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ON ϕ-SHARP RINGS

Ahmad Yousefian Darani and Mahdi Rahmatinia

Abstract. The purpose of this paper is to introduce some new class of rings that are closely related to the classes of sharp domains, pseudo-Dededkind domains, TV domains and finite character domains. A ring R is called a ϕ-sharp ring if whenever for nonnil ideals I, A, B of R with $I \supseteq AB$, then $I = A'B'$ for nonnil ideals A', B' of R where $A' \supseteq A$ and $B' \supseteq B$. We proof that a ϕ-Dedekind ring is a ϕ-sharp ring and we get some properties that by them a ϕ-sharp ring is a ϕ-Dedekind ring.

1. Introduction

We assume throughout this paper all rings are commutative with $1 \neq 0$ and all modules are unitary. An element x of an integral domain R is called primal if whenever $x \mid y_1y_2$, with $x, y_1, y_2 \in R$, then $x = z_1z_2$ where $z_1 \mid y_1$ and $z_2 \mid y_2$. Cohn in [11] introduced the concept of Schreier domains. An integral domain R is called a pre-Schreier domain if every nonzero element of R is primal. If in addition R is integrally closed, then R is called a Schreier domain. In [1], Z. Ahmad, T. Dumitrescu and M. Epure introduced the notion of sharp domains. A domain R is said to be a sharp domain if whenever $I \supseteq AB$ with I, A, B nonzero ideals of R, then there exist ideals $A' \supseteq A$ and $B' \supseteq B$ such that $I = A'B'$. Let R be a ring with identity and $Nil(R)$ be the set of nilpotent elements of R. Recall from [13] and [4], that a prime ideal P of R is called a divided prime ideal if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. Badawi in [3], [4], [5], [6], [7] and [8], investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring with $1 \neq 0$ and $Nil(R)$ is a divided prime ideal of $R\}$. Anderson and Badawi in [2] and [9] generalized the concept of Prüfer, Dedekind, Krull and Bezout domain to context of rings that are in the class $\mathcal{H}$. Lucas and Badawi in [10] generalized the concept of Mori domains to the context of rings that are in the class $\mathcal{H}$. Let R be a ring, $Z(R)$ the set of zero divisors of R and $S = R \setminus Z(R)$. Then $T(R) := S^{-1}R$ denoted the total quotient ring of R. We start by recalling some

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background material. A nonzero divisor of a ring R is called a regular element and an ideal of R is said to be regular if it contains a regular element. An ideal I of a ring R is said to be a nonnil ideal if $I \nsubseteq Nil(R)$. If I is a nonnil ideal of $R \in \mathcal{H}$, then $Nil(R) \subset I$. In particular, it holds if I is a regular ideal of a ring $R \in \mathcal{H}$. Recall from [2] that for a ring $R \in \mathcal{H}$, the map $\phi : T(R) \longrightarrow R_{Nil(R)}$ given by $\phi(a/b) = a/b$, for $a \in R$ and $b \in R \setminus Z(R)$, is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$ and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$ given by $\phi(x) = x/1$ for every $x \in R$.

For a nonzero ideal I of R let $I^{-1} = \{x \in T(R) : xI \subseteq R\}$. It is obvious that $II^{-1} \subseteq R$. An ideal I of R is called invertible, if $II^{-1} = R$. A ring R is called a Dedekind ring (res; Prüfer ring) in the sense of [15], if every regular ideal of R (res; every finitely generated regular ideal of R) is invertible. The ν-closure of I is the ideal $I_\nu = (I^{-1})^{-1}$ and I is called divisorial ideal (or ν-ideal) if $I_\nu = I$. A nonzero ideal I of R is called t-ideal if $I = I_t$ in which

$$I_t = \bigcup \{J_\nu \mid J \subseteq I \text{ is a nonzero finitely generated ideal of } R\}.$$

