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JORDAN HIGHER DERIVATIONS ON TRIVIAL EXTENSION ALGEBRAS

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Abstract. We first give the constructions of (Jordan) higher derivations on a trivial extension algebra and then we provide some sufficient conditions under which a Jordan higher derivation on a trivial extension algebra is a higher derivation. We then proceed to the trivial generalized matrix algebras as a special trivial extension algebra. As an application we characterize the construction of Jordan higher derivations on a triangular algebra. We also provide some illuminating examples of Jordan higher derivations on certain trivial extension algebras which are not higher derivations.

1. Introduction

Let $\mathcal{A}$ be an algebra (over a unital abelian ring) and let $\mathbb{N}$ stand for the set of all nonnegative integers. A sequence $D = \{\delta_k\}_{k\in\mathbb{N}}$ of additive mappings on $\mathcal{A}$ (with $\delta_0 = id_{\mathcal{A}}$) is said to be

- a higher derivation if for each $k \in \mathbb{N}$,

$$\delta_k(xy) = \sum_{i=0}^{k} \delta_i(x)\delta_{k-i}(y) \qquad (x, y \in \mathcal{A}); \tag{1.1}$$

- a Jordan higher derivation if for each $k \in \mathbb{N}$,

$$\delta_k(x^2) = \sum_{i=0}^{k} \delta_i(x)\delta_{k-i}(x) \qquad (x \in \mathcal{A}). \tag{1.2}$$

It is easy to verify that if $D = \{\delta_k\}$ is a Jordan higher derivation, then

$$\delta_k(xy + yx) = \sum_{i=0}^{k} \delta_i(x)\delta_{k-i}(y) + \delta_i(y)\delta_{k-i}(x) \qquad (x, y \in \mathcal{A}) \tag{1.3}$$

and the converse holds in the case where $\mathcal{A}$ is 2-torsionfree (that is, $2x = 0$ implies $x = 0$ for any $x \in \mathcal{A}$).

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It is obvious that if $\{\delta_k\}$ is a (Jordan) higher derivation, then δ_1 is a (Jordan) derivation. For a typical example of a (Jordan) higher derivation, one can consider $\{\frac{\delta^k}{k!}\}$, where $\delta : \mathcal{A} \to \mathcal{A}$ is a (Jordan) derivation. This kind of (Jordan) higher derivation is called an ordinary (Jordan) higher derivation.

It is also trivial that every higher derivation is a Jordan higher derivation, but the converse is not true in general. The standard problem is finding out whether a Jordan higher derivation is necessarily a higher derivation. It was shown by Herstein [5] that every Jordan derivation on a 2-torsionfree prime ring is a derivation. This result was extended by Brešar [2] to the case of semiprime rings. Zhang an Yu [15] proved that every Jordan derivation on a triangular algebra is a derivation. Benkovič and Širovnik [1] studied Jordan derivations on a unital algebra with a nontrivial idempotent. Li et al. [6] investigated the Jordan derivations on a generalized matrix algebra. In the context of (Jordan) higher derivations, the construction of a higher derivation on a general algebra has been studied by Mirzavaziri [7]. Xiao and Wei [13] showed that any Jordan higher derivation on a triangular algebra is a higher derivation. Jordan higher derivations on some class of operator algebras have also investigated by Xiao and Wei [14].

In this paper we shall study Jordan higher derivations on trivial extension algebras. Let us introduce a trivial extension algebra. Let $\mathcal{A}$ be an algebra and let $\mathcal{M}$ be an $\mathcal{A}-$module. Then the direct product $\mathcal{A} \times \mathcal{M}$ under its usual pairwise operations and the multiplication given by

$$(a, m)(b, n) = (ab, an + mb) \qquad (a, b \in \mathcal{A}, m, n \in \mathcal{M}),$$

is an algebra which is called the "*trivial extension*" of $\mathcal{A}$ by $\mathcal{M}$ and is denoted by $\mathcal{A} \ltimes \mathcal{M}$. This name comes from some cohomological properties of $\mathcal{A} \ltimes \mathcal{M}$. Indeed, Hochschild has noticed that $\mathcal{A} \ltimes \mathcal{M}$ corresponds to the "*trivial*" element in the second cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{M}$. This is related to the fact that there is a correspondence between derivations from $\mathcal{A}$ to $\mathcal{M}$ and the automorphisms of $\mathcal{A} \ltimes \mathcal{M}$, [11]. It should also be remarked that in functional analysis literature, algebras of this type were termed "module extension" algebras; see [16], in which some interesting performances of Banach algebras of this type are presented.

The main example of a trivial extension algebra is the so-called triangular algebra $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$, which was first introduced by Cheung [3]. Indeed, $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ can be identified to the trivial extension $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathcal{M}$. More generally, as we shall discuss in Section 3, every trivial generalized matrix algebra $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ can be identified to the trivial extension algebra $(\mathcal{A} \oplus \mathcal{B}) \ltimes (\mathcal{M} \oplus \mathcal{N})$.

