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CERTAIN SEQUENCE SPACES AND RELATED DUALS WITH RESPECT TO THE b-METRIC

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Abstract. The aim of this paper is to present the classical sets of sequences and related matrix transformations with respect to the b-metric. Also, we introduce the relationships between these sets and their classical forms with corresponding properties including convergence and completeness. Further we determine the duals of the new spaces and characterize matrix transformations on them into the sets of b-bounded, b-convergent and b-null sequences.

1. Introduction

By ω, we denote the space of all real valued sequences and any subspace of w is called a *sequence space*. We define the classical sets ℓ_∞^b, c^b, c_0^b and ℓ_p^b of all, bounded, convergent, null and absolutely p-summable sequences over the real field $\mathbb{R}$ with respect to the b-metric which correspond to the classical sets ℓ_∞, c, c_0 and ℓ_p over the real field $\mathbb{R}$, respectively. That is to say that

$$\ell_\infty^b := \left\{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} \left\{\rho(x_k, 0)\right\} < \infty\right\},$$

$$c^b := \left\{x = (x_k) \in \omega : \exists l \in \mathbb{R} \ni \ {}^b\lim_{k\to\infty} \rho(x_k, l) = 0\right\},$$

$$c_0^b := \left\{x = (x_k) \in \omega : \ {}^b\lim_{k\to\infty} \rho(x_k, 0) = 0\right\},$$

$$\ell_p^b := \left\{x = (x_k) \in \omega : \sum_{k=0}^{\infty} \rho(x_k, 0)^p < \infty\right\}, \ (0 < p < \infty)$$

where the distance function ρ is a b-metric for $s \geq 1$. One can show that ℓ_∞^b, c^b and c_0^b are b-complete metric spaces with ρ_∞ defined by

$$\rho_\infty(x, y) := \sup_{k\in\mathbb{N}} \left\{\rho(x_k, y_k)\right\}.$$

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Also, the set ℓ_p^b of absolutely p-summable sequence is a b-complete with ρ_p defined by

$$\rho_p(x,y) := \left\{ \sum_{k=0}^{\infty} \rho(x_k, y_k)^p \right\}^{1/p}, \ (p \geq 1)$$

where $x = (x_k)$, $y = (y_k)$ are the elements of the set ℓ_p^b. Secondly, we construct the sets bs^b, cs^b and cs_0^b of all bounded, convergent, null series by using b-metric ρ, as follows:

$$\begin{aligned}
bs^b &:= \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}} \rho \left(\sum_{k=0}^{n} x_k, 0 \right) < \infty \right\} \text{ or} \\
&:= \left\{ x = (x_k) \in \omega : \left(\sum_{k=0}^{n} x_k \right) \in \ell_\infty^b \right\}, \\
cs^b &:= \left\{ x = (x_k) \in \omega : \exists \ell \in \mathbb{R} \ni \ {}^{b}\lim_{n \to \infty} \rho \left(\sum_{k=0}^{n} x_k, \ell \right) = 0 \right\} \text{ or} \\
&:= \left\{ x = (x_k) \in \omega : \left(\sum_{k=0}^{n} x_k \right) \in c^b \right\}, \\
cs_0^b &:= \left\{ x = (x_k) \in \omega : \ {}^{b}\lim_{n \to \infty} \rho \left(\sum_{k=0}^{n} x_k, 0 \right) = 0 \right\} \text{ or} \\
&:= \left\{ x = (x_k) \in \omega : \left(\sum_{k=0}^{n} x_k \right) \in c_0^b \right\}.
\end{aligned}$$

One can conclude that bs^b, cs^b and cs_0^b are b-complete metric spaces with corresponding b-metric defined by

$$D_\infty^b(x,y) := \sup_{n \in \mathbb{N}} \left\{ \rho \Big(\sum_{k=0}^{n} x_k, \sum_{k=0}^{n} y_k \Big) \right\}, \ (s \geq 1)$$

where $x = (x_k)$, $y = (y_k)$ are the elements of the sets bs^b, cs^b and cs_0^b. Finally, we introduce the sets bv^b, bv_p^b and bv_∞^b of p-bounded variation of sequences by using b-metric, as follows:

$$\begin{aligned}
bv^b &:= \left\{ x = (x_k) \in \omega : \sum_k \rho \left[(\Delta x)'_k, 0 \right] < \infty \right\}, \\
\rho_\Delta(x,y) &:= \sum_{k=0}^{\infty} \left\{ \rho \big[(\Delta x)'_k, (\Delta y)'_k \big] \right\}, \\
bv_p^b &:= \left\{ x = (x_k) \in \omega : \sum_{k=0}^{\infty} \rho \left[(\Delta x)_k, 0 \right]^p < \infty \right\},
\end{aligned}$$

$$\rho_p^{\Delta}(x,y) := \left\{ \sum_{k=0}^{\infty} \rho\big[(\Delta x)_k, (\Delta y)_k\big]^p \right\}^{1/p}, \ (p \geq 1).$$

$$bv_{\infty}^b := \left\{ x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} \rho\left[(\Delta x)_k, 0\right] < \infty \right\},$$

$$\rho_{\infty}^{\Delta}(x,y) := \sup_{k\in\mathbb{N}} \left\{ \rho\big[(\Delta x)_k, (\Delta y)_k\big] \right\}.$$

One can easily see that bv^b, bv_p^b and bv_{∞}^b are b-complete metric spaces with b-metrics ρ_{Δ}, ρ_p^{Δ} and ρ_{∞}^{Δ}, respectively where $(\Delta x)_k' = x_k - x_{k-1}; x_{-1} = 0$ and $(\Delta x)_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Recently, many researchers have investigated on different contractive conditions in complete metric spaces with respect to a partial order and determined many fixed point results. For more details on fixed point results, especially comparison of different contractive conditions and related properties in ordered metric spaces we refer the reader to [1, 2, 7, 8, 18, 19, 20, 22]. Khamsi-Hussain [15] and Hussain et al. [10] studied the topology introduced by b-metric. For more results see also [6, 14, 17].

Furthermore, Kadak and Ozluk [13] have introduced the characterization of matrix transformations between some classical sets of sequences with respect to partial metric. Also Kadak [11] and Kadak and Efe [12] have determined the duals and matrix transformations over the non-Newtonian complex field.

The main focus of this work is to extend classical sequence spaces and related duals defined earlier to the sequence spaces with respect to b-metric and to characterize matrix transformations on them.

2. Preliminaries, bacground and notation

The concept of a b-metric space was introduced by Czerwik in [5]. After that, several results about the existence of a fixed point for single-valued and multi-valued operators in b-metric spaces have been given (see [4, 9, 23, 24]). Pacurar [21] introduced some results on sequences of almost contractions and fixed points in b-metric spaces. Consistent with [5] and [25], the following definitions and results will be needed in the sequel.

