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FINITE BCK-ALGEBRAS ARE SOLVABLE

Wieslaw A. Dudek

Abstract. We present a short and very elementary proof that each finite BCK-algebra is solvable. It is a positive answer to the question posed in [4].

Various types of BCK-algebras – as algebras strongly connected with non-classical propositional calculi – are studied by many authors. A short survey of basic results on BCK-algebras one can find in the book [2]. The list of some unsolved (rather hard) problems on BCK-algebras one can find in [1].

Let X be a non-empty set with a binary operation $\cdot$ denoted by juxtaposition and distinguished element 0. To save the simplicity of formulaes the part of brackets will be replaced by dots. For example, instead of $((xy)(xz))(yz) = ((xy)z)y$ we'll write $(xy \cdot xz) \cdot yz = (xy \cdot z)x$.

An algebra $(X, \cdot, 0)$ is called a *BCK-algebra* if for any $x, y, z \in X$ the following axioms:

(1) $(xy \cdot xz) \cdot zy = 0$,
(2) $(x \cdot xy)y = 0$,
(3) $xx = 0$,
(4) $0x = 0$,
(5) $xy = yx = 0 \longrightarrow x = y$

are satisfied.

One can prove (cf. for example [2]) that in any BCK-algebra X we have

(6) $xy \cdot z = xz \cdot y$,
(7) $x0 = x$,
(8) $xy \leq x$,

where $\leq$ is a partial order on X defined by $x \leq y \longleftrightarrow xy = 0$.

It is not difficult to see that $x \wedge y = y \cdot yx$ is a lower bound of x and y. A BCK-algebra which is a lower semilattice with respect to $\leq$ is called *commutative*. This name is motivated by the fact that such BCK-algebra is characterized by the identity $x \wedge y = y \wedge x$ (cf. [2] and [5]) or by the identity $[x, y] = 0$, where $[x, y] = (x \wedge y)(y \wedge x)$. Properties of the operation $[\ ,\ ]$ are described in [3] and [4].

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A BCK-algebra X is called *solvable* if there exists a natural number n such that $X^{(n)} = \{0\}$, where $X' = [X, X]$ is a product of a finite number of elements of the form $[x, y]$, where $x, y \in X$ and $X^{(k+1)} = [X^{(k)}, X^{(k)}]$. Then $X^{(k+1)}$ is a BCK-algebra contained in $X^{(k)}$ (cf. [4]). Moreover, a BCK-algebra X is commutative if and only if $X' = \{0\}$ (cf. [3]).

In [4] Najafi and Borumand Saied posed a question whether every finite BCK-algebra is solvable. Below we present a positive answer to this question.

Lemma 1. *In any BCK-algebra X for every $x, y \in X$*

(a) $[x, y] \le y \cdot yx \le x$,
(b) $[x, y] = x \longleftrightarrow x = 0$,
(c) $x \neq 0 \longrightarrow [x, y] < x$.

Proof. (a) Let $z = [x, y]$. Then $z(y\cdot yx) = (y\cdot yx)(x\cdot xy)\cdot(y\cdot yx) = 0\cdot(x\cdot xy) = 0$. Thus $z \le y \cdot yx \le x$.

(b) If $[x, y] = x$, then by (a) we obtain $x \le y \cdot yx \le x$. Thus $y \cdot yx = x$, whence $xy = (y \cdot yx)y = yy \cdot yx = 0 \cdot yx = 0$. So, $x \le y$. But in this case $[x, y] = (y \cdot yx)(x \cdot xy) = (y \cdot xy) \cdot x0 = (y \cdot yx)x = yx \cdot yx = 0$. Hence $x = 0$. The converse statement is obvious.

(c) It is a consequence of (a) and (b). □

Theorem 1. *Any finite BCK-algebra is solvable.*

Proof. Let X be a finite BCK-algebra and M be the set of all its maximal elements. Suppose that $|X| = n \ge 2$. Then M has at least one element and $[0, y] = 0 \notin M$ for every $y \in X$. For $x \neq 0$ there is $m \in M$ such that $x \le m$. This, by Lemma 1, implies $[x, y] < x \le m$, so $[x, y] \notin M$ for all $x, y \in X$. Since for any $u, v \in X \setminus M$ we have $uv \le u < m$, $uv \in X \setminus M$, which means that X' is a subalgebra contained in $X \setminus M$. Hence $|X| > |X'|$. By induction $|X| > |X'| > |X''| > \cdots > |X^{(k)}| = 1$ for some $k < n$. It is not difficult to see that k is less than the length of the longest sequence $0 < a_1 < a_2 < \cdots < a_m$ of elements of X. □

References

[1] W. A. Dudek, *Unsolved problems in BCK-algebras*, East Asian Math. J. **17** (2001), 115–128.
[2] J. Meng and Y. B. Jun, *BCK-Algebras*, Kyung Moon Sa Co. Seoul, Korea, 1994.
[3] A. Najafi, *Pseudo-commutators in BCK-algebras*, Pure Math. Sci. **2** (2013), 29–32.
[4] A. Najafi and A. Borumand Saied, *Solvable BCK-algebras*, Çankaya Univ. J. Sci. Engineering **11** (2014), 19–28.
[5] S. Tanaka, *A new class of algebras*, Math. Sem. Notes Kobe Univ. **3** (1975), no. 1, 37–43.

Faculty of Pure and Applied Mathematics
Wrocław University of Science and Technology
Wyb. Wyspiańskiego 27
50-370 Wrocław
Poland
E-mail address: wieslaw.dudek@pwr.edu.pl