Jordan derivations on $\mathcal{A} \ltimes \mathcal{M}$ are studied in [4]. In [10] (see also [8]), Lie derivations of $\mathcal{A} \ltimes \mathcal{M}$ are discussed. In this paper, our main aim is to provide some sufficient conditions under which a Jordan higher derivation on $\mathcal{A} \ltimes \mathcal{M}$ become a higher derivation.

The paper is organized as follows. Section 2 is devoted to the constructions of (Jordan) higher derivations on trivial extension algebras. We study some sufficient conditions under which a Jordan higher derivation on a trivial extension algebra is a higher derivation. In this respect, we consider those trivial extensions $\mathcal{A} \ltimes \mathcal{M}$ such that the $\mathcal{A}$-module $\mathcal{M}$ enjoys zero action from one side (Theorem 2.2). We include an illuminating example of a (non-ordinary) Jordan higher derivation on a trivial extension which is not a higher derivation (Example 2.3). In Section 3, we first show that every trivial generalized matrix algebra is a trivial extension algebra. We then explore the structure of (Jordan) higher derivations of a trivial generalized matrix algebra, intending to arrive at the "higher" version of some results of [6] and [1]. In this respect, we leave a conjecture, to the best of our knowledge, seems to be undecided. At the final part of Section 3, we employ our results to give the construction of Jordan higher derivations on a triangular algebra (Theorem 3.3). It, in particular, provides a direct proof for the fact that every Jordan higher derivation on a triangular algebra is a higher derivation, which has already proved in [13].

2. Jordan higher derivations on $\mathcal{A} \ltimes \mathcal{M}$

Let us proceed with the following result, characterizing the construction of a (Jordan) higher derivation on a trivial extension algebra $\mathcal{A} \ltimes \mathcal{M}$.

Proposition 2.1. *A sequence $\{\delta_k\}_{k\in\mathbb{N}}$ of additive mappings on $\mathcal{A} \ltimes \mathcal{M}$ can be presented as the form*

(2.1) $\delta_k(a,m) = (J_k(a) + T_k(m), K_k(a) + S_k(m)) \quad (a \in \mathcal{A}, m \in \mathcal{M}, k \in \mathbb{N}),$

where $J_k : \mathcal{A} \to \mathcal{A}, K_k : \mathcal{A} \to \mathcal{M}, T_k : \mathcal{M} \to \mathcal{A}$ and $S_k : \mathcal{M} \to \mathcal{M}$ are additive mappings. Moreover,

- *D is a higher derivation if and only if*

(1) $\{J_k\}_{k\in\mathbb{N}}$ *is a higher derivation on* $\mathcal{A}$;
(2) $K_k(ab) = \sum_{i=0}^{k} \big(J_i(a)K_{k-i}(b) + K_i(a)J_{k-i}(b)\big)$;
(3) $T_k(ma) = \sum_{i=0}^{k} T_i(m)J_{k-i}(a),\ T_k(am) = \sum_{i=0}^{k} J_i(a)T_{k-i}(m)$;
(4) $S_k(ma) = \sum_{i=0}^{k} \big(S_i(m)J_{k-i}(a) + T_i(m)K_{k-i}(a)\big)$,
$S_k(am) = \sum_{i=0}^{k} \big(J_i(a)S_{k-i}(m) + K_i(a)T_{k-i}(m)\big)$;
(5) $\sum_{i=0}^{k} T_i(m)T_{k-i}(n) = 0,\ \sum_{i=0}^{k} \big(T_i(m)S_{k-i}(n) + S_i(m)T_{k-i}(n)\big) = 0$

for all $a, b \in \mathcal{A}, m \in \mathcal{M}$.

- *D is a Jordan higher derivation if and only if*

(a) $\{J_k\}_{k\in\mathbb{N}}$ *is a Jordan higher derivation on* $\mathcal{A}$;
(b) $K_k(a^2) = \sum_{i=0}^{k} \big(J_i(a)K_{k-i}(a) + K_i(a)J_{k-i}(a)\big)$;
(c) $T_k(ma + am) = \sum_{i=0}^{k} \big(T_i(m)J_{k-i}(a) + J_i(a)T_{k-i}(m)\big)$;
(d) $S_k(ma + am) = \sum_{i=0}^{k} \big(S_i(m)J_{k-i}(a) + T_i(m)K_{k-i}(a) + J_i(a)S_{k-i}(m) + K_i(a)T_{k-i}(m)\big)$;
(e) $\sum_{i=0}^{k} T_i(m)T_{k-i}(m) = 0,\ \sum_{i=0}^{k} \big(T_i(m)S_{k-i}(m) + S_i(m)T_{k-i}(m)\big) = 0$

for all $a \in \mathcal{A}, m \in \mathcal{M}$.