Definition 2.1 ([24])**.** Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $\rho : X \times X \to [0, \infty)$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $\rho(x, y) = 0$ if and only if $x = y$,
(b_2) $\rho(x, y) = \rho(y, x)$,
(b_3) $\rho(x, z) \leq s[\rho(x, y) + \rho(y, z)]$.

The pair (X, ρ) is called a b-metric space.

It is pointed out that the class of b-metric spaces is effectively larger than of metric spaces, since a b-metric is a metric if and only if $s = 1$.

Example 2.2 ([24])**.** Let (X, d) be a metric space, and $\rho(x, y) = d(x, y)^p$, where $p \geq 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

However, (X, ρ) is not necessarily a metric space. For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b-metric, denoted by Euclidean b-metric on $\mathbb{R}$ with $s = 2$, but it is not a metric on $\mathbb{R}$.

Definition 2.3 ([4])**.** Let (X, ρ) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:

(a) b-convergent if and only if there exists $\ell \in X$ such that $\rho(x_n, \ell) \to 0$, as $n \to \infty$. In this case, we write ${}^b\lim_{n\to\infty} x_n = \ell$.
(b) b-Cauchy if and only if $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 2.4 ([4])**.** In a b-metric space (X, ρ) the following assertions hold:

(a) A b-convergent sequence has a unique limit.
(b) Each b-convergent sequence is b-Cauchy.
(c) In general, a b-metric is not continuous.

Also very recently N. Hussain et al. have presented an example of a b-metric which is not continuous (see Example 3 in [9]).

Definition 2.5 ([4])**.** (a) The b-metric space (X, ρ) is b-complete if every b-Cauchy sequence in X be b-converges.
(b) Let (X, ρ) be a b-metric space. If Y is a nonempty subset of X, then the closure $\overline{Y}$ of Y is the set of limits of all b-convergent sequences of points in Y, i.e.,

$$\overline{Y} := \Big\{x \in X : \text{there exists a sequence } \{x_n\} \text{ in } Y \text{ so that } \ {}^b\!\lim_{n\to\infty} x_n = x\Big\}.$$

Taking into account of the above definition, we have the following concepts.
(c) Let (X, ρ) be a b-metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\} \in Y$ which b-converges to an element x, we have $x \in Y$.
(d) Let (X, ρ) and (X', ρ') be two b-metric spaces. Then a function $f : X \to X'$ is b-continuous at a point $x \in X$ if and only if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x;$\{f(x_n)\}$ is b-convergent to $\{f(x)\}$.

Lemma 2.6 ([23])**.** *Let (X, ρ) be a b-metric space with $s \geq 1$, and suppose that each of sequence $\{x_n\}$ and $\{y_n\}$ is b-convergent to x and y, respectively. Then we have,*

$$\frac{1}{s^2}\ \rho(x, y) \leq \liminf_{n\to\infty} \rho(x_n, y_n) \leq \limsup_{n\to\infty} \rho(x_n, y_n) \leq s^2\ \rho(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} \rho(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s}\ \rho(x,z) \leq \liminf_{n\to\infty} \rho(x_n, z) \leq \limsup_{n\to\infty} \rho(x_n, z) \leq s^2\ \rho(x,z).$$

3. Completeness of new spaces

Lemma 3.1. *Define the relation ρ_∞ on the space λ by*

$$\begin{array}{rccl} \rho_\infty & : & \lambda \times \lambda & \longrightarrow & [0,\infty) \\ & & (x,y) & \longmapsto & \rho_\infty(x,y) = \sup_{k\in\mathbb{N}}\{\rho(x_k,y_k)\},\ (s \geq 1) \end{array}$$

where λ denotes any of the spaces ℓ_∞^b, c^b, c_0^b and $x = (x_k), y = (y_k) \in \lambda$. Then, (λ, ρ_∞) is a b-complete metric space.

Proof. Since the proof is similar for the spaces c^b and c_0^b, we prove the theorem only for the space ℓ_∞^b.

One can easily show by a routine verification that ρ_∞ satisfies the b-metric axioms for all $s \geq 1$ on the space ℓ_∞^b. So, we prove only the condition (b_3). Let $x = (x_k)$, $y = (y_k)$ and $z = (z_k) \in \ell_\infty^b$. Then,

(b_3) By using the axiom (b_3) in Definition 2.1 for all $s \geq 1$ we get

$$\begin{aligned} \rho_\infty(x_k, z_k) &= \sup_{k\in\mathbb{N}}\big\{\rho(x_k,z_k)\big\} \leq \sup_{k\in\mathbb{N}}\big\{s[\rho(x_k,y_k) + \rho(y_k,z_k)]\big\} \\ &\leq s\ [\sup_{k\in\mathbb{N}}\{\rho(x_k,y_k)\} + \sup_{k\in\mathbb{N}}\{\rho(y_k,z_k)\}] \\ &= s\ [\rho_\infty(x_k,y_k) + \rho_\infty(y_k,z_k)]. \end{aligned}$$

Therefore, one can conclude that $(\ell_\infty^b, \rho_\infty)$ is a b-metric space on ℓ_∞^b.

It remains to prove the b-completeness of the space ℓ_∞^b. Let $x_m = \Big\{x_1^{(m)}, x_2^{(m)}, \ldots\Big\}$ be a b-Cauchy sequence in ℓ_∞^b. Then, for any $\epsilon > 0$ there exists a positive integer m_0 such that

$$\rho_\infty(x_m, x_r) = \sup_{k\in\mathbb{N}} \rho(x_k^{(m)}, x_k^{(r)}) < \epsilon$$

for all $m, r > m_0$. A fortiori, for every fixed $k \in \mathbb{N}$ and for $m, r > m_0$ then

$$\Big\{\rho\left(x_k^{(m)}, x_k^{(r)}\right) : k \in \mathbb{N}\Big\} < \epsilon. \tag{1}$$

In this case for any fixed $k \in \mathbb{N}$, by using completeness of $\mathbb{R}$, we say that $x_k^{(m)} = \{x_k^{(1)}, x_k^{(2)}, \ldots\}$ is a b-Cauchy sequence and is b-convergent. Now, we suppose that ${}^b\lim_{m\to\infty} x_k^{(m)} = x_k$ and $x = (x_1, x_2, \ldots)$. We must show that ${}^b\lim_{m\to\infty} \rho_\infty(x_m, x) = 0$ and $x \in \ell_\infty^b$.