Let $R \in \mathcal{H}$. Then a nonnil ideal I of R is called ϕ-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. If every nonnil ideal of R is ϕ-invertible, then we say that R is a ϕ-Dedekind ring [9]. Also if every nonnil finitely generated ideal of R is ϕ-invertible, then we say that R is a ϕ-Prüfer ring [2]. A nonnil ideal I is ϕ-ν-ideal if $\phi(I)$ is a ν-ideal of $\phi(R)$ [10]. Observe that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}$, $Ker(\phi) \subset Nil(R)$, $Nil(T(R)) = Nil(R)$, $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{Nil(R)}$ is quasilocal with maximal ideal $Nil(\phi(R))$ and $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R))$. Therefore we have $x \in R \setminus Nil(R)$ if and only if $\phi(x) \in \phi(R) \setminus Z(\phi(R))$. Let $R \in H$. Then I is a finitely generated nonnil ideal of R if and only if $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$ [2, Lemma 2.1]. Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$ and let I be an ideal of R. Then I is an invertible ideal of R if and only if $I/Nil(R)$ is an invertible ideal of $R/Nil(R)$ [2, Lemma 2.3]. Let $R \in \mathcal{H}$ and let I be an ideal od R. Then I is a finitely generated nonnil ideal of R if and only if $I/Nil(R)$ is a finitely generated nonzero ideal of $R/Nil(R)$ [2, Lemma 2.4]. In [2, Lemma 2.5], it is shown that, if $R \in \mathcal{H}$ and P an ideal of R, then R/P is ring-isomorphic to $\phi(R)/\phi(P)$.

A ring R is called a sharp ring if whenever for regular ideals I, A, B of R with $I \supseteq AB$, then $I = A'B'$ for regular ideals A', B' of R where $A' \supseteq A$ and $B' \supseteq B$ [1]. In this paper, we define a ϕ-sharp ring. A ring R is called a ϕ-sharp ring if whenever for nonnil ideals I, A, B of R with $I \supseteq AB$, then $I = A'B'$ for nonnil ideals A', B' of R where $A' \supseteq A$ and $B' \supseteq B$. Let $R \in \mathcal{H}$. Then R is a ϕ-sharp ring if and only if $R/Nil(R)$ is a sharp domain (Theorem 2.2). In Theorem 2.3, we show that a ring $R \in \mathcal{H}$ is ϕ-sharp ring if and only if $\phi(R)$ is a sharp ring. In Corollary 2.7, we show that if a ring $R \in \mathcal{H}$ is ϕ-Dedekind ring, then R is a ϕ-sharp ring. Let $R \in \mathcal{H}$. If R is a ϕ-sharp ring, then R_P is a ϕ-sharp ring for every prime ideal P of R (Theorem 2.8). A ring R is called a

ϕ-pseudo-Dedekind ring if the ν-closure of each nonnil ideal of R is ϕ-invertible. Let $R \in \mathcal{H}$. Then R is a ϕ-pseudo-Dedekind ring if and only if $R/Nil(R)$ is a pseudo-Dedekind domain (Theorem 2.10). In Theorem 2.11, we proof that a ring $R \in \mathcal{H}$ is ϕ-pseudo-Dedekind ring if and only if $\phi(R)$ is a pseudo-Dedekind ring. In Corollary 2.15, we show that if a ring $R \in \mathcal{H}$ is ϕ-sharp ring, then R is a ϕ-pseudo-Dedekind ring. Let $R \in \mathcal{H}$ and let R is a ϕ-chained ring. Then R is a ϕ-sharp ring if and only if R is a ϕ-pseudo-Dedekind ring (Theorem 2.16). We say that an ideal I of R is a ϕ-t-ideal if $\phi(I)$ is a t-ideal of $\phi(R)$. A ring R is said to be a ϕ-TV ring in which every ϕ-t-ideal is a ϕ-ν-ideal. Let $R \in \mathcal{H}$. Then R is a ϕ-TV ring if and only if $R/Nil(R)$ is a TV domain (Theorem 2.20). Let $R \in \mathcal{H}$. Then R is a ϕ-TV ring if and only if $\phi(R)$ is a TV ring (Theorem 2.21). Let $R \in \mathcal{H}$. If R is a ϕ-sharp TV-ring, then R is a ϕ-Dedekind ring (Theorem 2.25). A ring R is said to be of ϕ-finite character if every nonnil element of R is contained in only finite many nonnil maximal ideals. In Theorem 2.27, we show that a ring $R \in \mathcal{H}$ is of ϕ-finite character if and only if $R/Nil(R)$ is of finite character. Let $R \in \mathcal{H}$. Then R is of ϕ-finite character if and only if $\phi(R)$ is of finite character (Theorem 2.28). In Corollary 2.31, we show that if a ring $R \in \mathcal{H}$ is a countable ϕ-pseudo-Dedekind Prüfer ring, then R is of ϕ-finite character. In Corollary 2.32, we show that if a ring $R \in \mathcal{H}$ is a countable ϕ-sharp ring, then R is a ϕ-Dedekind ring.

