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REGULARITY OF GENERALIZED DERIVATIONS IN BCI-ALGEBRAS

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Abstract. In this paper we study the regularity of inside (or outside) (θ, ϕ)-derivations in BCI-algebras X and prove that let $d_{(\theta,\phi)} : X \to X$ be an inside (θ, ϕ)-derivation of X. If there exists $a \in X$ such that $d_{(\theta,\phi)}(x) * \theta(a) = 0$, then $d_{(\theta,\phi)}$ is regular for all $x \in X$. It is also shown that if X is a BCK-algebra, then every inside (or outside) (θ, ϕ)-derivation of X is regular. Furthermore the concepts of θ-ideal, ϕ-ideal and invariant inside (or outside) (θ, ϕ)-derivations of X are introduced and their related properties are investigated. Finally we obtain the following result: If $d_{(\theta,\phi)} : X \to X$ is an outside (θ, ϕ)-derivation of X, then $d_{(\theta,\phi)}$ is regular if and only if every θ-ideal of X is $d_{(\theta,\phi)}$-invariant.

1. Introduction

Throughout the present paper X will denote a BCI-algebra unless otherwise mentioned. Jun and Xin [4] defined the notion of derivation on BCI-algebras as follows: A self map $d : X \to X$ is called a left-right derivation (briefly an (l, r)-derivation) of X if $d(x * y) = d(x) * y \wedge x * d(y)$ holds for all $x, y \in X$. Similarly, a self map $d : X \to X$ is called a right-left derivation (briefly an (r, l)-derivation) of X if $d(x*y) = x*d(y) \wedge d(x)*y$ holds for all $x, y \in X$. Moreover, if d is both (l, r)- and (r, l)-derivations, it is a derivation on X. Following [11], a self map $d_f : X \to X$ is said to be a left-right f-derivation or an (l, r)-f-derivation of X if it satisfies the identity $d_f(x * y) = d_f(x) * f(y) \wedge f(x) * d_f(y)$ for all $x, y \in X$. Similarly, a self map $d_f : X \to X$ is said to be a right-left f-derivation or an (r, l)-f-derivation of X if it satisfies the identity $d_f(x * y) = f(x)*d_f(y) \wedge d_f(x)*f(y)$ for all $x, y \in X$. Moreover, if d_f is both (l, r) and (r, l)-f-derivations, it is said that d_f is an f-derivation, where f is an endomorphism. Over the past decade, a number of research papers have been devoted to the study of various kinds of derivations in BCI-algebras (see for example, [2, 4, 7, 8, 9, 10, 11] where further references can be found).

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The purpose of this paper is to study the regularity of inside (or outside) (θ,ϕ)-derivations in BCI-algebras X and their useful properties. We prove that let $d_{(\theta,\phi)}: X \to X$ be an inside (θ,ϕ)-derivation of X and if there exists $a \in X$ such that $d_{(\theta,\phi)}(x) * \theta(a) = 0$, then $d_{(\theta,\phi)}$ is regular for all $x \in X$. It is also shown that if X is a BCK-algebra, then every inside (or outside) (θ,ϕ)-derivation of X is regular. Furthermore we introduce the concepts of θ-ideal, ϕ-ideal and invariant inside (or outside) (θ,ϕ)-derivations of X and investigated their related properties. We also prove that if $d_{(\theta,\phi)}: X \to X$ is an outside (θ,ϕ)-derivation of X, then $d_{(\theta,\phi)}$ is regular if and only if every θ-ideal of X is $d_{(\theta,\phi)}$-invariant.

2. Preliminaries

A nonempty set X with a constant 0 and a binary operation $*$ is called a *BCI-algebra* if for all $x, y, z \in X$ the following conditions hold:

(I) $((x*y)*(x*z))*(z*y) = 0$.
(II) $(x*(x*y))*y = 0$.
(III) $x*x = 0$.
(IV) $x*y = 0$ and $y*x = 0$ imply $x = y$.