The constant $m_0 \in \mathbb{N}$ for all $m > m_0$, taking the limit as $r \to \infty$ in (1), we obtain $\rho(x_k^{(m)}, x_k) < \epsilon$ for all $k \in \mathbb{N}$. Since $x_m = \big(x_k^{(m)}\big) \in \ell_\infty^b$, there exists a positive number $\delta > 0$ such that $\rho(x_k^{(m)}, 0) \leq \delta$. By taking into account b-metric axiom (b_3) we get

$$\rho(x_k, 0) \leq s[\rho(x_k, x_k^{(m)}) + \rho(x_k^{(m)}, 0)] < s(\epsilon + \delta) \tag{2}$$

for all $s \geq 1$. It is clear that (2) holds for every $k \in \mathbb{N}$ whose right-hand side does not involve k. This leads us to the consequence that $x = (x_k) \in \ell_\infty^b$. Also, we immediately deduce that the inequality

$$\rho_\infty(x_m, x) = \sup_{k\in\mathbb{N}} \rho\left(x_k^{(m)}, x_k\right) < \epsilon$$

holds for $m > m_0$. This shows that $\rho_\infty(x_m, x) \to 0$ as $m \to \infty$. Since (x_m) is an arbitrary b-Cauchy sequence, ℓ_∞^b is b-complete. □

Lemma 3.2. *Define the distance function ρ_p by*

$$\begin{array}{llll} \rho_p : \ell_p^b \times \ell_p^b & \longrightarrow & \mathbb{R}^+ & \\ (x, y) & \longmapsto & \rho_p(x, y) = \left\{ \displaystyle\sum_{k=0}^{\infty} \rho(x_k, y_k)^p \right\}^{1/p}, & (1 \leq p < \infty,\ s \geq 1) \end{array}$$

where $x = (x_k)$, $y = (y_k) \in \ell_p^b$. Then, (ℓ_p^b, ρ_p) is a b-complete metric space.

Proof. It is obvious that ρ_p satisfies the axioms (b_1) and (b_2). Let $x = (x_k)$, $y = (y_k)$ and $z = (z_k) \in \ell_p^b$. Then, we derive by applying the Minkowski's inequality that

$$\begin{aligned} \rho_p(x, z) = \left\{ \sum_{k=0}^{\infty} \rho(x_k, z_k)^p \right\}^{1/p} &\leq \left\{ \sum_{k=0}^{\infty} (s[\rho(x_k, y_k) + \rho(y_k, z_k)])^p \right\}^{1/p} \\ &\leq s \left\{ \left(\sum_{k=0}^{\infty} \rho(x_k, y_k)^p \right)^{1/p} + \left(\sum_{k=0}^{\infty} \rho(y_k, z_k)^p \right)^{1/p} \right\} \\ &= s\ [\rho_p(x, y) + \rho_p(y, z)]. \end{aligned}$$

This shows that the axiom (b_3) also holds. Therefore, one can conclude that (ℓ_p^b, ρ_p) is a b-metric space.

Since the proof is analogous for the cases $p = 1$ and $p = \infty$ we omit their detailed proof and we consider only case $1 < p < \infty$. It remains to prove the completeness of the space ℓ_p^b. Let $x_m = \left\{x_1^{(m)}, x_2^{(m)}, \ldots\right\}$ be any b-Cauchy sequence on ℓ_p^b. Then for every $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\rho_p(x_m, x_r) = \left\{ \sum_{k=0}^{\infty} \rho\left(x_k^{(m)}, x_k^{(r)}\right)^p \right\}^{1/p} < \epsilon \tag{3}$$

for all $m, r > m_0$. We obtain for each fixed $k \in \mathbb{N}$ from (3) that

$$\rho\left(x_k^{(m)}, x_k^{(r)}\right) < \epsilon \tag{4}$$

for all $m, r > m_0$. By using the completeness of $\mathbb{R}$, we say that the sequence $x_k^{(m)} = \left\{x_k^{(1)}, x_k^{(2)}, \dots\right\}$ is a b-Cauchy sequence and is b-convergent for each fixed $k \in \mathbb{N}$, say to $x_k \in \mathbb{R}$. Now, we suppose that $x_k^{(m)} \to x_k$ as $m \to \infty$ and $x = (x_k)$. We must show that ${}^b\lim_{m\to\infty} \rho_p(x_m, x) = 0$ and $x \in \ell_p^b$. From (4) for each $j \in \mathbb{N}$ and $m, r > m_0$ we get

$$\sum_{k=0}^{j} \rho\left(x_k^{(m)}, x_k^{(r)}\right)^p < \epsilon^p. \tag{5}$$

Take any $m > m_0$. Let us pass to b-limit firstly $r \to \infty$ and next $j \to \infty$ in (5) to obtain $\rho_p(x_m, x) < \epsilon$. By using the inclusion (2) and Minkowski's inequality for each $j \in \mathbb{N}$ that

$$\begin{aligned} \left\{\sum_{k=0}^{\infty} \rho(x_k, 0)^p\right\}^{1/p} &\leq s \left\{\left(\sum_{k=0}^{\infty} \rho\left(x_k^{(m)}, x_k\right)^p\right)^{1/p} + \left(\sum_{k=0}^{\infty} \rho\left(x_k^{(m)}, 0\right)^p\right)^{1/p}\right\} \\ &< \infty, \end{aligned}$$

which implies that $x \in \ell_q^b$. Since $\rho_p(x_m, x) \leq \epsilon$ for all $m > m_0$ it follows that ${}^b\lim_{m\to\infty} \rho_p(x_m, x) = 0$. Since (x_m) is an arbitrary b-Cauchy sequence, the space (ℓ_q^b, ρ_p) is b-complete. □

Lemma 3.3. *Define the relation D_∞^b on the space μ by*

$$\begin{aligned} D_\infty^b \quad : \quad \mu \times \mu \quad &\longrightarrow \quad [0, \infty) \\ (x, y) \quad &\longmapsto \quad D_\infty^b(x, y) = \sup_{n\in\mathbb{N}} \rho\left(\sum_{k=0}^{n} x_k, \sum_{k=0}^{n} y_k\right), \ (s \geq 1) \end{aligned}$$

where μ denotes any of the spaces bs^b, cs^b, cs_0^b for all $x = (x_k), y = (y_k) \in \mu$. Then, (μ, D_∞^b) is a b-complete metric space.

