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SOME RESULTS ON THE QUESTIONS OF KIT-WING YU

Sujoy Majumder

Abstract. The paper deals with the problem of meromorphic functions sharing a small function with its differential polynomials and improves the results of Liu and Gu [9], Lahiri and Sarkar [8], Zhang [13] and Zhang and Yang [14] and also answer some open questions posed by Kit-Wing Yu [16]. In this paper we provide some examples to show that the conditions in our results are the best possible.

1. Introduction, definition and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as the characteristic function $T(r,f)$, the counting function of the poles $N(r,\infty;f)$ and the proximity function $m(r,\infty;f)$ (see [10]). For a non-constant meromorphic function f we denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$, outside of a possible exceptional set of finite linear measure. Let $k\in\mathbb{N}$ and $a\in\mathbb{C}\cup\{\infty\}$. We use $N_{(k}(r,a;f)$ to denote the counting function of a-points of f with multiplicity $\geq k$, $N_{k)}(r,a;f)$ to denote the counting function of a-points of f with multiplicity $<k$. Similarly $\overline{N}_{(k}(r,a;f)$ and $\overline{N}_{k)}(r,a;f)$ are their reduced functions respectively.

For $a\in\mathbb{C}\cup\{\infty\}$ and $p\in\mathbb{N}$ we denote by $N_p(r,a;f)$ the sum

$$\overline{N}(r,a;f)+\overline{N}_{(2}(r,a;f)+\cdots+\overline{N}_{(p}(r,a;f).$$

Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

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For $a \in \mathbb{C} \cup \{\infty\}$ and $p \in \mathbb{N}$, we put

$$\delta_p(a;f) = 1 - \limsup_{r\longrightarrow\infty} \frac{N_p(r,a;f)}{T(r,f)}.$$

Clearly

$$0 \le \delta(a;f) \le \delta_p(a;f) \le \delta_{p-1}(a;f) \le \cdots \le \delta_2(a;f) \le \delta_1(a;f) = \Theta(a;f).$$

Let $a, b \in \mathbb{C} \cup \{\infty\}$ and $p \in \mathbb{N}$. We denote by $\overline{N}_{(p}(r,a;f \mid g = b)$ ($\overline{N}_{(p}(r,a;f \mid g \neq b)$) the reduced counting function of those a-points of f with multiplicities $\ge p$, which are the b-points (not the b-points) of g.

Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$.

A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

To the knowledge of the author perhaps Rubel and Yang [12] were the first authors to study the entire functions that share values with their derivatives and they proved the following result.

Theorem A. *If a non-constant entire function f share two distinct finite values CM with $f^{'}$, then $f \equiv f^{'}$.*

E. Mues and N. Steinmetz [11] have shown that "CM" can be replaced by "IM" in Theorem A (another proof of this result for nonzero shared values is in [2]).

In 1983, Gundersen [3] improved *Theorem A* and obtained the following result.

Theorem B. *Let f be a non-constant meromorphic function, a and b be two distinct finite values. If f and $f^{'}$ share the values a and b IM, then $f \equiv f^{'}$.*

In the aspect of one CM value, R. Brück [1] posed the following question:

Question. What results can be obtained if one assumes that f and $f^{'}$ share only one value CM plus some growth condition?

In this direction Brück [1] proposed the following conjecture:

Conjecture 1.1. *Let f be a non-constant entire function. Suppose*

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and $f^{'}$ share one finite value a CM, then

$$\frac{f^{'} - a}{f - a} = c$$

for some non-zero constant c.

The case that $a = 0$ and that $N(r, 0; f^{'}) = S(r, f)$ had been proved by Brück [1] while the case that f is of finite order had been proved by Gundersen-Yang [4]. However, the corresponding conjecture for meromorphic functions fails in general (see [4]).

In 2003, Yu [16] considered the case that a is a small function and obtained the following results.

Theorem C. *Let f be a non-constant entire function, let $k \in \mathbb{N}$ and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0; f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.*

Theorem D. *Let f be a non-constant non-entire meromorphic function, let $k \in \mathbb{N}$ and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$, f and a do not have any common pole. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $4\,\delta(0; f) + 2(8 + k)\,\Theta(\infty; f) > 19 + 2k$, then $f \equiv f^{(k)}$.*

In the same paper Yu [16] posed the following questions.