A BCI-algebra X has the following properties: For all $x, y, z \in X$

(a1) $x*0 = x$.
(a2) $(x*y)*z = (x*z)*y$.
(a3) $x \leq y$ implies $x*z \leq y*z$ and $z*y \leq z*x$.
(a4) $(x*z)*(y*z) \leq x*y$.
(a5) $x*(x*(x*y)) = x*y$.
(a6) $0*(x*y) = (0*x)*(0*y)$.
(a7) $x*0 = 0$ implies $x = 0$.

For a BCI-algebra X, denote the BCK-part (resp. the BCI-G part) of X by X_+ (resp. $G(X)$), i.e., X_+ is the set of all $x \in X$ such that $0 \leq x$ (resp. $G(X) := \{x \in X \mid 0*x = x\}$). Note that $G(X) \cap X_+ = \{0\}$ (see [3]). If $X_+ = \{0\}$, then X is called a *p-semisimple BCI-algebra.* In a p-semisimple BCI-algebra X, the following hold: For all $x, y, z, a, b \in X$

(a8) $(x*z)*(y*z) = x*y$.
(a9) $0*(0*x) = x$.
(a10) $x*(0*y) = y*(0*x)$.
(a11) $x*y = 0$ implies $x = y$.
(a12) $x*a = x*b$ implies $a = b$.
(a13) $a*x = b*x$ implies $a = b$.
(a14) $a*(a*x) = x$.

Let X be a p-semisimple BCI-algebra. We define addition "+" as $x + y = x*(0*y)$ for all $x, y \in X$. Then $(X, +)$ is an abelian group with identity 0 and $x - y = x*y$. Conversely let $(X, +)$ be an abelian group with identity 0 and let $x*y=x-y$. Then X is a p-semisimple BCI-algebra and $x+y=x*(0*y)$ for all $x,y\in X$ (see [6]).

For a BCI-algebra X we denote $x\wedge y=y*(y*x)$, in particular $0*(0*x)=a_x$, and $L_p(X):=\{a\in X \mid x*a=0 \;\Rightarrow\; x=a, \forall x\in X\}$. We call the elements of $L_p(X)$ the *p-atoms* of X. For any $a\in X$, let $V(a):=\{x\in X \mid a*x=0\}$, which is called the *branch* of X with respect to a. It follows that $x*y\in V(a*b)$ whenever $x\in V(a)$ and $y\in V(a)$ for all $x,y\in X$ and all $a,b\in L_p(X)$. Note that $L_p(X)=\{x\in X \mid a_x=x\}$, which is the p-semisimple part of X, and X is a p-semisimple BCI-algebra if and only if $L_p(X)=X$ (see [5, Proposition 3.2]). Note also that $a_x\in L_p(X)$, i.e., $0*(0*a_x)=a_x$, which implies that $a_x*y\in L_p(X)$ for all $y\in X$. It is clear that $G(X)\subset L_p(X)$, and $x*(x*a)=a$ and $a*x\in L_p(X)$ for all $a\in L_p(X)$ and all $x\in X$. For more details, refer to [1, 3, 5, 6].

3. Regularity of generalized derivations

To develop our main results we recall the following:

Definition 3.1 ([10])**.** Let θ and ϕ be two endomorphisms of X. A self map $d_{(\theta,\phi)}:X\to X$ is called

(1) an *inside* (θ,ϕ)*-derivation* of X if it satisfies:

$$(\forall x,y\in X)\left(d_{(\theta,\phi)}(x*y)=\left(d_{(\theta,\phi)}(x)*\theta(y)\right)\wedge\left(\phi(x)*d_{(\theta,\phi)}(y)\right)\right), \tag{3.1}$$

(2) an *outside* (θ,ϕ)*-derivation* of X if it satisfies:

$$(\forall x,y\in X)\left(d_{(\theta,\phi)}(x*y)=\left(\theta(x)*d_{(\theta,\phi)}(y)\right)\wedge\left(d_{(\theta,\phi)}(x)*\phi(y)\right)\right), \tag{3.2}$$

(3) a (θ,ϕ)*-derivation* of X if it is both an inside (θ,ϕ)-derivation and an outside (θ,ϕ)-derivation.