Proof. Since the proof is similar to Lemma 3.1, we omit the detail. □

Lemma 3.4. *Define the distance functions ρ_Δ, ρ_p^Δ and ρ_∞^Δ by*

$$\begin{aligned} \rho_\Delta(x, y) &:= \sum_{k=0}^{\infty} \rho[(\Delta x)_k', (\Delta y)_k'], \ (\Delta x)_k' = x_k - x_{k-1}, \ x_{-1} = 0, \\ \rho_p^\Delta(x, y) &:= \left\{\sum_{k=0}^{\infty} \rho[(\Delta x)_k, (\Delta y)_k]^p\right\}^{1/p}, (1 \leq p < \infty), \\ \rho_\infty^\Delta(x, y) &:= \sup_{k\in\mathbb{N}} \left\{\rho[(\Delta x)_k, (\Delta y)_k]\right\}, \ (\Delta x)_k = x_k - x_{k+1}, \end{aligned}$$

where $x = (x_k)$, $y = (y_k)$ are the element of any space μ, respectively with $s \geq 1$. Then, we can say the sets (bv^b, ρ_Δ), (bv_p^b, ρ_p^Δ) and $(bv_\infty^b, \rho_\infty^\Delta)$ are b-complete metric spaces.

Proof. Proof follows from Lemma 3.2. □

4. The duals of the sequence spaces with the Euclidean b-metric

In this section, following [3], we focus on the α-, β- and γ-duals of the classical sequence spaces with Euclidean b-metric on $\mathbb{R}$. For the sequence spaces λ, μ, the set $S(\lambda,\mu)$ defined by

$$S(\lambda,\mu) := \Big\{ w = (w_k) \in \omega : (w_k z_k) \in \mu \text{ for all } z = (z_k) \in \lambda \Big\}, \tag{6}$$

is called the multiplier space of λ and μ for all $k \in \mathbb{N}$. One can easily observe for a sequence space ν that the inclusions $S(\lambda,\mu) \subset S(\nu,\mu)$ if $\nu \subset \lambda$ and $S(\lambda,\mu) \subset S(\lambda,\nu)$ if $\mu \subset \nu$ hold. We define the α-, β- and γ-duals of a set $\lambda \subset \omega$ which are respectively denoted by $\{\lambda\}^\alpha$, $\{\lambda\}^\beta$ and $\{\lambda\}^\gamma$, as follows:

$$\{\lambda\}^\alpha := \Big\{ w = (w_k) \in \omega : (w_k z_k) \in \ell_1^b \text{ for all } z = (z_k) \in \lambda \Big\},$$
$$\{\lambda\}^\beta := \Big\{ w = (w_k) \in \omega : (w_k z_k) \in cs^b \text{ for all } z = (z_k) \in \lambda \Big\},$$
$$\{\lambda\}^\gamma := \Big\{ w = (w_k) \in \omega : (w_k z_k) \in bs^b \text{ for all } z = (z_k) \in \lambda \Big\},$$

where $(w_k z_k)$ the coordinatewise product of the sequence w and z for all $k \in \mathbb{N}$. Then $\{\lambda\}^\beta$ is called β-dual of λ or the set of all convergence factor sequences of λ in cs^b. Firstly, we give a remark concerning with the b-convergence factor sequences.

Remark 4.1. Let $\emptyset \neq \lambda \subset \omega$. Then the following statements are valid:

(a) $\{\lambda\}^\beta$ is a sequence space and $\varphi < \{\lambda\}^\beta < \omega$ ('$<$' stands for 'is a linear subspace of') where $\varphi := \{u = (u_k) : \exists N \in \mathbb{N}, \forall k \geq N, u_k = 0\}$.

(b) If $\lambda \subset \mu \subset \omega$ then $\{\mu\}^\beta < \{\lambda\}^\beta$.

(c) $\lambda \subset \{\lambda\}^{\beta\beta} := \big(\{\lambda\}^\beta\big)^\beta$.

Proof. Since the proof is trivial for the conditions (b) and (c), we prove only (a). Let $m = (m_k)$ and $n = (n_k) \in \{\lambda\}^\beta$.

(a) Let $l \in \lambda$. Then we get $(m_k l_k) \in cs^b$; $(n_k l_k) \in cs^b$ and $(m_k + n_k) l_k = (m_k l_k) + (n_k l_k) \in cs^b$. Since l is arbitrary, $m + n \in \{\lambda\}^\beta$. For any $\alpha \in \mathbb{R}$, $w = (w_k) \in \{\lambda\}^\beta$ we have $(\alpha w_k) l_k = \alpha (w_k l_k) \in cs^b$ and $\alpha w \in \{\lambda\}^\beta$. Therefore, $\{\lambda\}^\beta$ is a linear subspace of the space ω. □

Remark 4.2. In the proof of Remark 4.1 we use this fact that in a b-metric space if $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$. By taking into account that d is an Euclidean b-metric space such as $\rho(x,y) = (x-y)^2$ this claim is true. But it is not true for an arbitrary b-metric. Because of an arbitrary b-metric function $\rho(x,y)$ is not to be continuous in general case with $s > 1$ one can conclude that $x_n + y_n \nrightarrow x + y$ whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$. So $\{\lambda\}^\beta$ is not a linear subspace of the space ω.

Theorem 4.3. *The following statements hold*:

(a) $\{c_0^b\}^\beta = \{c^b\}^\beta = \{\ell_\infty^b\}^\beta = \ell_1^b$.
(b) $\{\ell_1^b\}^\beta = \ell_\infty^b$.

Proof. (a) Let ρ be an Euclidean b-metric on $\mathbb{R}$. Obviously $\{\ell_\infty^b\}^\beta \subset \{c^b\}^\beta \subset \{c_0^b\}^\beta$ by Remark 4.1(b). Then we must show that $\ell_1^b \subset \{\ell_\infty^b\}^\beta$ and $\{c_0^b\}^\beta \subset \ell_1^b$. Now, consider $w = (w_k) \in \ell_1^b$ and $z = (z_k) \in \ell_\infty^b$ be given. Then

$$\sum_{k=0}^{n} \rho(w_k z_k, 0) \leq \sup_k \rho(z_k, 0) \sum_{k=0}^{n} \rho(w_k, 0) < \infty$$

which implies that $wz \in cs^b$. So the condition $\ell_1^b \subset \{\ell_\infty^b\}^\beta$ holds.

Conversely, for a given $y = (y_k) \in \omega \setminus \ell_1^b$ we prove the existence of an $x \in c_0^b$ with $yx \notin cs^b$. According to $y \notin \ell_1^b$ we may take an index sequence (n_p) which is a strictly increasing real valued sequence with $n_0 = 0$ and $\sum_{k=n_{p-1}}^{n_p-1} \rho(y_k, 0) > p$ $(p \in \mathbb{N})$. If we define $x = (x_k) \in c_0^b$ by $x_k := ((sgny_k)/p)$ where the real signum function defined by

$$sgn(u) := \begin{cases} \frac{u}{|u|} & , \quad u \neq 0, \\ 0 & , \quad u = 0. \end{cases}$$

Thus, we get

$$\sum_{k=n_{p-1}}^{n_p-1} y_k x_k = \frac{1}{p} \sum_{k=n_{p-1}}^{n_p-1} y_k (sgny_k) = \frac{1}{p} \sum_{k=n_{p-1}}^{n_p-1} \rho(y_k, 0) \geq 1$$

for all $n_{p-1} \leq k < n_p$. Therefore $yx \notin cs^b$ and thus $y \notin \{c_0^b\}^\beta$. Hence $\{c_0^b\}^\beta \subset \ell_1^b$.