Question 1. Can a CM shared value be replaced by an IM shared value in Theorem C?

Question 2. Is the condition $\delta(0; f) > \frac{3}{4}$ sharp in Theorem C?

Question 3. Is the condition $4\,\delta(0; f) + 2(8 + k)\,\Theta(\infty; f) > 19 + 2k$ sharp in Theorem D?

Question 4. Can the condition "f and a do not have any common pole" be deleted in Theorem D?

In 2004, Liu and Gu [9] obtained the following results.

Theorem E. *Let $k \in \mathbb{N}$ and let f be a non-constant meromorphic function and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $f^{(k)}$ and a do not have any common pole of same multiplicity and $2\,\delta(0; f) + 4\,\Theta(\infty; f) > 5$, then $f \equiv f^{(k)}$.*

Theorem F. *Let $k \in \mathbb{N}$ and let f be a non-constant entire function and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0; f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.*

In 2001 an idea of gradation of sharing of values was introduced in [5, 6] which measures how close a shared value is to be shared CM or to be shared IM. This notion is known as weighted sharing and is defined as follows.

Definition 1.1 ([5, 6])**.** Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.

The definition implies that if f and g share a value a with weight k, then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

f and g share (a, k) means that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Lahiri and Sarkar [8] improved Theorem E with weighted shared values and obtained the following theorem.

Theorem G. *Let $f(z)$ be a non-constant meromorphic function, $k \in \mathbb{N}$, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If*

(i) *$a(z)$ has no zero (pole) which is also a zero (pole) of f or $f^{(k)}$ with the same multiplicity,*
(ii) *$f - a$ and $f^{(k)} - a$ share $(0, 2)$,*
(iii) $2\,\delta_{2+k}(0; f) + (4+k)\,\Theta(\infty; f) > 5 + k$,

then $f \equiv f^{(k)}$.

In 2005, Zhang [13] obtained the following result which is an improvement and complement of Theorem G.

Theorem H. *Let f be a non-constant meromorphic function, $k \in \mathbb{N}$ and $l \in \mathbb{N} \cup \{0\}$. Also let $a \equiv a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. Then $f \equiv f^{(k)}$ if one of the following conditions is satisfied,*

(i) *$l \geq 2$ and*
$$(3+k)\,\Theta(\infty; f) + 2\,\delta_{2+k}(0; f) > k + 4;$$
(ii) *$l = 1$ and*
$$(4+k)\,\Theta(\infty; f) + 3\,\delta_{2+k}(0; f) > k + 6;$$
(iii) *$l = 0$ and*
$$(6+2k)\,\Theta(\infty; f) + 5\,\delta_{2+k}(0; f) > 2k + 10.$$

It is natural to ask what happens if $f^{(k)}$ is replaced by a differential polynomial

$$L(f) = f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f \tag{1.1}$$

in the above Theorems E, F, where a_j $(j = 0, 1, \ldots, k-1)$ are polynomials. Corresponding to this question Zhang and Yang [14] obtained the following results.

Theorem I. *Let $k \in \mathbb{N}$, f be a non-constant meromorphic function, and let a be a small meromorphic function such that $a(z) \not\equiv 0, \infty$. Suppose that $L(f)$ is defined by* (1.1). *If $f - a$ and $L(f) - a$ share the value* 0 *IM and*

$$5\,\delta(0; f) + (2k + 6)\,\Theta(\infty; f) > 2k + 10,$$

then $f \equiv L(f)$.

Theorem J. *Let $k \in \mathbb{N}$, f be a non-constant meromorphic function, and let a be a small meromorphic function such that $a(z) \not\equiv 0, \infty$. Suppose that $L(f)$ is defined by* (1.1). *If $f - a$ and $L(f) - a$ share the value* 0 *CM and*

$$2\,\delta(0; f) + 3\,\Theta(\infty; f) > 4,$$

then $f \equiv L(f)$.

