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REGULARITY OF TRANSFORMATION SEMIGROUPS DEFINED BY A PARTITION

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Abstract. Let X be a nonempty set, and let $\mathscr{F} = \{Y_i : i \in I\}$ be a family of nonempty subsets of X with the properties that $X = \bigcup_{i\in I} Y_i$, and $Y_i \cap Y_j = \emptyset$ for all $i, j \in I$ with $i \neq j$. Let $\emptyset \neq J \subseteq I$, and let $T^{(J)}_{\mathscr{F}}(X) = \{\alpha \in T(X) : \forall i \in I \exists j \in J, Y_i\alpha \subseteq Y_j\}$. Then $T^{(J)}_{\mathscr{F}}(X)$ is a subsemigroup of the semigroup $T\left(X, Y^{(J)}\right)$ of functions on X having ranges contained in $Y^{(J)}$, where $Y^{(J)} := \bigcup_{i\in J} Y_i$. For each $\alpha \in T^{(J)}_{\mathscr{F}}(X)$, let $\chi^{(\alpha)} : I \to J$ be defined by $i\chi^{(\alpha)} = j \Leftrightarrow Y_i\alpha \subseteq Y_j$. Next, we define two congruence relations χ and $\widetilde{\chi}$ on $T^{(J)}_{\mathscr{F}}(X)$ as follows: $(\alpha, \beta) \in \chi \Leftrightarrow \chi^{(\alpha)} = \chi^{(\beta)}$ and $(\alpha, \beta) \in \widetilde{\chi} \Leftrightarrow \chi^{(\alpha)}|_J = \chi^{(\beta)}|_J$. We begin this paper by studying the regularity of the quotient semigroups $T^{(J)}_{\mathscr{F}}(X)/\chi$ and $T^{(J)}_{\mathscr{F}}(X)/\widetilde{\chi}$, and the semigroup $T^{(J)}_{\mathscr{F}}(X)$. For each $\alpha \in T_{\mathscr{F}}(X) := T^{(I)}_{\mathscr{F}}(X)$, we see that the equivalence class $[\alpha]$ of α under χ is a subsemigroup of $T_{\mathscr{F}}(X)$ if and only if $\chi^{(\alpha)}$ is an idempotent element in the full transformation semigroup $T(I)$. Let $I_{\mathscr{F}}(X)$, $S_{\mathscr{F}}(X)$ and $B_{\mathscr{F}}(X)$ be the sets of functions α in $T_{\mathscr{F}}(X)$ such that $\chi^{(\alpha)}$ is injective, surjective and bijective respectively. We end this paper by investigating the regularity of the subsemigroups $[\alpha]$, $I_{\mathscr{F}}(X)$, $S_{\mathscr{F}}(X)$ and $B_{\mathscr{F}}(X)$ of $T_{\mathscr{F}}(X)$.

1. Introduction and preliminaries

An element a of a semigroup $\mathcal{S}$ is called a *regular element* if there is an element b of $\mathcal{S}$ such that $aba = a$. Throughout this paper, for any semigroup $\mathcal{S}$, the set of all regular elements of $\mathcal{S}$ is denoted by $\mathcal{R}(\mathcal{S})$. A semigroup whose every element is regular is called a *regular semigroup.* For any nonempty set X, it is well-known that the semigroup $T(X)$ of full transformations on X under the composition is regular, that is, for every $\alpha \in T(X)$, there exists $\beta \in T(X)$ such that $\alpha\beta\alpha = \alpha$ (see [2], page 33). In fact, for each $\alpha \in T(X)$, the function β on X defined by $x\beta = a_x$ if $x \in X\alpha$ and $x\beta = a$ otherwise, where a is a point in X which is fixed and for each $x \in X\alpha$, a_x is a point in $x\alpha^{-1}$ which is also fixed, satisfies $\alpha\beta\alpha = \alpha$. There have been several works on studying

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subsemigroups of the regular semigroup $T(X)$. We mention some of them as follows.