(b) From the condition (c) of Remark 4.1 we have $\ell_\infty^b \subset \left(\{\ell_\infty^b\}^\beta\right)^\beta = \{\ell_1^b\}^\beta$ since $\{\ell_\infty^b\}^\beta = \ell_1^b$. Now we assume the existence of a $w = (w_n) \in \{\ell_1^b\}^\beta \setminus \ell_\infty^b$. Since w is unbounded, there exists a subsequence (w_{n_k}) of (w_n) and we can find a real number k^2 such that $\rho(w_{n_k}, 0) \geq k^2$ for all $k \in \mathbb{N}_1$. The sequence (x_n), defined by $x_n := (sgn(w_{n_k})/k^2)$ if $n = n_k$ and 0 otherwise. Then $x \in \ell_1^b$. However

$$\sum_n w_n x_n = \sum_k \frac{\rho(w_{n_k}, 0)}{k^2} \geq \sum_k 1 = \infty.$$

Hence $w \notin \{\ell_1^b\}^\beta$, which contradicts our assumption and $\{\ell_1^b\}^\beta \subset \ell_\infty^b$. □

Theorem 4.4. *The following statements hold*:

(a) $\{cs^b\}^\alpha = \{bv_1^b\}^\alpha = \{bv_0^b\}^\alpha = \{bs^b\}^\alpha = \ell_1^b$ $(bv_0^b = bv^b \cap c_0^b)$.
(b) $\{cs^b\}^\beta = bv_1^b$, $\{bv_1^b\}^\beta = cs^b$, $\{bv_0^b\}^\beta = bs^b$, $\{bs^b\}^\beta = bv_0^b$.
(c) $\{cs^b\}^\gamma = bv_1^b$, $\{bv_1^b\}^\gamma = bs^b$, $\{bv_0^b\}^\gamma = bs^b$, $\{bs^b\}^\gamma = bv_1^b$.

Proof. We prove (b) and (c) for the space cs^b and the proofs of all other cases are quite similar.

(b) Let $u = (u_k) \in \{cs^b\}^\beta$ and $w = (w_k) \in c_0^b$. Define the sequence $v = (v_k) \in cs^b$ by $v_k = (w_k - w_{k+1})$ for all $k \in \mathbb{N}$. Therefore, $\sum_k u_k v_k$ b-converges, but

$$\sum_{k=0}^{n}(w_k - w_{k+1})u_k = \left[\sum_{k=1}^{n-1} w_k(u_k - u_{k-1})\right] - w_{n+1}u_n \tag{7}$$

and the inclusion $\ell_1^b \subset cs^b$ yields that $(u_k) \in \{cs^b\}^\beta \subset \{\ell_1^b\}^\beta = \ell_\infty^b$. Then we derive by passing to the b-limit in (7) as $n \to \infty$ which implies that $\sum_{k=0}^{\infty}(w_k - w_{k+1})u_k = \sum_{k=0}^{\infty} w_k(u_k - u_{k-1})$ for every $k \in \mathbb{N}$. Hence $(u_k - u_{k-1}) \in \{c_0^b\}^\beta = \{c_0^b\}^\alpha = \ell_1^b$, i.e., $u \in bv_1^b$. Therefore, $\{cs^b\}^\beta \subseteq bv_1^b$.

Conversely, suppose that $u = (u_k) \in bv_1^b$. Then, $(u_k - u_{k-1}) \in \ell_1^b$. Further, if $v = (v_k) \in cs^b$, the sequence (w_n) defined by $w_n = \sum_{k=0}^{n} v_k$ for all $k \in \mathbb{N}$, is an element of the space c^b. Since $\{c^b\}^\alpha = \ell_1^b$, the series $\sum_k w_k(u_k - u_{k-1})$ is b-convergent. Also, we have

$$\sum_{k=m}^{n}(w_k - w_{k+1})u_k \leq \left[\sum_{k=m}^{n-1} w_k(u_k - u_{k-1})\right] + w_n u_n - w_{m-1}u_m. \tag{8}$$

Since $(w_n) \in c^b$ and $(u_k) \in bv^b \subset c^b$, the right-hand side of inequality (8) b-converges to zero as $m, n \to \infty$. Hence, the series $\sum_{k=0}^{\infty} u_k v_k$ b-converges and $bv_1^b \subseteq \{cs^b\}^\beta$.

(c) By using (a), it is known that $bv^b \subseteq \{cs^b\}^\beta$ and since $\{cs^b\}^\beta \subset \{cs^b\}^\gamma$, so $bv^b \subset \{cs^b\}^\gamma$. We need to show that $\{cs^b\}^\gamma \subset bv^b$. Let $u = (u_n) \in \{cs^b\}^\gamma$ and $v = (v_n) \in c_0^b$. Then, for the sequence $(w_n) \in cs^b$ defined by $w_n = (v_n - v_{n+1})$ for all $n \in \mathbb{N}$, we can find a number $K > 0$ such that $\rho\big(\sum_{k=0}^{n} u_k w_k, 0\big) \leq K$ for all $n \in \mathbb{N}$. Since $(v_n) \in c_0^b$ and $(u_n) \in \{cs^b\}^\gamma \subset \ell_\infty^b$, there exists a real number $M > 0$ such that $\rho(u_n v_n, 0) \leq M$ for all $n \in \mathbb{N}$. Therefore,

$$\sum_{k=0}^{n}(u_k - u_{k-1})v_k \leq \sum_{k=1}^{n+1} u_k(v_k - v_{k+1}) + v_{n+2}u_{n+1} \leq K + M.$$

Hence $(u_k - u_{k-1}) \in \{c_0^b\}^\gamma = \{c_0^b\}^\alpha = \ell_1^b$, i.e., $(u_n) \in bv^b$. Therefore, since the inclusion $\{cs^b\}^\gamma \subset bv^b$ holds, we conclude that $\{cs^b\}^\gamma = bv^b$. □

5. Some classes of matrix transformations

In this section, firstly we give some basic definitions which will be used in this article.

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_k) \in \omega$ be an infinite sequence. Then we obtain the sequence $(Ax)_n$, denoted by A-transform of x, as

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2k} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + & \cdots \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + & \cdots \\ \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + & \cdots \\ \vdots & \end{pmatrix} = \begin{pmatrix} (Ax)_1 \\ (Ax)_2 \\ \vdots \\ (Ax)_n \\ \vdots \end{pmatrix}.$$

In this case, we transform the sequence x into the sequence $Ax = \{(Ax)_n\}$ with

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k, \ n \in \mathbb{N}, \tag{9}$$

provided the series on the right hand side of (9) b-converges for each n.