Proof. As $\delta_0 = id_{\mathcal{A}\ltimes\mathcal{M}}$, we have $J_0 = id_{\mathcal{A}}, T_0 = 0, K_0 = 0$ and $S_0 = id_{\mathcal{M}}$. Fix $k \in \mathbb{N}$ and let $\delta_k : \mathcal{A} \ltimes \mathcal{M} \to \mathcal{A} \ltimes \mathcal{M}$ be an additive mapping. That δ_k has the presentation (2.1) is straightforward. Then D is a higher derivation if and only if the identity

(2.2)

$$\begin{aligned}&\big(J_k(ab) + T_k(an + mb), K_k(ab) + S_k(an + mb)\big)\\ = &\sum_{i=0}^{k} \big(J_i(a) + T_i(m), K_i(a) + S_i(m))(J_{k-i}(b) + T_{k-i}(n), K_{k-i}(b) + S_{k-i}(n)\big)\end{aligned}$$

holds for all $a, b \in \mathcal{A}, m, n \in \mathcal{M}$. One can directly check that (2.2) holds if and only if it is true for $(a, b, 0, 0), (a, 0, 0, m), (0, a, m, 0)$ and $(0, 0, m, n)$. We now have the following considerations for the equation (2.2).

It is true for
$(a, b, 0, 0)$ if and only if (1) and (2) hold;
$(a, 0, 0, m)$ if and only if a half of (3) and (4) hold (for $T_k(am)$ and $S_k(am)$);
$(0, a, m, 0)$ if and only if the other half of (3) and (4) hold (for $T_k(ma)$ and $S_k(ma)$);
$(0, 0, m, n)$ if and only if (5) holds.

A similar argument shows the result for Jordan higher derivations. □

By the virtue of Proposition 2.1(b), if we replace a with $a + b$ we arrive at the following identity which will be frequently used in the sequel.

(2.3)

$$\begin{aligned}&K_k(ab + ba)\\ = &\sum_{i=0}^{k} \big(J_i(a)K_{k-i}(b)+J_i(b)K_{k-i}(a)+K_i(a)J_{k-i}(b)+K_i(b)J_{k-i}(a)\big),\ (a, b\in\mathcal{A}).\end{aligned}$$

For an $\mathcal{A}$-module $\mathcal{M}$ we recall that:

$\mathcal{M}$ is left (resp. right) faithful if $a\mathcal{M} = 0$ (resp. $\mathcal{M}a = 0$) implies $a = 0$ for any $a \in \mathcal{A}$. If $\mathcal{M}$ is both left and right faithful, then it is called faithful.

In the case where $\mathcal{A}$ is unital, $\mathcal{M}$ is left (resp. right) unital if $1m = m$ (resp. $m1 = m$) for any $m \in \mathcal{M}$. If $\mathcal{M}$ is both left and right unital, then it is called unital.

We employ Proposition 2.1 for certain trivial extensions. We consider the case that the module operation on one side of $\mathcal{M}$ is trivial. We denote by $\mathcal{M}_0$ (resp. ${}_0\mathcal{M}$) specifically the $\mathcal{A}$-module with trivial right module action, (i.e., $ma = 0$ (resp. $am = 0$) for all $a \in \mathcal{A}, m \in \mathcal{M}$). The trivial extension algebras of this type are known as a fertile source of (counter-)examples in various situations in functional analysis. For example, they have been served for constructing certain counter-examples in the theory of weak amenability of

Banach algebras; [16, Example 7.5]. In the next result, we show that every Jordan higher derivation on either of the trivial extension algebras $\mathcal{A} \ltimes \mathcal{M}_0$ and $\mathcal{A} \ltimes {}_0\mathcal{M}$ is a higher derivation.

Theorem 2.2. *Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ be a left unital, left faithful $\mathcal{A}$-module. Then every Jordan higher derivation on $\mathcal{A}\ltimes\mathcal{M}_0$ is a higher derivation. The same fact holds for $\mathcal{A} \ltimes_0 \mathcal{M}$, in the case where $\mathcal{M}$ is a right unital, right faithful $\mathcal{A}$-module.*

Proof. Let $\{\delta_k\}$ be a Jordan higher derivation on $A\ltimes\mathcal{M}_0$ with the presentation as in (2.1). In order to show that $\{\delta_k\}$ is a higher derivation, it suffices to prove that the conditions (1) to (5) of Proposition 2.1 are fulfilled. First, let us show that $T_k = 0$ for all k. As $ma = 0$, for all $a \in \mathcal{A}, m \in \mathcal{M}$, from (c) we get the identity $T_k(am) = \sum_{i=0}^{k} \big(T_i(m)J_{k-i}(a) + J_i(a)T_{k-i}(m)\big)$. Applying the latter identity for $a = 1$ together with the fact that $J_k(1) = 0$ for each $k \geq 1$, one gets $T_k = 0$ for all k. From this the identities in (5) are fulfilled trivially and also the identities in (4) follow obviously from (d).