Example 3.2 ([10])**.** Consider a BCI-algebra $X=\{0,a,b\}$ with the following Cayley table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Define a map

$$d_{(\theta,\phi)}:X\to X,\; x\mapsto\begin{cases} b & \text{if } x\in\{0,a\},\\ 0 & \text{if } x=b,\end{cases}$$

and define two endomorphisms

$$\theta:X\to X,\; x\mapsto\begin{cases} 0 & \text{if } x\in\{0,a\},\\ b & \text{if } x=b,\end{cases}$$

and $\phi:X\to X$ such that $\phi(x)=x$ for all $x\in X$.

It is routine to verify that $d_{(\theta,\phi)}$ is both an inside (θ,ϕ)-derivation and an outside (θ,ϕ)-derivation of X.

Lemma 3.3 ([10]). *For any outside (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a BCI-algebra X, the following are equivalent*:

(1) $(\forall x \in X)\left(d_{(\theta,\phi)}(x) = \theta(x) \wedge d_{(\theta,\phi)}(x)\right)$.
(2) $d_{(\theta,\phi)}(0) = 0$.

Definition 3.4. Let $d_{(\theta,\phi)} : X \to X$ be an inside (or outside) (θ,ϕ)-derivation of a BCK/BCI-algebra X. Then $d_{(\theta,\phi)}$ is said to be *regular* if $d_{(\theta,\phi)}(0) = 0$.

Example 3.5. The inside (or outside) (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of X in Example 3.2 is not regular.

Proposition 3.6. *Let $d_{(\theta,\phi)}$ be a regular outside (θ,ϕ)-derivation of a BCI-algebra X. Then*

(1) *Both $\theta(x)$ and $d_{(\theta,\phi)}(x)$ belong to the same branch for all $x \in X$.*
(2) $(\forall x \in X)\left(d_{(\theta,\phi)}(x) \leq \theta(x)\right)$.
(3) $(\forall x, y \in X)\left(d_{(\theta,\phi)}(x) * \theta(y) \leq \theta(x) * d_{(\theta,\phi)}(y)\right)$.

Proof. (1) For any $x \in X$, we get

$$\begin{aligned}
0 &= d_{(\theta,\phi)}(0) = d_{(\theta,\phi)}(a_x * x) \\
&= \left(\theta(a_x) * d_{(\theta,\phi)}(x)\right) \wedge \left(d_{(\theta,\phi)}(a_x) * \phi(x)\right) \\
&= \left(d_{(\theta,\phi)}(a_x) * \phi(x)\right) * \left(\left(d_{(\theta,\phi)}(a_x) * \phi(x)\right) * \left(\theta(a_x) * d_{(\theta,\phi)}(x)\right)\right) \\
&= \theta(a_x) * d_{(\theta,\phi)}(x)
\end{aligned}$$

since $\theta(a_x) * d_{(\theta,\phi)}(x) \in L_p(X)$. Hence $\theta(a_x) \leq d_{(\theta,\phi)}(x)$, and so $d_{(\theta,\phi)}(x) \in V(\theta(a_x))$. Obviously, $\theta(x) \in V(\theta(a_x))$.

(2) Since $d_{(\theta,\phi)}$ is regular, $d_{(\theta,\phi)}(0) = 0$. It follows from Lemma 3.3 that

$$d_{(\theta,\phi)}(x) = \theta(x) \wedge d_{(\theta,\phi)}(x) \leq \theta(x).$$

(3) Since $d_{(\theta,\phi)}(x) \leq \theta(x)$ for all $x \in X$, we have

$$d_{(\theta,\phi)}(x) * \theta(y) \leq \theta(x) * \theta(y) \leq \theta(x) * d_{(\theta,\phi)}(y)$$

by (a3). □

If we take $\theta = \phi = f$ in Proposition 3.6, then we have the following corollary.