Let λ and μ be any two sequence spaces. If Ax exists and is in μ for every sequence $x = (x_k) \in \lambda$, then we say that A defines a matrix transformation from λ into μ, i.e., $A : \lambda \to \mu$. By $(\lambda : \mu)$, we denote the class of all matrices A from λ into μ.

Following Başar [3], the basic definition of summable sequences with respect to the b-metric can be given as follows:

Definition 5.1. A sequence $x = (x_k)$ is said to be summable A to a real number ℓ if the A-${}^b\lim$ of x is α, i.e.,

$$ {}^b\lim_{n\to\infty} \rho((Ax)_n, \ell) = 0, \tag{10}$$

where ρ is a b-metric with $s \geq 1$.

Basic Theorem 1. (i) *$A = (a_{nk}) \in (\ell_\infty^b : \ell_\infty^b)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k \rho(a_{nk}, 0) < \infty. \tag{11}$$

(ii) *$A = (a_{nk}) \in (c^b : \ell_\infty^b)$ if and only if* (11) *holds.*
(iii) *$A = (a_{nk}) \in (c_0^b : \ell_\infty^b)$ if and only if* (11) *holds.*
(iv) *$A = (a_{nk}) \in (\ell_p^b : \ell_\infty^b)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k \rho(a_{nk}, 0)^p < \infty \ (0 < p < \infty). \tag{12}$$

Proof. Since the proof can also be obtained in the similar way, we prove only case (i).

(i) Suppose that (11) holds and $x = (x_k) \in \ell_\infty^b$. In this case, the A-transform of x exists since $(a_{nk})_{k\in\mathbb{N}} \in \{\ell_\infty^b\}^\beta = \ell_1^b$ for every fixed $n \in \mathbb{N}$. By the hypothesis one can conclude that

$$\sup_{n\in\mathbb{N}} \rho((Ax)_n, 0) = \sup_{n\in\mathbb{N}} \rho\big(\sum_k a_{nk}x_k, 0\big) \leq \sup_{k\in\mathbb{N}} \rho(x_k, 0) \sup_{n\in\mathbb{N}} \sum_k \rho(a_{nk}, 0) < \infty.$$

Hence $Ax \in \ell_\infty^b$.

Now, in order to prove the converse, let us suppose that $A \in (\ell_\infty^b : \ell_\infty^b)$ and $x = (x_k) \in \ell_\infty^b$. Then the series $\sum_{k=1}^\infty a_{nk}x_k$ is b-convergent for each fixed n, since Ax exists. Hence $A_n = (a_{nk})_{k=0}^\infty \in \{\ell_\infty^b\}^\beta$ for all $n \in \mathbb{N}$ which implies that

$$\sum_k \rho\big(a_{nk}, 0\big) \leq \sum_k \rho\big(a_{nk}x_k, 0\big) \leq \sup_{n\in\mathbb{N}} \rho\big(\sum_k a_{nk}x_k, 0\big) < \infty.$$

Hence the sequence $\big\{\sum_k \rho\big(a_{nk}, 0\big)\big\}_{n\in\mathbb{N}}$ is b-bounded which means that (11) holds. □

Theorem 5.2. *$A = (a_{nk}) \in (c^b : c^b)$ if and only if (11) holds, and there exist $\alpha_k, l \in \mathbb{R}$ such that*

$$ {}^b\lim_{n\to\infty} \rho(a_{nk}, \alpha_k) = 0 \text{ for each } k \in \mathbb{N}, \tag{13}$$

$$ {}^b\lim_{n\to\infty} \rho\left(\sum_k a_{nk}, l\right) = 0. \tag{14}$$

Proof. Assume that $A = (a_{nk}) \in (c^b : c^b)$. Then Ax exists for every $x \in c^b$. Let $e = (e_k)$ and $e^{(n)} = (e_k^{(n)})$ be the sequences with $e_k = 1$ for all $k \in \mathbb{N}$. By taking $x = e^{(k)}$ and $x = e$, respectively, the necessity of (13) and (14) is trivial. Since $c^b \subset \ell_\infty^b$, the necessity of (11) is obtained from Basic Theorem (i).

Conversely, suppose that (11), (13) and (14) hold and $x = (x_k) \in c^b$ with $x_k \xrightarrow{\rho} s$ as $k \to \infty$. So that, obviously, the A-transform of x exists since $(a_{nk})_{k\in\mathbb{N}} \in \{c^b\}^\beta = \ell_1^b$ for each $n \in \mathbb{N}$. We now recall the following well-known sum:

$$\sum_{k=1}^n a_{nk}x_k = \sum_{k=1}^n a_{nk}(x_k - s) + s\sum_{k=1}^n a_{nk} \tag{15}$$

for each $n \in \mathbb{N}$. Taking the b-limit as $n \to \infty$ in (15), we get

$$ {}^b\lim_{n\to\infty} \sum_{k=1}^n a_{nk}x_k = {}^b\lim_{n\to\infty} \sum_{k=1}^n \alpha_k(x_k - s) + s\ {}^b\lim_{n\to\infty} \sum_{k=1}^n a_{nk} = sl.$$

Hence, $Ax \in c^b$, that is the conditions are sufficient. □

Corollary 5.3. *$A = (a_{nk}) \in (c_0^b : c^b)$ if and only if (11) holds and there exists $(\alpha_k) \in \mathbb{R}$ such that*

$$^b\lim_{n\to\infty} \rho(a_{nk}, \alpha_k) = 0 \tag{16}$$

for each $k \in \mathbb{N}$.

Example 5.4. Let $k, n, r \in \mathbb{N}$ and $r \geq 0$. The Cesaro means of order r is defined by the matrix $C_r = (c^r_{nk})$ as

$$(c^r_{nk}) = \begin{cases} \dfrac{\binom{n-k+r-1}{n-k}}{\binom{n+r}{n}} & ; \text{if } k \leq n \\ 0 & ; \text{otherwise.} \end{cases}$$

Taking $r = 2$ we see that

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & \cdots \\ \frac{3}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{2(n+1)}{(n+1)(n+2)} & \frac{2n}{(n+1)(n+2)} & \cdots & \frac{2}{(n+1)(n+2)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}.$$

Since $\sum_k \rho(c^2_{nk}, 0) < \infty$ where $\rho(x, y) = (x - y)^2$ then (11) and (14) hold. Therefore, by combining this with $^b\lim_{n\to\infty} \rho(c^2_{nk}, 0) = 0$ for each $k \in \mathbb{N}$ we deduce that (13) is satisfied. Therefore $C_2 \in (c^b : c^b)$.