Our next aim is to show that $\{J_k\}$ is a higher derivation. We prove by induction on k. Fix $a, b \in \mathcal{A}$ and $m \in \mathcal{M}$. From Proposition 2.1(4), we have

$$S_k(am) = \sum_{i=0}^{k} J_i(a)S_{k-i}(m). \tag{2.4}$$

It follows that $S_1(am) = J_1(a)m + aS_1(m)$, which gives

$$\begin{aligned} J_1(ab)m + abS_1(m) = S_1(abm) &= J_1(a)bm + aS_1(bm) \\ &= J_1(a)bm + aJ_1(b)m + abS_1(m), \end{aligned}$$

and the left faithfulness of $\mathcal{M}$ implies that $J_1(ab) = J_1(a)b + aJ_1(b)$. Suppose that the conclusion holds for any integer less than k. By (2.4), we arrive at

$$\begin{aligned} & J_k(ab)m + \sum_{i=0}^{k-1} J_i(ab)S_{k-i}(m) \\ = & \sum_{i=0}^{k} J_i(ab)S_{k-i}(m) \\ = & \, S_k(abm) \\ = & \sum_{i=0}^{k} J_i(a)S_{k-i}(bm) \\ = & \sum_{i=0}^{k} J_i(a)\Big(\sum_{j=0}^{k-i} J_j(b)S_{k-i-j}(m)\Big) \\ = & \sum_{i=0}^{k} J_i(a)J_{k-i}(b)m + \sum_{i=0}^{k-1} \Big(\sum_{j=0}^{i} J_j(a)J_{i-j}(b)\Big)S_{k-i}(m) \end{aligned}$$

$$= \sum_{i=0}^{k} J_i(a)J_{k-i}(b)m + \sum_{i=0}^{k-1} J_i(ab)S_{k-i}(m).$$

By the induction hypothesis and the fact that $\mathcal{M}$ is left faithful we conclude that $\{J_k\}$ is a higher derivation.

It remains to show that K_k satisfies (2). By (2.3), we have

$$K_k(ab+ba) = \sum_{i=0}^{k} \big(J_i(a)K_{k-i}(b) + J_i(b)K_{k-i}(a)\big). \tag{2.5}$$

Putting $b = 1$ in (2.5), we get

$$K_k(a) = \sum_{i=0}^{k} J_i(a)K_{k-i}(1). \tag{2.6}$$

Using the equation (2.6) for ab, as J_k is a higher derivation for each k, we have

$$\begin{aligned} K_k(ab) &= \sum_{i=0}^{k} J_i(ab)K_{k-i}(1) = \sum_{i=0}^{k} \big(\sum_{j=0}^{i} J_r(a)J_{i-j}(b)\big)K_{k-i}(1) \\ &= \sum_{i=0}^{k} J_i(a)\big(\sum_{j=0}^{k-i} J_r(b)K_{k-i-j}(1)\big) = \sum_{i=0}^{k} J_i(a)K_{k-i}(b). \end{aligned}$$

Thus K_k satisfies (2) and this completes the proof. □

The following example illustrates that, in contrast to the situation for triangular algebras (see [13] and Theorem 3.3 infra), a Jordan higher derivation on a trivial extension algebra may not be a higher derivation.

Example 2.3. Let $\mathcal{A}_2$ be the algebra of 2×2 upper triangular matrices on $\mathbb{R}$. We consider $\mathbb{R}$ as an $\mathcal{A}_2$-module equipped with the module operations $am = a_{22}m$ and $ma = ma_{11}$ $(a \in \mathcal{A}_2, m \in \mathbb{R})$. Then the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ defined by $\delta_0 = id_{\mathcal{A}_2 \ltimes \mathbb{R}}$ and

$$\begin{aligned} &\delta_k : \mathcal{A}_2 \ltimes \mathbb{R} \longrightarrow \mathcal{A}_2 \ltimes \mathbb{R} \\ &(a, m) \mapsto \Big(\begin{pmatrix} 0 & \frac{a_{12}}{k!} \\ 0 & 0 \end{pmatrix}, \frac{k-1}{k!}a_{12} \Big) \quad (a \in \mathcal{A}_2, m \in \mathbb{R}, k \in \mathbb{N}) \end{aligned}$$

is a Jordan higher derivation. Here

$$J_k(a) = \begin{pmatrix} 0 & \frac{a_{12}}{k!} \\ 0 & 0 \end{pmatrix}, \quad T_k(m) = 0, \quad K_k(a) = \frac{k-1}{k!}a_{12}, \quad S_k(m) = 0.$$

We prove that these mappings satisfy the conditions of Proposition 2.1. We have

(a) $\{J_k\}$ is a Jordan higher derivation on $\mathcal{A}$, since for $k \in \mathbb{N}$ we have

$$J_k(a^2) = \begin{pmatrix} 0 & \frac{a_{11}a_{12}+a_{12}a_{22}}{k!} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{a_{12}}{k!} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\begin{pmatrix} 0 & \frac{a_{12}}{k!} \\ 0 & 0 \end{pmatrix}$$
$$= J_k(a)a + aJ_k(a)$$
$$= \sum_{i=0}^{k} J_i(a)J_{k-i}(a).$$