For each nonempty set X and a nonempty subset Y of X, let

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\},$$

and

$$\overline{T}(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

These two subsemigroups of the full transformation semigroup $T(X)$ have been studied by many people (see [5], [10], [15], [16] for some related works). Here we list some necessary results on the regularity of $T(X,Y)$ from one of those ([10] of Nenthein et al.) for referring to in the sequel.

Theorem 1.1 ([10], Theorem 2.1, page 308)**.** *Let X and Y be nonempty sets such that $Y \subseteq X$. For any $\alpha \in T(X,Y)$, the following are equivalent*:

(1) $\alpha \in \mathcal{R}(T(X,Y))$;
(2) $X\alpha = Y\alpha$;
(3) $Y \cap \{z \in X : x\alpha = z\alpha\} \neq \emptyset$ *for all* $x \in X$;
(4) $x\alpha^{-1} \cap Y \neq \emptyset$ *for all* $x \in X\alpha$.

Theorem 1.2 ([10], Corollary 2.2, page 309)**.** *Let X and Y be nonempty sets such that $Y \subseteq X$. Then $T(X,Y)$ is regular if and only if $Y = X$ or $|Y| = 1$.*

Remark 1.3. In the proof of Theorem 1.2 mentioned above, the authors showed that if $|Y| > 1$ and $Y \neq X$, then $T(X,Y)$ is not regular by defining a function $\alpha \in T(X,Y)$ which is not regular as follows: $x\alpha = a$ if $x \in Y$ and $x\alpha = b$ otherwise, where a and b are two fixed different points in Y.

In this paper, we define some further subsemigroups of $T(X)$ and study their regularity. Over the years, there have also been a bunch of research works on semigroup regularity dealing with semigroups of functions on a set along with a mathematical structure, for instance, a vector space and a partially ordered set (see [1], [6], [7], [9], [13], [17], [18] for some references). Another interesting one is a set together with an equivalence relation. Let X be a nonempty set, and let $\mathcal{E}$ be an equivalence relation on X. Let

$$T_{\mathcal{E}}(X) = \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \mathcal{E} \Rightarrow (x\alpha, y\alpha) \in \mathcal{E}\}.$$

This subsemigroup of the full transformation semigroup $T(X)$ was defined in [11] by Huisheng. The author proved that $T_{\mathcal{E}}(X)$ is exactly the semigroup of all continuous functions on X equipped with the topology having the family of all equivalence classes as a base. The regularity of this semigroup was studied in [12] by the same author in 2005. He obtained that a function α in $T_{\mathcal{E}}(X)$ is regular if and only if for each equivalence class A, there exists an equivalence class B such that $A \cap X\alpha \subseteq B\alpha$. It was remarked that if the equivalence relation $\mathcal{E}$ on the set X is neither $\{(x,x) : x \in X\}$ nor $X \times X$, then the semigroup $T_{\mathcal{E}}(X)$ is not regular. To see this explicitly, the author defined a function $\alpha \in T_{\mathcal{E}}(X)$ which is not regular as follows: fix an $A \in X/\mathcal{E}$ such that

$A \neq X$ and $|A| > 1$, choose $a, b \in A$ with $a \neq b$, and let $\alpha : X \to X$ be defined by $x\alpha = a$ if $x \in A$ and $x\alpha = b$ otherwise. The nonregularity of α was deduced from the fact that $A \cap X\alpha$ is not a subset of $B\alpha$ for all $B \in X/\mathcal{E}$. From this, the following result was obtained.

Theorem 1.4 ([12], Proposition 2.4, page 111)**.** *For any nonempty set X and equivalence relation $\mathcal{E}$ on X, $T_{\mathcal{E}}(X)$ is regular if and only if $\mathcal{E} = \{(x, x) : x \in X\}$ or $\mathcal{E} = X \times X$.*

There have been a number of works extending the results of Huisheng mentioned above (see [4], [8], [14] for some references). Here we deal with another approach of the setting of Huisheng and extend it to be more general.

2. Semigroup of transformations defined by a partition

In this section, we define a subsemigroup of the full transformation semigroup $T(X)$, where X is a fixed nonempty set, and then study its regularity.