2. ϕ-sharp rings

Definition 2.1. A ring R is called a ϕ-sharp ring if whenever for nonnil ideals I, A, B of R with $I \supseteq AB$, then $I = A'B'$ for nonnil ideals A', B' of R where $A' \supseteq A$ and $B' \supseteq B$.

Theorem 2.2. *Let $R \in \mathcal{H}$. Then R is a ϕ-sharp ring if and only if $R/Nil(R)$ is a sharp domain.*

Proof. Let R be a ϕ-sharp ring and let $I/Nil(R), A/Nil(R), B/Nil(R)$ be nonzero ideals of $R/Nil(R)$ with $I/Nil(R) \supseteq (A/Nil(R))(B/Nil(R))$. Then, by [2, Lemma 2.4], I, A, B are nonnil ideals of R and $I \supseteq AB$. So $I = A'B'$ for nonnil ideals A', B' of R where $A' \supseteq A$ and $B' \supseteq B$. Therefore $I/Nil(R) = (A'/Nil(R))(B'/Nil(R))$ for nonzero ideals $A'/Nil(R)$ and $B'/Nil(R)$ of $R/Nil(R)$ by [2, Lemma 2.4], where $A'/Nil(R) \supseteq A/Nil(R)$ and $B'/Nil(R) \supseteq B/Nil(R)$. Thus $R/Nil(R)$ is a sharp domain. Conversely, let $R/Nil(R)$ be a sharp domain and I, A, B be nonnil ideals of R with $I \supseteq AB$. Then, by [2, Lemma 2.4], $I/Nil(R), A/Nil(R), B/Nil(R)$ are nonzero ideals of $R/Nil(R)$ and $I/Nil(R) \supseteq (A/Nil(R))(B/Nil(R))$. Therefore $I/Nil(R) = (A'/Nil(R))(B'/Nil(R))$ for nonzero ideals $A'/Nil(R)$ and $B'/Nil(R)$ of $R/Nil(R)$, where $A'/Nil(R) \supseteq A/Nil(R)$ and $B'/Nil(R) \supseteq B/Nil(R)$. So $I = A'B'$ for nonnil ideals A', B' of R by [2, Lemma 2.4], where $A' \supseteq A$ and $B' \supseteq B$. Thus R is a ϕ-sharp ring. □

Theorem 2.3. *Let $R \in \mathcal{H}$. Then R is a ϕ-sharp ring if and only if $\phi(R)$ is a sharp ring.*