(b) $K_k(a^2) = \sum_{i=0}^{k}(K_i(a)J_{k-i}(a) + J_i(a)K_{k-i}(a))$, since

$$K_k(a^2) = \frac{k-1}{k!}(a_{11}a_{12} + a_{12}a_{22})$$

and for $i = 1, \ldots, k-1$, $K_i(a)J_{k-i}(a) = 0 = J_i(a)K_{k-i}(a)$. Further,

$$K_k(a)J_0(a) = \frac{k-1}{k!}a_{12}\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \frac{k-1}{k!}a_{12}a_{11},$$
$$J_k(a)K_0(a) = 0,$$

and

$$K_0(a)J_k(a) = 0,$$
$$J_0(a)K_k(a) = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\frac{k-1}{k!}a_{12} = a_{22}\frac{k-1}{k!}a_{12}.$$

The conditions (c), (d), (e) in Proposition 2.1 clearly hold, and so $\{\delta_k\}$ is a Jordan higher derivation.

Moreover, $\{\delta_k\}$ is not a higher derivation, since

$$\delta_2\Big(\big(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},0\big)\big(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},0\big)\Big) = \delta_2\Big(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},0\Big) = \Big(\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix},\frac{1}{2}\Big);$$

while,

$$\delta_2\Big(\big(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},0\big)\big(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},0\big)\Big) + 2\delta_1\Big(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},0\Big)\delta_1\Big(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},0\Big)$$
$$+ \Big(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},0\Big)\delta_2\Big(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},0\Big)$$
$$= \Big(\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix},\frac{1}{2}\Big)\Big(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},0\Big) + 2\Big(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},0\Big)(0,0)$$
$$+ \Big(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},0\Big)\Big(0,0\Big)$$
$$= \Big(\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix},0\Big).$$

Nevertheless, δ_1 is a derivation, since for every $a, b \in \mathcal{A}_2, m, n \in \mathbb{R}$,

$$\delta_1((a,m)(b,n)) = \delta_1(ab, an + mb) = \Big(\begin{pmatrix} 0 & (ab)_{12} \\ 0 & 0 \end{pmatrix},0\Big)$$

$$=\Big(\begin{pmatrix} 0 & a_{11}b_{12}+a_{12}b_{22} \\ 0 & 0 \end{pmatrix},0\Big),$$

and

$$\begin{aligned}
&\delta_1(a,m)(b,n)+(a,m)\delta_1(b,n)\\
=&\Big(\begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix},0\Big)(b,n)+(a,m)\Big(\begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix},0\Big)\\
=&\Big(\begin{pmatrix} 0 & a_{12}b_{22} \\ 0 & 0 \end{pmatrix},0\Big)+\Big(\begin{pmatrix} 0 & a_{11}b_{12} \\ 0 & 0 \end{pmatrix},0\Big).
\end{aligned}$$

It should be remarked that, in the above example $\{\delta_k\}_{k\in\mathbb{N}}$ is not an ordinary Jordan higher derivation, since $\delta_2 \neq \frac{1}{2}\delta_1$. In the next example we present an ordinary Jordan higher derivation on the same trivial extension algebra $\mathcal{A}_2 \ltimes \mathbb{R}$ which is not a higher derivation.

Example 2.4. Let $\mathcal{A}_2 \ltimes \mathbb{R}$ be the trivial extension algebra as given in Example 2.3. Then, a direct verification reveals that, the map $\delta : \mathcal{A}_2 \ltimes \mathbb{R} \longrightarrow \mathcal{A}_2 \ltimes \mathbb{R}$ defined by $\delta(a,m) = (0, a_{12})$ is a Jordan derivation that is not a derivation. Consequently, the ordinary Jordan higher derivation $\{\frac{\delta^k}{k!}\}$ is not a higher derivation on $\mathcal{A}_2 \ltimes \mathbb{R}$.

3. Application to trivial generalized matrix algebras and triangular algebras

Trivial generalized matrix algebras: This kind of algebras, as the natural generalization of triangular algebras, were first introduced by Sands [12], where he studied various radicals of algebras occurring in Morita contexts. Roughly speaking, a generalized matrix algebra has the following presentation

$$\mathcal{G}(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B}) = \Big\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} :\ a\in\mathcal{A}, m\in\mathcal{M}, n\in\mathcal{N}, b\in\mathcal{B} \Big\},$$

where $\mathcal{A}$ and $\mathcal{B}$ are unital algebras and $\mathcal{M},\mathcal{N}$ are $(\mathcal{A},\mathcal{B})$-module and $(\mathcal{B},\mathcal{A})$-module, respectively, such that at least one of $\mathcal{M}$ and $\mathcal{N}$ is distinct from zero. The algebra operations of $\mathcal{G}(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ are the usual matrix-like operations, in which the symbolic products $mn = \Phi_{\mathcal{MN}}(m\otimes n) \in \mathcal{A}$ and $nm = \Psi_{\mathcal{NM}}(n \otimes m) \in \mathcal{B}$, $(m \in \mathcal{M}, n \in \mathcal{N})$, come from certain module homomorphisms $\Phi_{\mathcal{MN}} : \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \longrightarrow \mathcal{A}$ and $\Psi_{\mathcal{NM}} : \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \longrightarrow \mathcal{B}$. However, for our purpose here, we further assume that $mn = 0$ and $nm = 0$ for all $m \in \mathcal{M}, n \in \mathcal{N}$, that is, $\Phi_{\mathcal{MN}}$ and $\Psi_{\mathcal{NM}}$ are both zero. Such a generalized matrix algebra is called a "*trivial generalized matrix algebra*".