Corollary 3.7 ([11]). *If d_f is a regular (r,l)-f-derivation of a BCI-algebra X, then both $f(x)$ and $d_f(x)$ belong to the same branch for all $x \in X$.*

Now we provide conditions for an inside (or outside) (θ,ϕ)-derivation to be regular.

Theorem 3.8. *Let $d_{(\theta,\phi)}$ be an inside (θ,ϕ)-derivation of a BCI-algebra X. If there exists $a \in X$ such that $d_{(\theta,\phi)}(x) * \theta(a) = 0$ for all $x \in X$, then $d_{(\theta,\phi)}$ is regular.*

Proof. Assume that there exists $a \in X$ such that $d_{(\theta,\phi)}(x) * \theta(a) = 0$ for all $x \in X$. Then

$$\begin{aligned} 0 &= d_{(\theta,\phi)}(x * a) * a = \big(\big(d_{(\theta,\phi)}(x) * \theta(a)\big) \wedge \big(\phi(x) * d_{(\theta,\phi)}(a)\big)\big) * a \\ &= \big(0 \wedge \big(\phi(x) * d_{(\theta,\phi)}(a)\big)\big) * a = 0 * a, \end{aligned}$$

and so $d_{(\theta,\phi)}(0) = d_{(\theta,\phi)}(0 * a) = \big(d_{(\theta,\phi)}(0) * \theta(a)\big) \wedge \big(\phi(0) * d_{(\theta,\phi)}(a)\big) = 0$. Hence $d_{(\theta,\phi)}$ is regular. □

Theorem 3.9. *If X is a BCK-algebra, then every inside* (*or outside*) (θ, ϕ)-*derivation of X is regular.*

Proof. Let $d_{(\theta,\phi)}$ be an inside (θ, ϕ)-derivation of a BCK-algebra X. Then

$$\begin{aligned} d_{(\theta,\phi)}(0) &= d_{(\theta,\phi)}(0 * x) \\ &= \big(d_{(\theta,\phi)}(0) * \theta(x)\big) \wedge \big(\phi(0) * d_{(\theta,\phi)}(x)\big) \\ &= \big(d_{(\theta,\phi)}(0) * \theta(x)\big) \wedge 0 = 0. \end{aligned}$$

If $d_{(\theta,\phi)}$ is an outside (θ, ϕ)-derivation of a BCK-algebra X, then

$$\begin{aligned} d_{(\theta,\phi)}(0) &= d_{(\theta,\phi)}(0 * x) \\ &= \big(\theta(0) * d_{(\theta,\phi)}(x)\big) \wedge \big(d_{(\theta,\phi)}(0) * \phi(x)\big) \\ &= 0 \wedge \big(d_{(\theta,\phi)}(0) * \phi(x)\big) = 0. \end{aligned}$$

Hence $d_{(\theta,\phi)}$ is regular. □

To prove our next results, we define the following notions:

Definition 3.10. For an inside (or outside) (θ, ϕ)-derivation $d_{(\theta,\phi)}$ of a BCK/BCI-algebra X, we say that an ideal A of X is a *θ-ideal* (resp. *ϕ-ideal*) if $\theta(A) \subseteq A$ (resp. $\phi(A) \subseteq A$).

Definition 3.11. For an inside (or outside) (θ, ϕ)-derivation $d_{(\theta,\phi)}$ of a BCK/BCI-algebra X, we say that an ideal A of X is *$d_{(\theta,\phi)}$-invariant* if $d_{(\theta,\phi)}(A) \subseteq A$.

Example 3.12. Let $d_{(\theta,\phi)}$ be an outside (θ, ϕ)-derivation of X which is described in Example 3.2. We know that $A := \{0, a\}$ is both a θ-ideal and a ϕ-ideal of X. But $A := \{0, a\}$ is an ideal of X which is not $d_{(\theta,\phi)}$-invariant.