The main purpose of this paper is to improve Theorems F-J. Further in this paper we provide some examples to show that the conditions in our results are the best possible.

Henceforth throughout this paper we use the following notation.

$$L_1(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f, \tag{1.2}$$

where a_j $(j = 0, 1, \ldots, k)$ are small meromorphic functions of f such that $a_k(z) \not\equiv 0$.

The following theorems are the main results of the paper.

Theorem 1.1. *Let f be a non-constant meromorphic function and $k \in \mathbb{N}$, $l \in \mathbb{N} \cup \{0\}$. Also let $a \equiv a(z) (\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $L_1(f) - a$ share $(0, l)$, where $L_1(f)$ is defined by* (1.2). *If $l (\geq 1)$ and*

$$\frac{l+1}{l}\,\Theta(\infty; f) + \frac{1}{l}\,\Theta(0; f) + \delta_{k+1}(0; f) > \frac{l+2}{l} \tag{1.3}$$

or $l = 0$ and

$$(k + 1)\,\Theta(\infty; f) + \Theta(0; f) + \delta_k(0; f) > k + 2, \tag{1.4}$$

then $f \equiv L_1(f)$.

Corollary 1.1. *Let f be a non-constant meromorphic function and $k \in \mathbb{N}$. Also let $a \equiv a(z) (\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $L_1(f) - a$ share* 0 *CM, where $L_1(f)$ is defined by* (1.2). *If*

$$\Theta(\infty; f) + \delta_{k+1}(0; f) > 1,$$

then $f \equiv L_1(f)$.

Remark 1.1. Since $2\,\delta_{2+k}(0;f)+(k+4)\,\Theta(\infty;f)>k+5$ or equivalently

$$\begin{aligned}&\frac{3}{2}\,\Theta(\infty;f)+\frac{1}{2}\,\Theta(0;f)+\delta_{k+1}(0;f)\\&>2+(k+\frac{5}{2})\,(1-\Theta(\infty;f))+(\delta_{k+1}(0;f)-\delta_{k+2}(0;f))\\&\quad+\frac{1}{2}\,(\Theta(0;f)-\delta_{k+2}(0;f))+\frac{1}{2}\,(1-\delta_{k+2}(0;f)),\end{aligned}$$

Theorem 1.1 improves Theorem G.

Remark 1.2. Also Theorem 1.1 improves Theorem H.

Remark 1.3. Since $5\,\delta(0;f)+(2k+6)\,\Theta(\infty;f)>2k+10$ or equivalently

$$\begin{aligned}&(k+1)\,\Theta(\infty;f)+\Theta(0;f)+\delta_k(0;f)\\&>k+2+(k+5)\,(1-\Theta(\infty;f))+3\,(1-\delta(0;f))\\&\quad+(\Theta(0;f)-\delta(0;f))+(\delta_k(0;f)-\delta(0;f)),\end{aligned}$$

Theorem 1.1 improves Theorem I.

Remark 1.4. Since $2\,\delta(0;f)+3\,\Theta(\infty;f)>4$ or equivalently

$$\begin{aligned}\Theta(\infty;f)+\delta_{k+1}(0;f)>1&+2\,(1-\Theta(\infty;f))+(1-\delta(0;f))\\&+(\delta_{k+1}(0;f)-\delta(0;f)),\end{aligned}$$

Corollary 1.1 improves Theorem J.

Corollary 1.2. *Let f be a non-constant entire function and $k\in\mathbb{N}$. Also let $a\equiv a(z)(\not\equiv 0,\infty)$ be a meromorphic small function. Suppose that $f-a$ and $L_1(f)-a$ share 0 CM, where $L_1(f)$ is defined by* (1.2)*. If*

$$\delta_{k+1}(0;f)>0,$$

then $f\equiv L_1(f)$.

Remark 1.5. Clearly Corollary 1.2 improves Theorem F.

Remark 1.6. It is easy to see that the condition

$$\Theta(\infty;f)+\delta_{k+1}(0;f)>1$$

in *Corollary 1.1* is sharp by the following examples.

