}}\right)+N\left(r, \frac{Q[f]}{f^{\overline{d}(Q)}}\right)+S(r,f)\\
&\leq (\overline{d}(Q)-\underline{d}(Q))\left[T(r,f)-\{N(r,0;f\mid\leq k)+N(r,0;f\mid\geq k+1)\}\right]\\
&\quad +(\overline{d}(Q)-\underline{d}(Q))N(r,0;f\mid\geq k+1)+\mu\,\overline{N}(r,0;f\mid\geq k+1)\\
&\quad +\overline{d}(Q)N(r,0;f\mid\leq k)+S(r,f)\\
&\leq (\overline{d}(Q)-\underline{d}(Q))T(r,f)+\underline{d}(Q)N(r,0;f\mid\leq k)+S(r,f).
\end{aligned}
$$

From (3.11) it follows that

$$nT(r,f)\leq S(r,f),$$

which is absurd.

If $P[f]$ is a differential polynomial, then we consider the following two subcases.

Subcase 2.1.

If $C=-1$, then from (3.5) we get $FG\equiv 1$, i.e., $P[f]Q[f]\equiv a^2$. It is clear that $\overline{N}(r,\infty;P[f])=\overline{N}(r,\infty;Q[f])=S(r,f)$.

First we observe that since each monomial of $Q[f]$ contains a term involving a power of f, we have $N(r,0;f)=S(r,f)$. So from the first fundamental theorem, Lemma 2.5 and noting that $m\left(r,\frac{1}{f}\right)\leq\frac{1}{\underline{d}(Q)}m(r,\frac{1}{Q[f]}))$ we have

$$
\begin{aligned}
T(r,Q[f]) &\leq T(r,P[f])+S(r,f)\\
&\leq m(r,\frac{P[f]}{f^{\overline{d}(P)}})+\overline{d}(P)m(r,f)+S(r,f)\\
&\leq (\overline{d}(P)-\underline{d}(P))m(r,\frac{1}{f})+\overline{d}(P)m(r,f)+S(r,f)\\
&\leq \frac{(\overline{d}(P)-\underline{d}(P))}{\underline{d}(Q)}m(r,\frac{1}{Q[f]})+\overline{d}(P)\{m(r,\frac{1}{f})+N(r,0;f)\}+S(r,f)\\
&\leq \frac{(\overline{d}(P)-\underline{d}(P))}{\underline{d}(Q)}m(r,\frac{1}{Q[f]})+\frac{\overline{d}(P)}{\underline{d}(Q)}m(r,\frac{1}{Q[f]})+S(r,f),
\end{aligned}
$$

which is a contradiction as $\underline{d}(Q)>2\overline{d}(P)-\underline{d}(P)$.

Subcase 2.2.

Next we assume $C\neq -1$.

Then from (3.9) we have

$$\overline{N}(r,1+\frac{1}{C};F)=\overline{N}(r,\infty;G)=S(r,f).$$

So again noticing the fact that each monomial of $Q[f]$ contains a term involving a power of f, by the second fundamental theorem, *Lemma 2.9* we get

$$\begin{aligned}(3.12)\qquad & T(r,P[f])\\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1+\frac{1}{C};F)+S(r,f)\\ &\leq N(r,0;P[f])+S(r,f)\\ &\leq \frac{\overline{d}(P)-\underline{d}(P)}{\underline{d}(Q)}T(r,P[f])+S(r,f),\end{aligned}$$

i.e.,

$$\frac{\underline{d}(Q)+\underline{d}(P)-\overline{d}(P)}{\underline{d}(Q)}T(r,P[f])\leq S(r,f). \tag{3.13}$$

Since by the given condition $\underline{d}(Q)>2\overline{d}(P)-\underline{d}(P)>\overline{d}(P)-\underline{d}(P)$, (3.13) leads to a contradiction.

Hence $D=0$ and so $\frac{G-1}{F-1}=C$ or $\frac{Q[f]-a}{P[f]-a}=C$. This proves the theorem. □

4. Concluding remark and an open question

From the statement of Theorem 1.1 one can see that when (ii) happens one can not obtain the conclusion of Brück conjecture as a special case. We also see from (3.6) that if $\overline{N}(r,\infty;f)\not=S(r,f)$, then conclusion of Brück conjecture is satisfied for any two arbitrary differential polynomials $P[f]$ and $Q[f]$ where $Q[f]$ contains at least one derivative. The problem arises for those class of meromorphic functions whose poles are relatively small in numbers such as entire functions and thus poles have a vital contributions in this perspective. We point out that the counter examples (1.9)-(1.13), which demonstrate the indispensability of the conditions in (ii), have also been formed for entire functions. So the following question still remain open for further investigations.

Can Brück type conclusion be solely obtained for two arbitrary differential polynomials $P[f]$ and $Q[f]$ generated by the class of meromorphic functions containing relatively small number of poles sharing a small function $a\equiv a(z)$ $(\not\equiv 0,\infty)$ IM ?

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