Proof. Let R be a ϕ-sharp ring. Then, by Theorem 2.2, $R/Nil(R)$ is a sharp domain and so by [2, Lemma 2.5], $\phi(R)/Nil(\phi(R))$ is a sharp domain. Let $\phi(I) \supseteq \phi(A)\phi(B)$ for nonnil ideals I, A, B of R. Then we have

$$\phi(I)/Nil(\phi(R)) \supseteq (\phi(A)/Nil(\phi(R)))(\phi(B)/Nil(\phi(R)))$$

for nonzero ideals $\phi(I)/Nil(\phi(R))$, $\phi(A)/Nil(\phi(R))$, $\phi(B)/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$. Therefore

$$\phi(I)/Nil(\phi(R)) = (\phi(A')/Nil(\phi(R)))(\phi(B')/Nil(\phi(R)))$$

for nonzero ideals $\phi(A')/Nil(\phi(R))$ and $\phi(B')/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ where

$$\phi(A')/Nil(\phi(R)) \supseteq \phi(A)/Nil(\phi(R)) \text{ and}$$
$$\phi(B')/Nil(\phi(R)) \supseteq \phi(B)/Nil(\phi(R)).$$

So $\phi(I) = \phi(A')\phi(B')$ for nonnil ideals A', B' of R where $\phi(A') \supseteq \phi(A)$ and $\phi(B') \supseteq \phi(B)$. Therefore $\phi(R)$ is a sharp ring. Conversely, let $\phi(R)$ be a sharp ring and $I \supseteq AB$ for nonnil ideals I, A, B of R. Then $\phi(I) \supseteq \phi(A)\phi(B)$ for regular ideals $\phi(I)$, $\phi(A)$, $\phi(B)$ of $\phi(R)$, by [2, Lemma 2.1]. So $\phi(I)/Nil(\phi(R)) \supseteq (\phi(A)/Nil(\phi(R)))(\phi(B)/Nil(\phi(R)))$ for nonzero ideals $\phi(I)/Nil(\phi(R))$, $\phi(A)/Nil(\phi(R))$, $\phi(B)/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$. Since $\phi(I) = \phi(A')\phi(B')$ for regular ideals $\phi(A')$, $\phi(B')$ of $\phi(R)$ where $\phi(A') \supseteq \phi(A)$ and $\phi(B') \supseteq \phi(B)$, so $\phi(I)/Nil(\phi(R)) = (\phi(A')/Nil(\phi(R)))(\phi(B')/Nil(\phi(R)))$ for nonzero ideals $\phi(A')/Nil(\phi(R))$ and $\phi(B')/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$, where

$$\phi(A')/Nil(\phi(R)) \supseteq \phi(A)/Nil(\phi(R)) \text{ and}$$
$$\phi(B')/Nil(\phi(R)) \supseteq \phi(B)/Nil(\phi(R)).$$

Thus $\phi(R)/Nil(\phi(R))$ is a sharp domain and so by [2, Lemma 2.5], $R/Nil(R)$ is a sharp domain. Therefore, by Theorem 2.2, R is a ϕ-sharp ring. □

Corollary 2.4. *Let $R \in \mathcal{H}$. The following are equivalent*:

(1) *R is a ϕ-sharp ring*;
(2) *$\phi(R)$ is a sharp ring*;
(3) *$R/Nil(R)$ is a sharp domain*;
(4) *$\phi(R)/Nil(\phi(R))$ is a sharp domain.*

Theorem 2.5. *Let $R \in \mathcal{H}$. If R is a ϕ-sharp, then R is a sharp ring.*

Proof. Suppose that R is a ϕ-sharp ring. Then, by Theorem 2.2, $R/Nil(R)$ is a sharp domain. Let I, A, B be regular ideals of R with $I \supseteq AB$. Then $I/Nil(R), A/Nil(R), B/Nil(R)$ are nonzero ideals of $R/Nil(R)$ and $I/Nil(R) \supseteq (A/Nil(R))(B/Nil(R))$. So $I/Nil(R) = (A'/Nil(R))(B'/Nil(R))$ for nonzero ideals $A'/Nil(R), B'/Nil(R)$ of $R/Nil(R)$ where $A'/Nil(R) \supseteq A/Nil(R)$ and

$B'/Nil(R) \supseteq B/Nil(R)$. Therefore $I = A'B'$ for regular ideals A', B' of R where $A' \supseteq A$ and $B' \supseteq B$. Thus R is a sharp ring. □