Example 1.1. Let

$$f(z)=\frac{e^z}{e^{e^z}-1}$$

and $L_1(f)=(-e^{-z})f'+(e^{-z}-1)f$. Then f and $L_1(f)$ share the value e^z CM and

$$\Theta(\infty;f)+\delta_2(0;f)=0+1=1,$$

but $f\not\equiv L_1(f)$.

Example 1.2. Let

$$f(z) = \frac{z}{e^{-z} + 1}.$$

Then f and f' share the value 1 CM and $\Theta(\infty; f) + \delta_2(0; f) = 0 + 1 = 1$, but $f \not\equiv f'$.

Example 1.3. Let

$$f(z) = \frac{z + 1}{1 + e^{-z}}.$$

Then f and f' share the value 1 CM and $\Theta(\infty; f) + \delta_2(0; f) = 0 + 1 = 1$, but $f \not\equiv f'$.

Example 1.4. Let

$$f(z) = e^{e^z} + 1,$$

where $a(z) = \frac{1}{1-e^{-z}}$. Then f and f' share the value a CM and $\Theta(\infty; f) + \delta_2(0; f) = 1 + 0 = 1$, but $f \not\equiv f'$.

Remark 1.7. It is easy to see that the condition

$$\delta_{k+1}(0; f) > 0$$

in *Corollary 1.2* is sharp by the following examples.

Example 1.5. Let

$$f(z) = e^{c_1 z} + c_2,$$

where $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1 \neq 1$. Then f and f' share the value c_3 CM, where $c_1 c_2 = c_3(c_1 - 1)$ and $\delta_2(0; f) = 0$, but $f \not\equiv f'$.

Example 1.6. Let

$$f(z) = e^{3z} + \frac{2z}{3} + \frac{2}{9}.$$

Note that

$$f' - z = 3(f - z).$$

Then $f - z$ and $f' - z$ share 0 CM and $\delta_2(0; f) = 0$, but $f \not\equiv f'$.

Theorem 1.2. *Let f be a non-constant meromorphic function and $k \in \mathbb{N}$. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $L_1(f) - a$ share 0 IM, where $L_1(f)$ is defined by* (1.2). *If*

$$(k - 1)\,\Theta(\infty; f) + \delta_2(\infty; f) + \delta_k(0; f) + \delta_{k+1}(0; f) > k + 1, \tag{1.5}$$

then $f \equiv L_1(f)$.

Remark 1.8. It is easy to see that the condition

$$(k - 1)\,\Theta(\infty; f) + \delta_2(\infty; f) + \delta_k(0; f) + \delta_{k+1}(0; f) > k + 1$$

in *Theorem 1.2* is sharp by the following example.

Example 1.7. Let

$$f(z) = \frac{2A}{1 - Be^{-2z}},$$

where A and B are nonzero constants.

Note that

$$N(r, \infty; f) \sim T(r, f)$$

and so $\delta_2(\infty; f) = 0$. Also

$$\Theta(0; f) = \delta_2(0; f) = 1.$$

Then f and $f^{'}$ share the value A IM and

$$\delta_2(\infty; f) + \Theta(0; f) + \delta_2(0; f) = 2,$$

but $f \not\equiv f^{'}$.

2. Lemmas

Lemma 2.1 ([7])**.** *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N_{k)}(r, 0; f) + k\overline{N}_{(k}(r, 0; f) + S(r, f).$$

Lemma 2.2 ([15])**.** *Let f be a non-constant meromorphic function and let $a_n(z)$ ($\not\equiv 0$), $a_{n-1}(z)$,..., $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

3. Proofs of the theorem

Proof of Theorem 1.1. Let $F = \frac{f}{a}$ and $G = \frac{L_1(f)}{a}$, where $L_1(f)$ is defined by (1.2). Then $F - 1 = \frac{f-a}{a}$ and $G - 1 = \frac{L_1(f)-a}{a}$. Since $f - a$ and $L_1(f) - a$ share $(0, l)$, it follows that F and G share $(1, l)$ except the zeros and poles of $a(z)$.

Suppose $F \not\equiv G$.