Definition 2.1. Let X be a nonempty set, and let $\mathscr{F} = \{Y_i : i \in I\}$ be a family of nonempty subsets of X. We call $\mathscr{F}$ a partition of X if $X = \bigcup_{i \in I} Y_i$ and $Y_i \cap Y_j = \emptyset$ for all i, j with $i \neq j$. Let $\Sigma_X = \{X\}$ and $\Lambda_X = \{\{a\} : a \in X\}$. Each of these two partitions is called a *trivial partition* of X.

Throughout the rest of this paper, let X be a nonempty set which is arbitrarily fixed. For any $\alpha \in T(X)$, if $X\alpha = \{a_i : i \in J\}$ with $a_i \neq a_j$ for all $i \neq j$, then we write α in a form of matrix by

$$\alpha = \begin{pmatrix} a_i\alpha^{-1} \\ a_i \end{pmatrix}.$$

This notation was introduced by Clifford and Preston in [3] (see page 241). We also make use of the following notation: if $\{Y_i : i \in I\}$ is a partition of the set X, then for each $\alpha \in T(X)$, we write

$$\alpha = \begin{pmatrix} Y_i \\ \alpha_i \end{pmatrix},$$

where for each $i \in I$, α_i is the restriction of α to Y_i. Next, we consider another approach of the setting of Huisheng originally defined in [11]. Here, we begin with a fixed partition of the set X. It is well-known that any partition of a set induces naturally an equivalence relation on that set. Henceforth, in addition to the set X, let $\mathscr{F} = \{Y_i : i \in I\}$ be an arbitrarily fixed partition of X. Define

$$T_{\mathscr{F}}(X) = \{\alpha \in T(X) : \forall i \in I \exists j \in I, Y_i\alpha \subseteq Y_j\}.$$

It is evident that $T_{\mathscr{F}}(X)$ is a submonoid of the full transformation semigroup $T(X)$. The semigroup $T_{\mathscr{F}}(X)$ can be generalized by fixing, in addition to the partition $\mathscr{F}$ of X, a nonempty subset J of the index set I as follows. Let $J \subseteq I$ with $J \neq \emptyset$, and let

$$T^{(J)}_{\mathscr{F}}(X) = \{\alpha \in T(X) : \forall i \in I \exists j \in J, Y_i\alpha \subseteq Y_j\}.$$

Let $Y^{(J)} = \bigcup_{i\in J} Y_i$. Then we can easily see that $T^{(J)}_{\mathscr{F}}(X)$ is a subsemigroup of $T\left(X, Y^{(J)}\right)$.

Proposition 2.2. (1) *$T^{(J)}_{\mathscr{F}}(X) = T\left(X, Y^{(J)}\right)$ if and only if $|J| = 1$ or $\mathscr{F} = \Lambda_X$.*

(2) *$T^{(J)}_{\mathscr{F}}(X) = T(X)$ if and only if $J = I$ and $\mathscr{F}$ is trivial.*

Proof. (1) The necessity is obviously true. We are now going to prove the sufficiency. Suppose that $|J| \geq 2$, and that $\mathscr{F} \neq \Lambda_X$. Then there are $\nu, \mu \in J$ and $i \in I$ such that $\nu \neq \mu$ and $|Y_i| \geq 2$. Next, we fix $a \in Y_\mu$, $b \in Y_\nu$ and $c \in Y_i$ and then define a function $\alpha : X \to X$ as follows:

$$\alpha = \begin{pmatrix} Y_i \setminus \{c\} & (X \setminus Y_i) \cup \{c\} \\ a & b \end{pmatrix}.$$

Clearly, $\alpha \in T\left(X, Y^{(J)}\right) \setminus T^{(J)}_{\mathscr{F}}(X)$.