Theorem 3.13. *Let $d_{(\theta,\phi)}$ be a regular outside (θ, ϕ)-derivation of a BCI-algebra X. Then every θ-ideal of X is $d_{(\theta,\phi)}$-invariant.*

Proof. Let A be a θ-ideal of X. Since $d_{(\theta,\phi)}$ is regular, it follows from Lemma 3.3 that $d_{(\theta,\phi)}(x) = \theta(x) \wedge d_{(\theta,\phi)}(x) \leq \theta(x)$ for all $x \in X$. Let $y \in X$ be such that $y \in d_{(\theta,\phi)}(A)$. Then $y = d_{(\theta,\phi)}(x)$ for some $x \in A$. Thus

$$y * \theta(x) = d_{(\theta,\phi)}(x) * \theta(x) = 0 \in A.$$

Note that $\theta(x) \in \theta(A) \subseteq A$. Since A is an ideal of X, it follows that $y \in A$ so that $d_{(\theta,\phi)}(A) \subseteq A$. Therefore A is $d_{(\theta,\phi)}$-invariant. □

If we take $\theta = \phi = 1_X$ in Theorem 3.13 where 1_X is the identity map, then we have the following corollary.

Corollary 3.14 ([4]). *Let d be a regular (r,l)-derivation of a BCI-algebra X. Then every ideal of X is d-invariant.*

If we take $\theta = \phi = f$ in Theorem 3.13, then we have the following corollary.

Corollary 3.15 ([11]). *Let d_f be a regular (r,l)-f-derivation of a BCI-algebra X. Then every f-ideal of X is d_f-invariant.*

Theorem 3.16. *Let $d_{(\theta,\phi)}$ be an outside (θ,ϕ)-derivation of a BCI-algebra X. If every θ-ideal of X is $d_{(\theta,\phi)}$-invariant, then $d_{(\theta,\phi)}$ is regular.*

Proof. Assume that every θ-ideal of X is $d_{(\theta,\phi)}$-invariant. Since the zero ideal $\{0\}$ is clearly θ-ideal and $d_{(\theta,\phi)}$-invariant, we have $d_{(\theta,\phi)}(\{0\}) \subseteq \{0\}$, and so $d_{(\theta,\phi)}(0) = 0$. Hence $d_{(\theta,\phi)}$ is regular. □

Combining Theorems 3.13 and 3.16, we have a characterization of a regular outside (θ,ϕ)-derivation.

Theorem 3.17. *For an outside (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a BCI-algebra X, the following are equivalent*:

(1) *$d_{(\theta,\phi)}$ is regular.*
(2) *Every θ-ideal of X is $d_{(\theta,\phi)}$-invariant.*

If we take $\theta = \phi = 1_X$ in Theorem 3.17 where 1_X is the identity map, then we have the following corollary.

Corollary 3.18 ([4]). *Let d be an (r,l)-derivation of a BCI-algebra X. Then d is regular if and only if every ideal of X is d-invariant.*

If we take $\theta = \phi = f$ in Theorem 3.17, then we have the following corollary.

Corollary 3.19 ([11]). *For an (r,l)-f-derivation d_f of a BCI-algebra X, the following are equivalent*:

(1) *d_f is regular.*
(2) *Every f-ideal of X is d_f-invariant.*

Conclusion

In the present paper, we have considered the notions of regular inside (or outside) (θ,ϕ)-derivation, θ-ideal, ϕ-ideal and invariant inside (or outside) (θ,ϕ)-derivation of a BCK/BCI-algebra, and investigated related properties. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as subtraction algebras, B-algebras, MV-algebras, d-algebras, Q-algebras etc. In future we can study the notion of regular (θ,ϕ)-derivations on various algebraic structures which may have a lot of applications in different branches of theoretical physics, engineering and computer science.

It is our hope that this work would serve as a foundation for the further study in the theory of derivations of BCK/BCI-algebras.

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