Now we consider the following cases.

Case 1. Let $l \geq 1$.

$$\begin{aligned} H &= \frac{1}{F}\left(\frac{G^{'}}{G-1} - \frac{F^{'}}{F-1}\right) \\ &= \frac{G}{F}\left(\frac{G^{'}}{G-1} - \frac{G^{'}}{G}\right) - \left(\frac{F^{'}}{F-1} - \frac{F^{'}}{F}\right). \end{aligned} \tag{3.1}$$

Now from Nevanlinna's fundamental estimate of the logarithmic derivative we obtain

$$\begin{aligned} m(r,\infty;\frac{L_1(f)}{f}) &= m(r,\infty;\sum_{j=0}^{k} a_j \frac{f^{(j)}}{f}) \\ &\le \sum_{j=0}^{k} m(r,\infty;a_j\frac{f^{(j)}}{f}) + O(1) \\ &\le \sum_{j=0}^{k} m(r,\infty;\frac{f^{(j)}}{f}) + S(r,f) = S(r,f), \end{aligned}$$

where we define $f^{(0)}(z) = f(z)$.

Suppose $H \equiv 0$. Then from (3.1) we find that

$$G - 1 \equiv c(F-1), \tag{3.2}$$

where $c \in \mathbb{C} \setminus \{0\}$.

Let z_{11} be a pole of $f(z)$ with multiplicity $p_{11} \ge 1$ such that $a_i(z_{11}), a(z_{11}) \ne 0, \infty$, where $i = 0, 1, 2, \ldots, k$. Clearly z_{11} is a pole of $G-1$ with multiplicity $p_{11}+k$ and a pole of $F-1$ with multiplicity p_{11}. Now from (3.2) we see that $p_{11}+k = p_{11}$, which is impossible. Hence either $a_i(z_{11}) = 0, \infty$ (for at least one $i \in \{0, 1, 2, \ldots, k\}$) or $a(z_{11}) = 0, \infty$. Therefore

$$\overline{N}(r,\infty;f) = S(r,f)$$

and so

$$\Theta(\infty;f) = 1.$$

From (1.3) we know

$$\frac{1}{l}\ \Theta(0;f) + \delta_{k+1}(0;f) > \frac{1}{l}. \tag{3.3}$$

Now (3.3) yields $\delta_{k+1}(0;f) > 0$.

If $c = 1$, then $F \equiv G$, a contradiction. Therefore $c \ne 1$ and so

$$\frac{1}{F} = \frac{1}{1-c}\Big(\frac{G}{F} - c\Big). \tag{3.4}$$

It can be easily calculated that the possible poles of $\frac{L_1(f)}{f}$ occur at (i) poles of f and (ii) zeros of f.

Let z^*_{11} be a zero of f with multiplicity $p^*_{11} \ge k+1(p^*_{11} \le k)$ such that $a(z^*_{11}), a_j(z^*_{11}) \ne 0, \infty$, where $j = 0, 1, \ldots, k$. Then z^*_{11} will be a pole of $\frac{L_1(f)}{f}$ with multiplicity $k(p^*_{11})$. Hence

$$\begin{aligned} N(r,\infty;\frac{L_1(f)}{f}) &\le k\,\overline{N}(r,\infty;f) + N_{k+1)}(r,0;f) + k\,\overline{N}_{(k+1}(r,0;f) + S(r,f) \\ &\le N_{k+1)}(r,0;f) + (k+1)\,\overline{N}_{(k+1}(r,0;f) + S(r,f) \\ &= \overline{N}(r,0;f) + \overline{N}_{(2}(r,0;f) + \cdots + \overline{N}_{(k+1}(r,0;f) + S(r,f) \end{aligned}$$

$$= N_{k+1}(r, 0; f) + S(r, f).$$

Now from (3.4) it is clear that

$$\begin{aligned} T(r, f) = T(r, F) + S(r, f) &\le T(r, \frac{G}{F}) + S(r, f) \\ &= T(r, \frac{L_1(f)}{f}) + S(r, f) \\ &\le N(r, \infty; \frac{L_1(f)}{f}) + m(r, \infty; \frac{L_1(f)}{f}) + S(r, f) \\ &\le N_{k+1}(r, 0; f) + S(r, f) \\ &\le (1 - \delta_{k+1}(0; f) + \varepsilon) T(r, f) + S(r, f), \end{aligned}$$

which is a contradiction as $\delta_{k+1}(0; f) > 0$. Hence $H \not\equiv 0$.