Theorem 2.6. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-sharp ring if and only if R is a sharp ring.*

Proof. Observe that in this case, $\phi(R) = R$. □

Corollary 2.7. *Let $R \in \mathcal{H}$. If R is a ϕ-Dedekind ring, then R is a ϕ-sharp ring.*

Proof. Suppose that R is a ϕ-Dedekind ring. Then, by [9, Theorem 2.5], $R/Nil(R)$ is a Dedekind domain. So, by [1, Corollary 3], $R/Nil(R)$ is a sharp domain. Therefore, by Theorem 2.2, R is a ϕ-sharp ring. □

Theorem 2.8. *Let $R \in \mathcal{H}$. If R is a ϕ-sharp ring, then R_P is a ϕ-sharp ring for every prime ideal P of R.*

Proof. Set $D = R/Nil(R)$. Since D is a sharp domain, by Theorem 2.2, we conclude that $D_P/Nil(R)$ is a sharp domain for each prime ideal P of R, by [1, Proposition 7]. Since $D_P/Nil(R)$ is ring-homomorphic to $R_P/Nil(R)R_P = R_P/Nil(R_P)$ and $R_P \in \mathcal{H}$, then by Theorem 2.2, we conclude that R_P is a ϕ-sharp ring. □

Definition 2.9. A ring R is called a ϕ-pseudo-Dedekind ring if the ν-closure of each nonnil ideal of R is ϕ-invertible.

Theorem 2.10. *Let $R \in \mathcal{H}$. Then R is a ϕ-pseudo-Dedekind ring if and only if $R/Nil(R)$ is a pseudo-Dedekind domain.*

Proof. Let R be a ϕ-pseudo-Dedekind ring and $I/Nil(R)$ a nonzero ideal of $R/Nil(R)$. Then, by [2, Lemma 2.4], I is a nonnil ideal of R. So I_ν is a ϕ-invertible ideal of R. Therefore, by [10, Lemma 2.4], $(I/Nil(R))_\nu = I_\nu/Nil(R)$ is an invertible ideal of $R/Nil(R)$. Thus $R/Nil(R)$ is a pseudo-Dedekind domain. Conversely, let $R/Nil(R)$ be a pseudo-Dedekind domain and I a nonnil ideal of R. Then, by [2, Lemma 2.4], $I/Nil(R)$ is a nonzero ideal of $R/Nil(R)$. So $(I/Nil(R))_\nu = I_\nu/Nil(R)$ is an invertible ideal of $R/Nil(R)$. Therefore, by [10, Lemma 2.4], I_ν is a ϕ-invertible ideal of R. Thus R is a ϕ-pseudo-Dedekind ring. □

Note that $\phi(I_\nu) = \phi((I^{-1})^{-1}) = (\phi(I)^{-1})^{-1} = \phi(I)_\nu$.

Theorem 2.11. *Let $R \in \mathcal{H}$. Then R is a ϕ-pseudo-Dedekind ring if and only if $\phi(R)$ is a pseudo-Dedekind ring.*

Proof. Let R be a ϕ-pseudo-Dedekind ring and $\phi(I)$ a regular ideal of $\phi(R)$. Then, by [2, Lemma 2.1], I is a nonnil ideal of R. So I_ν is a ϕ-invertible ideal of R. Therefore $\phi(I_\nu) = \phi(I)_\nu$ is an invertible ideal of $\phi(R)$. Thus $\phi(R)$ is a pseudo-Dedekind ring. Conversely, let $\phi(R)$ be a pseudo-Dedekind ring and I a nonnil ideal of R. Then, by [2, Lemma 2.1], $\phi(I)$ is a regular ideal of R. So

$\phi(I_\nu)$ is an invertible ideal of $\phi(R)$. Therefore I_ν is a ϕ-invertible ideal of R. Thus R is a ϕ-pseudo-Dedekind ring. □