It is worth to notice that every trivial generalized matrix algebra is a trivial extension algebra. Indeed, it can be readily verified that $\mathcal{G}(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ is isomorphic to the trivial extension algebra $(\mathcal{A}\oplus\mathcal{B}) \ltimes (\mathcal{M}\oplus\mathcal{N})$, where the algebra $\mathcal{A} \oplus \mathcal{B}$ has its usual pointwise operations and $\mathcal{M} \oplus \mathcal{N}$ as an $(\mathcal{A} \oplus \mathcal{B})$-module is equipped with the module operations

$$(a \oplus b)(m \oplus n) = am \oplus bn \quad \text{and} \quad (m \oplus n)(a \oplus b) = mb \oplus na,$$
$$(a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}, n \in \mathcal{N}).$$

In the case where $\mathcal{N} = 0$ we arrive at the so-called triangular algebra $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$, which is isomorphic to the trivial extension algebra $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathcal{M}$. It is known that every Jordan higher derivation on $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a higher derivation [13]. In particular, every Jordan derivation on $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a derivation [15].

In contrast to the situation for triangular algebras, as the following example demonstrates, a Jordan (higher) derivation on a trivial generalized matrix algebra need not be a (higher) derivation, in general.

Example 3.1 (See [6, Example 3.5])**.** Set $\mathcal{A} = \mathcal{B} = \mathbb{R}$, equipped with its usual algebra operations, $\mathcal{M} = \mathcal{N} = \mathbb{R}$ with the multiplication as $\mathbb{R}$-module operations and suppose that $mn = 0 = nm$ for each $m \in \mathcal{M}, n \in \mathcal{N}$. Then a direct verification reveals that the map δ on the trivial generalized matrix algebra $(\mathcal{A} \oplus \mathcal{B}) \ltimes (\mathcal{M} \oplus \mathcal{N})$ defined by

$$(a \oplus b, m \oplus n) \mapsto (0, m + n \oplus m - n) \quad (a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}, n \in \mathcal{N}),$$

is a Jordan derivation, but not a derivation. It follows that the ordinary Jordan higher derivation $\{\frac{\delta^k}{k!}\}$ is not a higher derivation on $(\mathcal{A} \oplus \mathcal{B}) \ltimes (\mathcal{M} \oplus \mathcal{N})$.

In the context of Jordan derivations, it has been shown in [6, Theorem 3.11] (see also [1, Corollary 4.2]) that, under some mild conditions, every Jordan derivation on a trivial generalized matrix algebra $(\mathcal{A} \oplus \mathcal{B}) \ltimes (\mathcal{M} \oplus \mathcal{N})$ can be expressed as the sum of a derivation and an antiderivation; and the involved antiderivation is identically zero in the case where $\mathcal{N} = 0$. For instance, one can directly check that, in the setting of Example 3.1, the maps $(a \oplus b, m \oplus n) \mapsto (0, m \oplus -n)$ and $(a \oplus b, m \oplus n) \mapsto (0, n \oplus m)$ are the desired derivation and antiderivation, respectively.

In [4, Theorem 2.1] it has been shown that, under some very technical conditions on a trivial extension algebra, every Jordan derivation is the sum of a derivation and an antiderivation. However, a careful look at the proposed conditions reveals that the discussed trivial extension algebra is a trivial generalized matrix algebra.

As generalizations of [6, Propositions 3.1, 3.2], one can employ Proposition 2.1 for $(\mathcal{A} \oplus \mathcal{B}) \ltimes (\mathcal{M} \oplus \mathcal{N})$, (of course through tedious computations) to give the construction of (Jordan) higher derivations of a trivial generalized matrix algebra. The obtained constructions can apply for exploring those sufficient conditions expressing a Jordan higher derivation $\{\delta_k\}$ on a trivial generalized matrix algebra as the sum of a higher derivation $\{d_k\}$ and some "suitable" sequence $\{d'_k\}$ of additive mappings. It should be noticed that, in contrast to the case $k = 1$, the tail sequence $\{d'_k\}$ is not a higher antiderivation, (i.e., does not satisfy the equation $d'_k(xy) = \sum_{i=0}^{k} d'_i(y)d'_{k-i}(x)$ for all x, y), in general.