Now from the fundamental estimate of logarithmic derivative it follows that

$$m(r, H) = S(r, f). \tag{3.5}$$

If z_0 is a pole of f with multiplicity $p \ge 1$ such that $a(z_0), a_j(z_0) \ne 0, \infty$, where $j = 0, 1, \dots, k$, then

$$H(z) = O((z - z_0)^{p-1}). \tag{3.6}$$

Let z_1 be a zero of f with multiplicity $q \ge k+1 (q \le k)$ such that $a(z_1), a_j(z_1) \ne 0, \infty$, where $j = 0, 1, \dots, k$. Then z_1 will be a pole of H with multiplicity $k + 1 (q)$.

Also if z_2 is a common zero of $F - 1$ and $G - 1$ with different multiplicities, then z_2 will be a pole of H. Thus

$$\begin{aligned} N(r, \infty; H) &\le \overline{N}_*(r, 1; F, G) + N_{k+1}(r, 0; f) + S(r, f) \\ &\le \overline{N}_{(l+1}(r, 1; F) + N_{k+1}(r, 0; f) + S(r, f). \end{aligned} \tag{3.7}$$

Then from (3.1), (3.5) and (3.7) we get

$$\begin{aligned} & N(r, \infty; f) - \overline{N}(r, \infty; f) \\ \le\ & N(r, 0; H) + S(r, f) \\ \le\ & T(r, \frac{1}{H}) - m(r, \frac{1}{H}) + S(r, f) \\ \le\ & T(r, H) - m(r, \frac{1}{H}) + S(r, f) \\ =\ & N(r, \infty; H) + m(r, H) - m(r, \frac{1}{H}) + S(r, f) \\ \le\ & \overline{N}_{(l+1}(r, 1; F) + N_{k+1}(r, 0; f) - m(r, \frac{1}{H}) + S(r, f). \end{aligned} \tag{3.8}$$

On the other hand it follows from (3.1) that

$$m(r, f) \le m(r, \frac{1}{H}) + S(r, f). \tag{3.9}$$

Now Lemma 2.1, (3.8) and (3.9) yield

(3.10)

$$\begin{aligned} T(r,f) &\le \overline{N}_{(l+1}(r,1;F) + N_{k+1}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\le \frac{1}{l}\, N(r,0;F^{'}|F\neq 0) + N_{k+1}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\le \frac{1}{l}\{\overline{N}(r,0;F) + \overline{N}(r,\infty;F)\} + N_{k+1}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\le \frac{1}{l}\{\overline{N}(r,0;f) + \overline{N}(r,\infty;f)\} + N_{k+1}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\le \Big(\frac{2l+2}{l} - \frac{l+1}{l}\,\Theta(\infty;f) - \frac{1}{l}\,\Theta(0;f) - \delta_{k+1}(0;f) + \varepsilon\Big)T(r,f) \\ &\quad + S(r,f), \end{aligned}$$

i.e.,

$$\Big(-\frac{l+2}{l} + \frac{l+1}{l}\,\Theta(\infty;f) + \frac{1}{l}\,\Theta(0;f) + \delta_{k+1}(0;f) - \varepsilon\Big)T(r,f) \le S(r,f). \tag{3.11}$$

Thus if (1.3) holds, then we arrive at a contradiction from (3.11).

Hence $F \equiv G$, i.e., $f \equiv L_1(f)$.

Case 2. Let $l = 0$.