Corollary 2.12. *Let $R \in \mathcal{H}$. The following are equivalent*:

(1) *R is a ϕ-pseudo-Dedekind ring*;
(2) *$\phi(R)$ is a pseudo-Dedekind ring*;
(3) *$R/Nil(R)$ is a pseudo-Dedekind domain*;
(4) *$\phi(R)/Nil(\phi(R))$ is a pseudo-Dedekind domain.*

Theorem 2.13. *Let $R \in \mathcal{H}$. If R is a ϕ-pseudo-Dedekind ring, then R is a pseudo-Dedekind ring.*

Proof. Let R be a ϕ-pseudo-Dedekind ring. Then, by Theorem 2.10, $R/Nil(R)$ is a pseudo-Dedekind domain. Let I be a regular ideal of R. Then $I/Nil(R)$ is a nonzero ideal of $R/Nil(R)$. So $(I/Nil(R))_\nu = I_\nu/Nil(R)$ is an invertible ideal of $R/Nil(R)$. Therefore, I_ν is an invertible ideal of R. Thus R is a pseudo-Dedekind ring. □

Theorem 2.14. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-pseudo-Dedekind ring if and only if R is a pseudo-Dedekind ring.*

Proof. In this case, $\phi(R) = R$. □

Corollary 2.15. *Let $R \in \mathcal{H}$. If R is a ϕ-sharp ring, then R is a ϕ-pseudo-Dedekind ring.*

Proof. Suppose that R is a ϕ-sharp ring. Then, by Theorem 2.2, $R/Nil(R)$ is a sharp domain. So, by [1, Proposition 4], $R/Nil(R)$ is a pseudo-Dedekind domain. Therefore, by Theorem 2.10, R is a ϕ-pseudo-Dedekind ring. □

Theorem 2.16. *Let $R \in \mathcal{H}$ and let R be a ϕ-chained ring. Then R is a ϕ-sharp ring if and only if R is a ϕ-pseudo-Dedekind ring.*

Proof. Let R be a ϕ-chained ring, then by [2, Theorem 2.7], $R/Nil(R)$ is a valuation domain. So, by [1, Proposition 6], $R/Nil(R)$ is a sharp domain if and only if $R/Nil(R)$ is a pseudo-Dedekind domain. Therefore, by Theorem 2.2 and Theorem 2.10, R is a ϕ-sharp ring if and only if R is a ϕ-pseudo-Dedekind ring. □

Recall that an ideal I of R is a ϕ-ν-ideal (or ϕ-divisorial ideal) if $\phi(I)$ is a ν-ideal (divisorial ideal) of $\phi(R)$, [10].

Definition 2.17. An ideal I of R is called ϕ-t-ideal if $\phi(I)$ is a t-ideal of $\phi(R)$.

In [12], it is shown that $(I/Nil(R))_\nu = I_\nu/Nil(R)$. Also it is shown that, for each nonnil ideal I of $R \in \mathcal{H}$, I is ϕ-*nu*-ideal if and only if $I/Nil(R)$ is a ν ideal of $R/Nil(R)$, [10, Lemma 2.4].

Lemma 2.18. *Let $R \in \mathcal{H}$ and I an ideal of R. Then I is a ϕ-t-ideal if and only if $I/Nil(R)$ is a t-ideal of $R/Nil(R)$.*

Proof. It is clear by [10, Lemma 2.4]. □

Definition 2.19. A ring R is called a ϕ-TV ring in which every ϕ-t-ideal is a ϕ-ν-ideal.