(2) It is obvious that the necessity is true. To prove the sufficiency, we assume that $T^{(J)}_{\mathscr{F}}(X) = T(X)$. Then $T^{(J)}_{\mathscr{F}}(X) = T\left(X, Y^{(J)}\right) = T(X)$. Since $T\left(X, Y^{(J)}\right) = T(X)$, it follows that $X = Y^{(J)}$. Hence $J = I$. And since $T^{(J)}_{\mathscr{F}}(X) = T\left(X, Y^{(J)}\right)$, we have by (1) that $|I| = |J| = 1$ or $\mathscr{F} = \Lambda_X$, which yields that $\mathscr{F}$ is trivial. □

Form the above proposition, we obtain immediately that $T_{\mathscr{F}}(X) = T(X)$ if and only if $\mathscr{F}$ is trivial. By the assumption that $\mathscr{F}$ is a partition of X, we have for each $\alpha \in T^{(J)}_{\mathscr{F}}(X)$ that for every $i \in I$, there is a unique $j_i \in J$ such that $Y_i\alpha \subseteq Y_{j_i}$. Thus we can define a function $\chi^{(\alpha)} : I \to J$ corresponding to α as follows:

$$\chi^{(\alpha)} = \begin{pmatrix} i \\ j_i \end{pmatrix}.$$

The function $\chi^{(\alpha)}$ is called the *character* of α.

Lemma 2.3. *For all $\alpha, \beta \in T^{(J)}_{\mathscr{F}}(X)$, $\chi^{(\alpha\beta)} = \chi^{(\alpha)}\chi^{(\beta)}$.*

Proof. Let $\alpha, \beta \in T^{(J)}_{\mathscr{F}}(X)$, and let $i \in I$, $i\chi^{(\alpha)} = j$, and $j\chi^{(\beta)} = k$. Then $Y_i\alpha\beta = (Y_i\alpha)\beta \subseteq Y_j\beta \subseteq Y_k$. Thus $i\chi^{(\alpha\beta)} = k = j\chi^{(\beta)} = i\chi^{(\alpha)}\chi^{(\beta)}$. The proof is complete. □

The notion of character provided above leads us to define two relations χ and $\widetilde{\chi}$ on $T^{(J)}_{\mathscr{F}}(X)$ as follows:

$$(\alpha, \beta) \in \chi \Leftrightarrow \chi^{(\alpha)} = \chi^{(\beta)},$$

and

$$(\alpha, \beta) \in \widetilde{\chi} \Leftrightarrow \chi^{(\alpha)}|_J = \chi^{(\beta)}|_J.$$

It is obvious that $\chi \subseteq \widetilde{\chi}$, and that $\chi = \widetilde{\chi}$ if and only if $I = J$. It is also clear that they both are equivalence relations on $T^{(J)}_{\mathscr{F}}(X)$. And by Lemma 2.3, they are furthermore congruence relations. Thus both $T^{(J)}_{\mathscr{F}}(X)/\chi$ and $T^{(J)}_{\mathscr{F}}(X)/\widetilde{\chi}$

are semigroups. For each $\alpha \in T^{(J)}_{\mathscr{F}}(X)$, let $[\alpha]$ and $\widetilde{[\alpha]}$ denote the equivalence classes of α under the equivalence relations χ and $\widetilde{\chi}$ respectively.

Theorem 2.4. *(1) $T^{(J)}_{\mathscr{F}}(X)/\chi \cong T(I,J)$ by the isomorphism $[\alpha] \mapsto \chi^{(\alpha)}$.*

(2) $T^{(J)}_{\mathscr{F}}(X)/\widetilde{\chi} \cong T(J)$ by the isomorphism $\widetilde{[\alpha]} \mapsto \chi^{(\alpha)}|_J$.

Proof. (1) By the definition of χ, we immediately have that the function $[\alpha] \mapsto \chi^{(\alpha)}$ is well-defined and injective. And by Lemma 2.3, it is a homomorphism. To see that it is surjective, let $\gamma \in T(I,J)$. For each $i \in I$, fix an $a_i \in Y_{i\gamma}$, and then define a function $\alpha : X \to X$ by

$$\alpha = \begin{pmatrix} Y_i \\ a_i \end{pmatrix}.$$

It is evident that $\alpha \in T^{(J)}_{\mathscr{F}}(X)$ and $\chi^{(\alpha)} = \gamma$.