Note that

$$\begin{aligned} \overline{N}(r,1;F) &\le \overline{N}(r,1;\frac{G}{F}) + S(r,f) \\ &\le T(r,\frac{G}{F}) + S(r,f) \\ &\le N(r,\infty;\frac{G}{F}) + m(r,\infty;\frac{G}{F}) + S(r,f) \\ &= N(r,\infty;\frac{L_1(f)}{f}) + m(r,\infty;\frac{L_1(f)}{f}) + S(r,f) \\ &\le k\,\overline{N}(r,\infty;f) + N_k(r,0;f) + S(r,f). \end{aligned} \tag{3.12}$$

Now using (3.12), we get from the second fundamental theorem that

(3.13)

$$\begin{aligned} T(r,f) &= T(r,F) + S(r,f) \\ &\le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) + S(r,F) \\ &\le (k+1)\,\overline{N}(r,\infty;f) + \overline{N}(r,0;f) + N_k(r,0;f) + S(r,f) \\ &\le \Big(k+3-(k+1)\,\Theta(\infty;f) - \Theta(0;f) - \delta_k(0;f) + \varepsilon\Big)\,T(r,f) + S(r,f), \end{aligned}$$

i.e.,

$$\Big(-k-2+(k+1)\,\Theta(\infty;f) + \Theta(0;f) + \delta_k(0;f) - \varepsilon\Big)\,T(r,f) \le S(r,f). \tag{3.14}$$

Thus if (1.4) holds, then we arrive at a contradiction from (3.14).

Hence $F \equiv G$, i.e., $f \equiv L_1(f)$.
This completes the proof. □

Proof of Theorem 1.2. Let $F = \frac{f}{a}$ and $G = \frac{L_1(f)}{a}$, where $L_1(f)$ is defined by (1.2). Then $F-1 = \frac{f-a}{a}$ and $G-1 = \frac{L_1(f)-a}{a}$. Since $f-a$ and $L_1(f)-a$ share 0 IM, it follows that F and G share 1 IM except the zeros and poles of $a(z)$.

Suppose $F \not\equiv G$.

Let

$$\Phi = \frac{1}{F}\left(\frac{G^{'}}{G-1} - (k+1)\frac{F^{'}}{F-1}\right)$$

$$= \frac{G}{F}\left(\frac{G^{'}}{G-1} - \frac{G^{'}}{G}\right) - (k+1)\left(\frac{F^{'}}{F-1} - \frac{F^{'}}{F}\right). \tag{3.15}$$

Suppose $\Phi \equiv 0$. Then from (3.15) we find that

$$G - 1 \equiv d(F-1)^{k+1}, \tag{3.16}$$

where $d \in \mathbb{C} \setminus \{0\}$.

Let z_{12} be a pole of $f(z)$ with multiplicity $p_{12} \geq 1$ such that $a_i(z_{12}), a(z_{12}) \neq 0, \infty$, where $i = 0, 1, 2, \ldots, k$ (otherwise the counting function of those poles of $f(z)$ which are the zeros or poles of $a_i(z), a(z)$ is equal to $S(r,f)$). Clearly z_{12} is a pole of $G-1$ with multiplicity $p_{12}+k$ and a pole of $(F-1)^{k+1}$ with multiplicity $(k+1)p_{12}$. Now from (3.16) we see that $p_{12}+k = (k+1)p_{12}$. If $p_{12} \geq 2$, then we arrive at a contradiction and so

$$N_{(2}(r,\infty;f) = S(r,f). \tag{3.17}$$

Also from (1.5) we know

$$(k-1)\,\Theta(\infty;f) + \delta_2(\infty;f) + \delta_k(0;f) + \delta_{k+1}(0;f) > k+1. \tag{3.18}$$

Now (3.18) yields

$$(k-1)\,\Theta(\infty;f) + \delta_2(\infty;f) + \delta_k(0;f) > 1. \tag{3.19}$$

By Lemma 2.2 we get from (3.16) that

$$\begin{aligned}
&(k+1)\,T(r,f)\\
&= (k+1)\,T(r,F) + S(r,f)\\
&\leq T(r,(F-1)^{k+1}) + S(r,f)\\
&\leq T(r,G) + S(r,f)\\
&\leq T(r,\frac{G}{F}) + T(r,F) + S(r,f)\\
&\leq N(r,\infty;\frac{L_1(f)}{f}) + m(r,\infty;\frac{L_1(f)}{f}) + T(r,f) + S(r,f)\\
&\leq k\,\overline{N}(r,\infty;f) + N_k(r,0;f) + T(r,f) + S(r,f)\\
&\leq (k-1)\,\overline{N}(r,\infty;f) + N_2(r,\infty;f) + N_k(r,0;f) + T(r,f) + S(r,f)
\end{aligned}$$