It would be more desirable if one studies those conditions under which the tail sequence $\{d'_k\}$ is identically zero. However, the existent examples support to conjecture that:

Conjecture 3.2. If every Jordan higher derivation on a trivial generalized matrix algebra $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is a higher derivation, then either $\mathcal{M} = 0$ or $\mathcal{N} = 0$; (which makes $\mathcal{G}$ into a triangular algebra).

Triangular algebras: The rest of this section is devoted to the case that $\mathcal{N} = 0$. In this case we arrive at the so-call triangular algebra $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. In the following result, we apply Proposition 2.1 to give the construction of (Jordan) higher derivations on $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. This, in particular, provides a direct proof for the main result of [13], stating that every Jordan higher derivation on a triangular algebra is a higher derivation, which was proved by a quite different method.

Theorem 3.3. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are 2-torsion free unital algebras and $\mathcal{M}$ is faithful as an $(\mathcal{A}, \mathcal{B})$-module. Then every Jordan higher derivation $\{\delta_k\}$ on the triangular algebra* $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *can be presented in the form*

$$\delta_k \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} p_k(a) & \sum_{i=1}^k \left(p_{k-i}(a)m_i - m_i q_{k-i}(b)\right) + S_k(m) \\ 0 & q_k(b) \end{pmatrix}$$
$$(a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}),$$

where $\{p_k\}$ and $\{q_k\}$ are higher derivations on $\mathcal{A}$ and $\mathcal{B}$, respectively; $\{m_k\}$ is a sequence in $\mathcal{M}$ and $\{S_k\}$ is a sequence of additive mappings on $\mathcal{M}$ satisfying:

$$S_k(am) = \sum_{i=0}^{k} p_i(a) S_{k-i}(m) \quad \text{and} \quad S_k(mb) = \sum_{i=0}^{k} S_i(m) q_{k-i}(b)$$
$$(a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}).$$

In particular, every Jordan higher derivation on the triangular algebra $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *is a higher derivation.*

Proof. We recall that the triangular algebra $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ can be identified to the trivial extension algebra $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathcal{M}$, where the algebra $(\mathcal{A} \oplus \mathcal{B})$ acts on $\mathcal{M}$ via the operations $(a \oplus b)m = am$ and $m(a \oplus b) = mb$. Let $\{\delta_k\}$ be a Jordan higher derivation on $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathcal{M}$. By Proposition 2.1, we can write

$$\delta_k(a \oplus b, m) = (J_k(a \oplus b) + T_k(m), K_k(a \oplus b) + S_k(m)), \quad ((a \oplus b) \in \mathcal{A} \oplus \mathcal{B}, m \in \mathcal{M}),$$

where $J_k : \mathcal{A} \oplus \mathcal{B} \longrightarrow \mathcal{A} \oplus \mathcal{B}$, $K_k : \mathcal{A} \oplus \mathcal{B} \to \mathcal{M}$, $T_k : \mathcal{M} \to \mathcal{A} \oplus \mathcal{B}$ and $S_k : \mathcal{M} \to \mathcal{M}$ are additive mappings satisfying (a) to (e) of Proposition 2.1. In order to prove that $\{\delta_k\}$ is a higher derivation, it is enough to show that conditions (1) to (5) of Proposition 2.1 are satisfied.

Fix $a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}$ and $k \in \mathbb{N}$. By induction on k we shall prove that $J_k(a \oplus b) = p_k(a) \oplus q_k(b)$, where $\{p_k\}$ and $\{q_k\}$ are Jordan higher derivations on $\mathcal{A}$ and $\mathcal{B}$, respectively. To this end, we write J_k in the form $J_k(a \oplus b) =$

$(p_k(a)+p'_k(b))\oplus(q_k(b)+q'_k(a))$ for some additive mappings $p_k:\mathcal{A}\longrightarrow\mathcal{A}$, $q_k:\mathcal{B}\to\mathcal{B}$, $p'_k:\mathcal{B}\to\mathcal{A}$ and $q'_k:\mathcal{A}\to\mathcal{B}$. We then trivially have $p_0(a)=a, q_0(b)=b, p'_0(b)=0, q'_0(a)=0$. Using the identity (1.2) for J_k at $(1,0)$ we conclude that $p'_k(1)=0$, $p_k(1)=0$. One more time applying (1.2) for J_k at $a\oplus 0$, shows that $\{p_k\}$ is a Jordan higher derivation on $\mathcal{A}$. Using the identity (1.3) for $J_k\big((a\oplus 0)(1\oplus 0)+(1\oplus 0)(a\oplus 0)\big)$ together with 2-faithfulness of $\mathcal{A}$ show that $q'_k(a)=0$ for each k. Similarly, $\{q_k\}$ is a Jordan higher derivation and $p'_k=0$ for each k.