(2) Similarly to (1), the function $\widetilde{[\alpha]} \mapsto \chi^{(\alpha)}|_J$ is a well-defined injective homomorphism. To get that it is surjective, let $\gamma \in T(J)$. For each $i \in J$, fix an $a_i \in Y_{i\gamma}$. Also, we fix an $a \in Y^{(J)}$, and then define a function $\alpha : X \to X$ by

$$\alpha = \begin{pmatrix} Y_i & X \setminus Y^{(J)} \\ a_i & a \end{pmatrix}_{i \in J}.$$

Obviously, $\alpha \in T^{(J)}_{\mathscr{F}}(X)$ and $\chi^{(\alpha)}|_J = \gamma$. □

From Theorem 1.2 and Theorem 2.4, the following corollary is immediately obtained.

Corollary 2.5. (1) *The following are equivalent*:

(a) *The quotient semigroup $T^{(J)}_{\mathscr{F}}(X)/\chi$ is regular*;

(b) *the semigroup $T(I,J)$ is regular*;

(c) *$J = I$ or $|J| = 1$.*

In particular, the quotient semigroup $T_{\mathscr{F}}(X)/\chi$ is regular.

(2) *The quotient semigroup $T^{(J)}_{\mathscr{F}}(X)/\widetilde{\chi}$ is regular.*

We now turn our attention to investigate the regularity of the semigroup $T^{(J)}_{\mathscr{F}}(X)$.

Theorem 2.6. *The semigroup $T^{(J)}_{\mathscr{F}}(X)$ is regular if and only if $\left|T^{(J)}_{\mathscr{F}}(X)\right| = 1$ or $T^{(J)}_{\mathscr{F}}(X) = T(X)$.*

Proof. The necessity is obviously true. To prove the sufficiency, we suppose that $\left|T^{(J)}_{\mathscr{F}}(X)\right| > 1$ and $T^{(J)}_{\mathscr{F}}(X) \neq T(X)$, and divide the proof into two cases.

Case 1 $J = I$: In this case, by the assumption that $T^{(J)}_{\mathscr{F}}(X) \neq T(X)$, we have $\mathscr{F}$ is not trivial. Hence there are $i, j \in I$ such that $i \neq j$ and $|Y_i| \geq 2$. Let a and b be two different points in Y_i, and define a function $\alpha : X \to X$ as follows:

$$\alpha = \begin{pmatrix} Y_j & X \setminus Y_j \\ a & b \end{pmatrix}.$$

Clearly, $\alpha \in T_{\mathscr{F}}(X)$. Next, let $\beta \in T_{\mathscr{F}}(X)$. If $i \in j\left(\chi^{(\beta)}\right)^{-1}$, then we obtain for any point $x \in X \setminus Y_j$ that $x\alpha\beta\alpha = a \neq b = x\alpha$. For the case where $i \notin j\left(\chi^{(\beta)}\right)^{-1}$, we have for each point $x \in Y_j$ that $x\alpha\beta\alpha = b \neq a = x\alpha$.
Case 2 $J \neq I$: We have in this case that $Y^{(J)} \neq X$, which yields that $T_{\mathscr{F}}^{(J)}(X) \subseteq T\left(X, Y^{(J)}\right) \subsetneq T(X)$. And by the assumption that $\left|T_{\mathscr{F}}^{(J)}(X)\right| > 1$, we have $\left|Y^{(J)}\right| > 1$. Let α be the nonregular element of the semigroup $T\left(X, Y^{(J)}\right)$ defined in Remark 1.3. It is clear that $\alpha \in T_{\mathscr{F}}^{(J)}(X)$, which yields that $T_{\mathscr{F}}^{(J)}(X)$ is not regular. $\square$

Remark 2.7. Case 1 in the proof of the above theorem can actually be deduced from Theorem 1.4 of Huisheng mentioned in the introduction. Here, we prove again by our own way. Our proof is just straightforward from the definition of regularity.

3. Some further subsemigroups of $T_{\mathscr{F}}(X)$

In this section, we introduce some new subsemigroups of the semigroup $T_{\mathscr{F}}(X)$ and study their regularity. According to Theorem 2.4, for any $\gamma \in T(I)$, the equivalence class of $\alpha \in T_{\mathscr{F}}(X)$ having the character γ may be denoted for convenience by $\overline{\gamma}$. Clearly, for each $\gamma \in T(I)$, the set $\overline{\gamma}$ is a subsemigroup of $T_{\mathscr{F}}(X)$ if and only if γ is an idempotent element. The following result on characterizations of idempotents in the full transformation semigroup $T(Z)$, for any set Z, is elementary. We state here for completeness and self-containiness of the contents in this paper.