Theorem 2.20. *Let $R \in \mathcal{H}$. Then R is a ϕ-TV ring if and only if $R/Nil(R)$ is a TV domain.*

Proof. Let R be a ϕ-TV ring and let $I/Nil(R)$ be a t-ideal of $R/Nil(R)$. Then, by Lemma 2.18, I is a ϕ-t-ideal of R. So I is a ϕ-ν-ideal of R. Thus, by [10, Lemma 2.4], $I/Nil(R)$ is a ν-ideal of $R/Nil(R)$. Therefore, $R/Nil(R)$ is a TV-domain. Conversely, let $R/Nil(R)$ be a TV domain and I be a ϕ-t-ideal of R. Then, by Lemma 2.18, $I/Nil(R)$ is a t-ideal of $R/Nil(R)$. So $I/Nil(R)$ is a ν-ideal of $R/Nil(R)$. Thus, by [10, Lemma 2.4], I is a ϕ-ν-ideal of R. Therefore, R is a ϕ-TV ring. □

Theorem 2.21. *Let $R \in \mathcal{H}$. Then R is a ϕ-TV ring if and only if $\phi(R)$ is a TV ring.*

Proof. Let R be a ϕ-TV ring and let $\phi(I)$ be a t-ideal of $\phi(R)$. Then I is a ϕ-t-ideal and so I is a ϕ-ν-ideal of R. Therefore, $\phi(I)$ is a ν-ideal of R. Thus $\phi(R)$ is a TV-ring. Conversely, let $\phi(R)$ be a TV-ring and I be a ϕ-t-ideal of R. Then $\phi(I)$ is a t-ideal and so is a ν-ideal of $\phi(R)$. Therefore, I is a ϕ-ν-ideal of R. Thus R is a ϕ-TV ring. □

Corollary 2.22. *Let $R \in \mathcal{H}$. The following are equivalent*:

(1) *R is a ϕ-TV ring*;
(2) *$\phi(R)$ is a TV ring*;
(3) *$R/Nil(R)$ is a TV domain*;
(4) *$\phi(R)/Nil(\phi(R))$ is a TV domain.*

Theorem 2.23. *Let $R \in \mathcal{H}$. If R is a ϕ-TV ring, then R is a TV ring.*

Proof. Let R be a ϕ-TV ring. Then, by Theorem 2.20, $R/Nil(R)$ is a TV domain. If I is a regular t-ideal of R, then $I/Nil(R)$ is a nonzero t-ideal of $R/Nil(R)$. So, $I/Nil(R)$ is a ν-ideal of $R/Nil(R)$. Therefore, by [12, Lemma 2.10], I is a ν-ideal of R. Thus R is a TV ring. □

Theorem 2.24. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-TV ring if and only if R is a TV ring.*

Proof. In this case, $\phi(R) = R$. □

Note that a ϕ-sharp TV-ring is a ϕ-sharp ring and is also a ϕ-TV-ring.

Theorem 2.25. *Let $R \in \mathcal{H}$. If R is a ϕ-sharp TV-ring, then R is a ϕ-Dedekind ring.*

Proof. Let R be a ϕ-sharp TV-ring. Then, by Theorem 2.2 and Theorem 2.20, $R/Nil(R)$ is a sharp TV domain. So, by [1, Corollary 12], $R/Nil(R)$ is a Dedekind domain. Therefore, by [9, Theorem 2.5], R is a ϕ-Dedekind ring. □

Definition 2.26. A ring R is said to be of ϕ-finite character if every nonnil element of R is contained in only finite many nonnil maximal ideals.

Theorem 2.27. *Let $R \in \mathcal{H}$. Then R is of ϕ-finite character if and only if $R/Nil(R)$ is of finite character.*

Proof. Let R be of ϕ-finite character and $x + Nil(R)$ an element of $R/Nil(R)$. Then x is a nonnil element of R. So, there exist only finite many nonnil maximal ideals, say $M_1, \ldots, M_n$, such that x is contained in M_i for $i = 1, \ldots, n$. Therefore, there exist only finitely many maximal ideals, namely $M_1/Nil(R), \ldots, M_n/Nil(R)$, such that $x + Nil(R)$ is contained in $M_i/Nil(R)$ for $i = 1, \ldots, n$. Thus $R/Nil(R)$ is of finite character. Conversely, let $R/Nil(R)$ be of finite character and x an element of R. Then $x + Nil(R)$ is an element of $R/Nil(R)$. So, there exist only finitely many maximal ideals, say $M_1/Nil(R), \ldots, M_n/Nil(R)$, such that $x + Nil(R)$ is contained in $M_i/Nil(R)$ for $i = 1, \ldots, n$. Therefore, there exist only finite many nonnil maximal ideals, namely $M_1, \ldots, M_n$, such that x is contained in M_i for $i = 1, \ldots, n$. Thus R is of ϕ-finite character. □