Our next aim is to show that $T_k=0$. To this end, we may write $T_k:\mathcal{M}\longrightarrow\mathcal{A}\oplus\mathcal{B}$ in the form $T_k(m)=t_k(m)\oplus t'_k(m)$ for some additive mappings $t_k:\mathcal{M}\longrightarrow\mathcal{A}$, $t'_k:\mathcal{M}\longrightarrow\mathcal{B}$. Using Proposition 2.1(c) for T_k we have,

$$\begin{aligned}&t_k(m)\oplus t'_k(m)\\ &=T_k(m)=T_k(m(1\oplus 0)+(1\oplus 0)m)\\ &=\sum_{i=0}^{k}((t_i(m)\oplus t'_i(m))(p_{k-i}(1)\oplus 0)+(p_i(1)\oplus 0)(t_{k-i}(m)\oplus t'_{k-i}(m)))\\ &=2t_k(m)\oplus 0;\end{aligned}$$

and this implies that $T_k(m)=0$.

For $S_k:\mathcal{M}\longrightarrow\mathcal{M}$ we have

$$\begin{aligned}S_k(am)&=S_k(m(a\oplus 0)+(a\oplus 0)m)\\ &=\sum_{i=0}^{k}\big(S_i(m)(p_{k-i}(a)\oplus 0)+(p_i(a)\oplus 0)S_{k-i}(m)\big)\\ &=\sum_{i=0}^{k}p_i(a)S_{k-i}(m).\end{aligned}$$

Similarly $S_k(mb)=\sum_{i=0}^{k}S_i(m)q_{k-i}(b)$. In particular, S_k satisfies (4).

As $\mathcal{M}$ is faithful as an $\mathcal{A}\oplus\mathcal{B}$-module, similar to that in the proof of Theorem 2.2, $J_k(a\oplus b)=p_k(a)\oplus q_k(b)$ is a higher derivation on $\mathcal{A}\oplus\mathcal{B}$, which in turn implies that $\{p_k\}$ and $\{q_k\}$ are higher derivations on $\mathcal{A}$ and $\mathcal{B}$, respectively.

Next, we can write $K_k:\mathcal{A}\oplus\mathcal{B}\longrightarrow\mathcal{M}$ in the form $K_k(a\oplus b)=f_k(a)+g_k(b)$ for some additive mappings $f_k:\mathcal{A}\longrightarrow\mathcal{M}$ and $g_k:\mathcal{B}\longrightarrow\mathcal{M}$. Applying the identity of Proposition 2.1(b) at $1\oplus 1$, one can easily deduce that $f_k(1)=-g_k(1)$. Set $m_k=f_k(1)$ for each $k\in\mathbb{N}$. By using (2.3), we see that

$$\begin{aligned}2f_k(a)=K_k(2a\oplus 0)&=K_k\big((a\oplus 0)(1\oplus 0)\oplus(1\oplus 0)(a\oplus 0)\big)\\ &=\sum_{i=0}^{k}\big((p_i(a)\oplus 0)f_{k-i}(1)+(p_i(1)\oplus 0)f_{k-i}(a)\big)\\ &=\sum_{i=0}^{k}\big(p_i(a)m_{k-i}\big)+f_k(a).\end{aligned}$$

That is $f_k(a) = \sum_{i=0}^{k} p_i(a)m_{k-i}$. Similarly $g_k(b) = -\sum_{i=0}^{k} m_{k-i}q_i(b)$. We thus have $K_k(a,b) = \sum_{i=0}^{k}(p_i(a)m_{k-i} - m_{k-i}q_i(b))$. Now a direct verification, based on the fact that $\{p_k\}, \{q_k\}$ are higher derivations, reveals that K_k satisfies Proposition 2.1(2), as required. The proof is now complete. □

The case $k = 1$ of the characterization given in Theorem 3.3 shows that every (Jordan) derivation on the triangular algebra $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ has the presentation

$$\delta \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} p(a) & am_1 - m_1 b + S(m) \\ 0 & q(b) \end{pmatrix},$$

where $m_1 \in \mathcal{M}$ and $p : \mathcal{A} \to \mathcal{A},\ q : \mathcal{B} \to \mathcal{B}$ are derivations and $S : \mathcal{M} \to \mathcal{M}$ is an additive mapping satisfying $S(am) = aS(m)+p(a)m$ and $S(mb) = S(m)b+ mq(b)$; for all $a \in \mathcal{A}, b \in \mathcal{B}$ and $m \in \mathcal{M}$. In particular, δ is a derivation, (see [3] and also [15, Theorem 2.1]).

Furthermore, as a consequence of Theorem 3.3, one can show that the ordinary higher derivation $\{\frac{\delta^k}{k!}\}$ (induced by δ) on $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is in the form

$$\left(\frac{\delta^k}{k!}\right)\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} \left(\frac{p^k}{k!}\right)(a) & \sum_{i=1}^{k}\left(\left(\frac{p^{k-i}}{(k-i)!}\right)(a)m_i - m_i\left(\frac{q^{k-i}}{(k-i)!}\right)(b)\right) + \left(\frac{S^k}{k!}\right)(m) \\ 0 & \left(\frac{q^k}{k!}\right)(b) \end{pmatrix},$$

where $\{m_i\}$ is a sequence in $\mathcal{M}$ (see [9]).

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