Proposition 3.1. *Let Z be a nonempty set, and let $\gamma \in T(Z)$. Then the following are equivalent*:

(1) *γ is an idempotent element*;
(2) *$\gamma|_{Z\gamma}$ is the identity on $Z\gamma$*;
(3) *there is a partition $\{Z_j : j \in E\}$ of the set Z such that*
$$\gamma = \begin{pmatrix} Z_j \\ z_j \end{pmatrix},$$
where z_j is a fixed element in Z_j for all $j \in E$.

In this situation, the partition $\{Z_j : j \in E\}$ and the subset $\{z_j : j \in E\}$ of the set Z are uniquely determined by γ.

Theorem 3.2. *Let $\gamma \in T(I)$ be an idempotent element, and let $\{I_j : j \in E\}$ be the partition of I with the element i_j in I_j for all $j \in E$ such that*
$$\gamma = \begin{pmatrix} I_j \\ i_j \end{pmatrix}.$$
For each $j \in E$, let $W_j = \bigcup_{i \in I_j} Y_i$. Then, for every $\alpha \in \overline{\gamma}$, $\alpha \in \mathcal{R}(\overline{\gamma})$ if and only if $\alpha|_{W_j} \in \mathcal{R}\left(T(W_j, Y_{i_j})\right)$ for all $j \in E$.

Proof. Let $\alpha \in \overline{\gamma}$. Suppose that $\alpha \in \mathcal{R}(\overline{\gamma})$. Then there is $\beta \in \overline{\gamma}$ such that $\alpha\beta\alpha = \alpha$. For each $j \in E$, let $\alpha_j = \alpha|_{W_j}$ and $\beta_j = \beta|_{W_j}$. Since $\chi^{(\alpha)} = \chi^{(\beta)} = \gamma$, we have that both α_j and β_j belong to $T(W_j, Y_{i_j})$ for all $j \in E$. And since $\alpha\beta\alpha = \alpha$, it follows for each $j \in E$ that $\alpha_j\beta_j\alpha_j = \alpha_j$. Therefore $\alpha_j \in \mathcal{R}\left(T(W_j, Y_{i_j})\right)$ for all $j \in E$. Conversely, suppose that $\alpha_j := \alpha|_{W_j} \in \mathcal{R}\left(T(W_j, Y_{i_j})\right)$ for all $j \in E$. Then for each $j \in E$, there is $\beta_j \in T(W_j, Y_{i_j})$ such that $\alpha_j\beta_j\alpha_j = \alpha_j$. Since $\{W_j : j \in E\}$ is a partition of X, the following function $\beta : X \to X$:

$$\beta = \begin{pmatrix} W_j \\ \beta_j \end{pmatrix},$$

is well-defined. It is obvious that $\beta \in \overline{\gamma}$, and that $\alpha\beta\alpha = \alpha$, which yields that α is a regular element of $\overline{\gamma}$. The proof is complete. □

Remark 3.3. From Theorem 3.2, some further characterizations of the regularity of elements in $\overline{\gamma}$ for any idempotent γ in $T(I)$ can immediately be deduced from Theorem 1.1 of Nenthein et al. [10] mentioned in the introduction.