Theorem 2.28. *Let $R \in \mathcal{H}$. Then R is of ϕ-finite character if and only if $\phi(R)$ is of finite character.*

Proof. Let R be of ϕ-finite character and $\phi(x)$ a regular element of $\phi(R)$. Then x is a nonnil element of R. So, there exist only finitely many nonnil maximal ideals $M_1, \ldots, M_n$ of R such that x is contained in M_i for $i = 1, \ldots, n$. Therefore, there exist only finitely many regular maximal ideal $\phi(M_1), \ldots, \phi(M_n)$ of $\phi(R)$ such that $\phi(x)$ is contained in $\phi(M_i)$ for $i = 1, \ldots, n$. Thus $\phi(R)$ is of finite character. Conversely, let $\phi(R)$ be of finite character and x a nonnil element of R. Then $\phi(x)$ is a regular element of $\phi(R)$. So, there exist only finitely many regular maximal ideal $\phi(M_1), \ldots, \phi(M_n)$ of $\phi(R)$ such that $\phi(x)$ is contained in $\phi(M_i)$ for $i = 1, \ldots, n$. Therefore, there exist only finitely many nonnil maximal ideals $M_1, \ldots, M_n$ of R such that x is contained in M_i for $i = 1, \ldots, n$. Thus R is of ϕ-finite character. □

Corollary 2.29. *Let $R \in \mathcal{H}$. The following are equivalent*:

(1) *R is of ϕ-finite character*;

(2) *$\phi(R)$ is of finite character*;

(3) *$R/Nil(R)$ is of finite character*;

(4) *$\phi(R)/Nil(\phi(R))$ is of finite character.*

Theorem 2.30. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is of ϕ-finite character if and only if R is of finite character.*

Proof. It is obvious. □

Note that a ϕ-pseudo-Dedekind Prüfer ring is a ring such that is a ϕ-pseudo-Dedekind ring and is also a ϕ-Prüfer ring.

Corollary 2.31. *Let $R \in \mathcal{H}$. If R is a countable ϕ-pseudo-Dedekind Prüfer ring, then R is of ϕ-finite character.*

Proof. Suppose that R is a countable ϕ-pseudo-Dedekind ring. Then, by Theorem 2.10 and [2, Theorem 2.6], $R/Nil(R)$ is a countable pseudo-Dedekind Prüfer domain. So, by [1, Proposition 16], $R/Nil(R)$ is of finite character. Therefore, by Theorem 2.27, R is of ϕ-finite character. □

Corollary 2.32. *Let $R \in \mathcal{H}$. If R is a countable ϕ-sharp ring, then R is a ϕ-Dedekind ring.*

Proof. Suppose that R is a countable ϕ-sharp ring. Then, by Theorem 2.2, $R/Nil(R)$ is a countable sharp domain. So, by [1, Corollary 17], $R/Nil(R)$ is a dedekind domain. Therefore, by [9, Theorem 2.5], R is a ϕ-Dedekind ring. □

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Ahmad Yousefian Darani
Department of Mathematics and Applications
University of Mohaghegh Ardabili
P. O. Box 179, Ardabil, Iran
E-mail address: yousefian@uma.ac.ir, youseffian@gmail.com

Mahdi Rahmatinia
Department of Mathematics and Applications
University of Mohaghegh Ardabili
P. O. Box 179, Ardabil, Iran
E-mail address: m.rahmati@uma.ac.ir, mahdi.rahmatinia@gmail.com