$$\leq (k+2-(k-1)\,\Theta(\infty;f)-\delta_2(\infty;f)-\delta_k(0;f)+\varepsilon)T(r,f)+S(r,f),$$

which contradicts (3.19). Hence $\Phi \not\equiv 0$.

Now from the fundamental estimate of logarithmic derivative it follows that

$$m(r,\Phi)=S(r,f). \tag{3.20}$$

If z_p is a pole of f with multiplicity $p \geq 1$ such that $a(z_p), a_j(z_p) \neq 0, \infty$, where $j=0,1,\ldots,k$, then

$$\Phi(z)=\begin{cases} O((z-z_p)), & if\ \ p=1 \\ O((z-z_p)^{p-1}), & if\ \ p\geq 2. \end{cases} \tag{3.21}$$

Thus from (3.15) we get

(3.22)

$$\begin{aligned} N(r,\infty;\Phi) &\leq \overline{N}(r,1;F)+N_{k+1}(r,0;f)+S(r,f)\\ &\leq N(r,0;\frac{F-G}{F})+N_{k+1}(r,0;f)+S(r,f)\\ &\leq T(r,\frac{G}{F})+N_{k+1}(r,0;f)+S(r,f)\\ &\leq N(r,\infty;\frac{L_1(f)}{f})+m(r,\infty;\frac{L_1(f)}{f})+N_{k+1}(r,0;f)+S(r,f)\\ &\leq k\,\overline{N}(r,\infty;f)+N_k(r,0;f)+N_{k+1}(r,0;f)+S(r,f). \end{aligned}$$

Then from (3.15), (3.20) and (3.22) we get

(3.23)

$$\begin{aligned} N(r,\infty;f)-\overline{N}_{(2}(r,\infty;f) &\leq N(r,0;\Phi)+S(r,f)\\ &\leq T(r,\frac{1}{\Phi})-m(r,\frac{1}{\Phi})+S(r,f)\\ &\leq T(r,\Phi)-m(r,\frac{1}{\Phi})+S(r,f)\\ &= N(r,\infty;\Phi)+m(r,\Phi)-m(r,\frac{1}{\Phi})+S(r,f)\\ &\leq k\,\overline{N}(r,\infty;f)+N_k(r,0;f)+N_{k+1}(r,0;f)\\ &\quad -m(r,\frac{1}{\Phi})+S(r,f). \end{aligned}$$

On the other hand it follows from (3.15) that

$$m(r,f)\leq m(r,\frac{1}{\Phi})+S(r,f). \tag{3.24}$$

Now (3.23) and (3.24) yield

(3.25)

$$\begin{aligned} &T(r,f)\\ &\leq (k-1)\,\overline{N}(r,\infty;f)+N_2(r,\infty;f)+N_k(r,0;f)+N_{k+1}(r,0;f)+S(r,f) \end{aligned}$$

$$\leq \Big(k+2-(k-1)\,\Theta(\infty;f)-\delta_2(\infty;f)-\delta_k(0;f)-\delta_{k+1}(0;f)+\varepsilon\Big)T(r,f) + S(r,f),$$

i.e.,

$$\Big(-k-1+(k-1)\,\Theta(\infty;f)+\delta_2(\infty;f)+\delta_k(0;f)+\delta_{k+1}(0;f)-\varepsilon\Big)T(r,f) \leq S(r,f),$$

which contradicts (3.18). Hence $F \equiv G$, i.e., $f \equiv L_1(f)$.

This completes the proof. □

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Department of Mathematics
Katwa College
Burdwan, 713130, India
E-mail address: sujoy.katwa@gmail.com, sr.koshigram@gmail.com,
smajumder05@yahoo.in