Corollary 5.5. *$A = (a_{nk}) \in (c_0^b : c_0^b)$ if and only if (11) holds and (16) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Theorem 5.6. *$A = (a_{nk}) \in (\ell_p^b : c^b)$ if and only if (12) and (16) hold.*

Proof. Suppose that (12) and (16) hold, and $x = (x_k) \in \ell_p^b$. Then, since Ax exists and $A_n \in \{\ell_p^b\}^\beta = \ell_1^b$ for each $n \in \mathbb{N}$. We thus find from (12) and Hölder's inequality that

$$\begin{aligned} \sum_{k=1}^m \rho(\alpha_k x_k, 0) = {} & {}^b\lim_{n\to\infty} \sum_{k=1}^m \rho(a_{nk} x_k, 0) \\ & \leq \left\{ \sum_k \rho(x_k, 0)^p \right\}^{1/p} \left\{ \sup_{n\in\mathbb{N}} \sum_k \rho(a_{nk}, 0)^q \right\}^{1/q} < \infty \end{aligned}$$

for $1 \leq p$, $1/p + 1/q = 1$ for all $n \in \mathbb{N}$ which says us that $(\alpha_k x_k) \in \ell_1^b$. That is to say that $(\alpha_k) \in \ell_q^b$ whenever $(x_k) \in \ell_p^b$. Since $x = (x_k) \in \ell_p^b$, one can choose a $k_0 \in \mathbb{N}$ for $\varepsilon > 0$ such that

$$\sum_{k=k_0+1}^\infty [\rho(x_k, 0)]^p < \left(\frac{\varepsilon}{2[(\sum_k \rho(a_{nk}, 0)^q)^{1/q} + \{\sum_k \rho(\alpha_k, 0)^q\}^{1/q}]} \right)^p$$

for each fixed $k \geq k_0$. Additionally, from (16) we write ${}^b\lim_n \rho(a_{nk}x_k, \alpha_k x_k) = 0$ for each fixed $k \in \mathbb{N}$. Hence there exists an $N = N(k_0) \in \mathbb{N}$ such that

$$\sum_{k=0}^{k_0} \rho(a_{nk}x_k, \alpha_k x_k) < \frac{\varepsilon}{2}$$

for all $n \geq N$. Then, we obtain

$$\begin{aligned}
&\rho\left(\sum_k a_{nk}x_k, \sum_k \alpha_k x_k\right) \\
&\leq \sum_k \rho(a_{nk}x_k, \alpha_k x_k) \\
&= \sum_{k=0}^{k_0} \rho(a_{nk}x_k, \alpha_k x_k) + \sum_{k=k_0+1}^{\infty} \rho(a_{nk}x_k, \alpha_k x_k) \\
&\leq \frac{\varepsilon}{2} + \sum_{k=k_0+1}^{\infty} [\rho(a_{nk}x_k, 0) + \rho(\alpha_k x_k, 0)] \\
&\leq \frac{\varepsilon}{2} + \sum_{k=k_0+1}^{\infty} \rho(a_{nk}, 0)\rho(x_k, 0) + \sum_{k=k_0+1}^{\infty} \rho(\alpha_k, 0)\rho(x_k, 0) \\
&\leq \frac{\varepsilon}{2} + \left\{\sum_{k=k_0+1}^{\infty} \rho(x_k, 0)^p\right\}^{1/p} \left[\left\{\sum_{k=k_0+1}^{\infty} \rho(a_{nk}, 0)^q\right\}^{1/q} + \left\{\sum_{k=k_0+1}^{\infty} \rho(\alpha_k, 0)^q\right\}^{1/q}\right] \\
&\leq \frac{\varepsilon}{2} + \varepsilon \frac{\left(\sum_k \rho(a_{nk}, 0)^q\right)^{1/q} + \left(\sum_k \rho(\alpha_k, 0)^q\right)^{1/q}}{2[(\sum_k \rho(a_{nk}, 0)^q)^{1/q} + \{\sum_k \rho(\alpha_k, 0)^q\}^{1/q}]} = \varepsilon
\end{aligned}$$

for all $n \geq N$. Hence $\sum_k a_{nk}x_k$ b-converges for each $n \in \mathbb{N}$ and $\rho\big(\sum_k a_{nk}x_k, \sum_k \alpha_k x_k\big) \to 0$ as $n \to \infty$. This means that $Ax \in c^b$.

Conversely, Let $A = (a_{nk}) \in (\ell_p^b : c^b)$ and $x = (x_k) \in \ell_p^b$. Then, since Ax exists and the inclusion $(\ell_p^b : c^b) \subset (\ell_p^b : \ell_\infty^b)$ holds, the necessity of (12) is trivial by (iv) of Basic Theorem. Given $x^{(n)} = \{x_k^{(n)}\} \in \ell_p^b$ with $x = e^{(k)}$ for each fixed $k \in \mathbb{N}$, the necessity of (16) is obvious since $(Ax)_n \in c^b$. $\square$

Corollary 5.7. *$A = (a_{nk}) \in (\ell_p^b : c_0^b)$ if and only if (12) holds and (16) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Theorem 5.8. *$A = (a_{nk}) \in (cs^b : c^b)$ if and only if (13) holds, and*

$$\sup_{n\in\mathbb{N}} \sum_k \rho(\Delta a_{nk}, 0) < \infty \text{ where } \Delta a_{nk} = a_{nk} - a_{n,k+1} \tag{17}$$

for all $n, k \in \mathbb{N}$.

Proof. Let $x = (x_k) \in cs^b$ with $(\sum_{k=1}^n x_k) \xrightarrow{\rho} s$ as $n \to \infty$ and $y_k = \sum_{i=0}^k x_i$ for all $k \in \mathbb{N}$. Given infinite matrix $B = (b_{nk})$ by $b_{nk} = \Delta a_{nk}$ for all $n, k \in \mathbb{N}$

and let $A \in (cs^b : c^b)$. Then, Ax exists for every $x \in cs^b$ and is in c^b. It is easy to prove the necessity of (13) that for a given $x = e^{(k)} \in cs^b$ for each fixed $k \in \mathbb{N}$. Also by using Abel's partial summation for m^{th}-partial sums of the series $\sum_k a_{nk}x_k$ we write

$$\begin{aligned} \sum_{k=0}^{m} a_{nk}x_k &= \sum_{k=0}^{m-1} \Delta a_{nk}(y_k - s) + s\sum_{k=0}^{m-1} \Delta a_{nk} + a_{nm}y_m \\ &= \sum_{k=0}^{m-1} \Delta a_{nk}(y_k - s) + s(a_{n0} - a_{nm}) + a_{nm}y_m \text{ for all } m, n \in \mathbb{N}. \end{aligned} \tag{18}$$

Then, by considering $(a_{nk})_{k\in\mathbb{N}} \in \{cs^b\}^\beta = bv^b \subset \ell_\infty^b$ for every fixed $n \in \mathbb{N}$ and taking the b-limit for $m \to \infty$ in (18) we get

$$(Ax)_n = \sum_k a_{nk}x_k = \sum_k b_{nk}(y_k - s) + sa_{n0} \tag{19}$$

for all $n \in \mathbb{N}$. Combining this with ${}^b\lim_n (Ax)_n$ exists and $a_{n0} \xrightarrow{\rho} \alpha_0$, then ${}^b\lim_n \sum_k b_{nk}(y_k - s)$ also exists for $n \to \infty$ in (19). Since $y - s \in c^b$ whenever $x \in cs^b$ then $B \in (c_0^b : c^b)$. Hence, the matrix $B = (b_{nk})$ satisfies the condition (17) which is necessary.

Conversely, (13) and (17) hold. From (17) it is obvious that $A_n = (a_{nk})_{k\in\mathbb{N}} \in \{cs^b\}^\beta = bv^b \subset \ell_\infty^b$ for every fixed $n \in \mathbb{N}$. This leads us Ax exists for every $x \in cs^b$. Also (13) and (17) imply by Corollary 5.5 that $B = (b_{nk}) \in (c_0^b : c_0^b)$. It follows by (19) that ${}^b\lim_n \rho(\sum_k a_{nk}x_k, \alpha_0 s) = 0$ which shows that $A = (a_{nk}) \in (cs^b : c^b)$. □

Theorem 5.9. *$A = (a_{nk}) \in (cs^b : cs^b)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k \rho\left(\sum_{j=0}^{n} \Delta a_{jk}, 0\right) < \infty, \tag{20}$$

$$ {}^b\lim_{n\to\infty} \rho\left(\sum_n a_{nk}, \alpha_k\right) = 0, \tag{21}$$

where $\alpha_k \in \mathbb{R}$ for each $k \in \mathbb{N}$.

Proof. Let $x = (x_k) \in cs^b$ and define the infinite matrix $C = (c_{nk})$ by $c_{nk} = \sum_{j=0}^{n} a_{jk}$, i.e.,

$$C = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0k} & \cdots \\ a_{00} + a_{10} & a_{01} + a_{11} & a_{02} + a_{12} & \cdots & a_{0k} + a_{1k} & \cdots \\ a_{00} + a_{10} + a_{20} & a_{01} + a_{11} + a_{21} & a_{02} + a_{12} + a_{22} & \cdots & a_{0k} + a_{1k} + a_{2k} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ a_{00} + \cdots + a_{n0} & a_{01} + \cdots + a_{n1} & a_{02} + \cdots + a_{n2} & \cdots & a_{0k} + \cdots + a_{nk} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix}$$

for all $k, n \in \mathbb{N}$. Let $A = (a_{nk}) \in (cs^b : cs^b)$. Then, Ax exists for every $x = (x_k) \in cs^b$ and is in cs^b. By choosing $x = e^{(k)} \in cs^b$ for each fixed $k \in \mathbb{N}$ that (21) is necessary. By using the n^{th} and m^{th}-partial sums of the double series $\sum_j \sum_k a_{jk} x_k$, it is clear that

$$\sum_{j=0}^{n}\sum_{k=0}^{m} a_{jk}x_k = \sum_{k=0}^{m}\sum_{j=0}^{n} a_{jk}x_k = \sum_{k=0}^{m} c_{nk}x_k \tag{22}$$

holds for all $m, n \in \mathbb{N}$. Thus, by $m \to \infty$ in (22) we have

$$\sum_{j=0}^{n} (Ax)_j = (Cx)_n \text{ for all } n \in \mathbb{N}. \tag{23}$$

From the hypothesis ${}^b\lim_n \sum_{j=0}^{n} (Ax)_j$ exists and $C = (c_{nk}) \in (cs^b : c^b)$. Then, we deduce by the matrix $C = (c_{nk})$ that the inclusion (17) holds which is equivalent to the condition (20).

Conversely, consider (20) and (21) hold. Hence the existence of the A-transform of $x \in cs^b$ is trivial. Then, since (23) also holds, the matrix C satisfies the conditions of Theorem 5.8. Hence, ${}^b\lim_n (Cx)_n$ exists which implies that $Ax \in cs^b$. □

Theorem 5.10. $A = (a_{nk}) \in (c^b : cs^b)$ *if and only if* (21) *holds and*

$$\sup_{n\in\mathbb{N}} \sum_k \rho\left(\sum_{j=0}^{n} a_{jk}, 0\right) < \infty, \tag{24}$$

$$\sum_n \sum_k a_{nk} \textit{ is b-convergent} \tag{25}$$

for all $k, n \in \mathbb{N}$.

Proof. Let $x = (x_k) \in c^b$ and define the matrix $C = (c_{nk})$ as in the proof of Theorem 5.9. Let $A = (a_{nk}) \in (c^b : cs^b)$. Then, Ax exists for every $x \in c^b$ and is in cs^b. This yields for $x = e^{(k)} \in c^b$ and $x = e \in c^b$ which give the necessity of the conditions (21) and (25), respectively. By applying the same way used in the proof of Theorem 5.9, we get the inclusion (23). Then, since $Ax \in cs^b$, that is $\sum_j (Ax)_j$ b-converges. By the hypothesis ${}^b\lim_n \sum_{j=0}^{n} (Ax)_j$ exists, from (23), we say that $C = (c_{nk}) \in (c^b : c^b)$. Thus, the condition (i) of Basic Theorem is satisfied which is equivalent to the condition (24).

Conversely, assume that (21), (24) and (25) hold, then the existence of the A-transform of $x \in c^b$ is clear. Since (23) also holds then $C \in (c^b : c^b)$ which yields that $Ax \in cs^b$, as was desired. □

Concluding remarks

The idea of dual sequence space which plays an important role in the representation of linear functionals and the characterization of matrix transformations between sequence spaces, was introduced by Köthe and Toeplitz [16], whose main results concerned α-duals.

In this paper we have introduced the sequence spaces ℓ_∞^b, c^b, c_0^b, ℓ_p^b, bs^b, cs^b, cs_0^b, bv^b, bv_p^b and bv_∞^b as a generalization of the sets ℓ_∞, c, c_0, ℓ_p, bs, cs, cs_0, bv, bv_p and bv_∞ of sequences. Our main purpose is to determine the Kothe-Toeplitz duals of the new spaces and related matrix transformations on them. As a future work we will try to obtain other characterizations of the classes of infinite matrices via this metric.

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