Corollary 3.4. *Let $\gamma \in T(I)$ be an idempotent element, and let $\{I_j : j \in E\}$ be the partition of I with the element i_j in I_j for all $j \in E$ such that*

$$\gamma = \begin{pmatrix} I_j \\ i_j \end{pmatrix}.$$

Let $W_j = \bigcup_{i \in I_j} Y_i$ for all $j \in E$ and $E_0 = \{j \in E : |Y_{i_j}| = 1\}$. If $E_0 \neq \emptyset$, let $W = \bigcup_{j \in E_0} W_j$, $I_0 = \bigcup_{j \in E_0} I_j$, and define $\alpha : W \to W$ as follows:

$$\alpha = \begin{pmatrix} W_j \\ y_j \end{pmatrix},$$

where y_j is the only element of Y_{i_j} for each $j \in E_0$. Then the following are equivalent:

(1) *$\overline{\gamma}$ is regular*;
(2) *$T(W_j, Y_{i_j})$ is regular for all $j \in E$*;
(3) *for every $j \in E$, if $j \notin E_0$, then $W_j = Y_{i_j}$*;
(4) *γ is the identity function on I, or $\overline{\gamma} = \{\alpha\}$, or $\overline{\gamma}$ is exactly the set of all $\beta \in T_{\mathscr{F}}(X)$ such that $\beta|_W = \alpha$ and $\chi^{(\beta)}|_{I \setminus I_0}$ is the identity function on $I \setminus I_0$.*

Proof. (1) ⇔ (2) and (2) ⇔ (3) immediately follow from Theorem 3.2 and Theorem 1.2 respectively.

We now prove (3) ⇒ (4). Suppose that (3) holds. There are three cases to be considered.

<u>Case 1</u> $E_0 = \emptyset$: We have in this case that $W_j = Y_{i_j}$ for all $j \in E$, which yields that $I_j = \{i_j\}$ for all $j \in E$. Thus $\{I_j : j \in E\} = \Lambda_I$, and therefore γ is the identity function on I.

<u>Case 2</u> $E_0 = E$: In this case, it is clear that if $\beta \in \overline{\gamma}$, then $\beta|_{W_j} = \alpha|_{W_j}$ for all $j \in E$. This implies that $\beta = \alpha$ for all $\beta \in \overline{\gamma}$. Hence $\overline{\gamma} = \{\alpha\}$.

Case 3 $\emptyset \neq E_0 \neq E$: In this case, we have that $\emptyset \neq W \neq X$ and $\emptyset \neq E \setminus E_0 \neq E$. Similarly to Case 1, we have $\{I_j : j \in E \setminus E_0\} = \Lambda_{I \setminus I_0}$. This yields that γ is the identity function on $I \setminus I_0$. Thus, for any $\beta \in \overline{\gamma}$, we get that $\chi^{(\beta)}|_{I \setminus I_0}$ is the identity function on $I \setminus I_0$. Since $W \neq \emptyset$, the function α can be considered in this case. For each $\beta \in \overline{\gamma}$, since $\chi^{(\beta)}|_{I_0} = \gamma|_{I_0}$, similarly to Case 2, we obtain that $\beta|_{W_j}$ is exactly $\alpha|_{W_j}$ for all $j \in E_0$. Accordingly, $\beta|_W = \alpha$ for all $\beta \in \overline{\gamma}$.

Finally, we will prove (4) $\Rightarrow$ (3). Suppose that (4) holds. Then we have three cases to consider.

Case 1 γ is the identity function on I: In this case, we have that $I_j = \{i_j\}$, which yields that $W_j = Y_{i_j}$ for all $j \in E$. So (3) holds.

Case 2 $\overline{\gamma} = \{\alpha\}$: We have in this case that $E_0 = E$. Thus (3) is true.

Case 3 $\overline{\gamma}$ is exactly the set of all $\beta \in T_{\mathscr{F}}(X)$ such that $\beta|_W = \alpha$ and $\chi^{(\beta)}|_{I \setminus I_0}$ is the identity function on $I \setminus I_0$: To prove that (3) is true, let $j \in E$ and assume that $j \notin E_0$. Then $i_j \in I \setminus I_0$. Since $\gamma|_{I \setminus I_0}$ is the identity on $I \setminus I_0$, we have that $I_j = \{i_j\}$, which yields that $W_j = Y_{i_j}$. □

Let $I_{\mathscr{F}}(X)$, $S_{\mathscr{F}}(X)$ and $B_{\mathscr{F}}(X)$ be the sets of elements in $T_{\mathscr{F}}(X)$ whose characters are injective, surjective and bijective respectively. Then $B_{\mathscr{F}}(X) = I_{\mathscr{F}}(X) \cap S_{\mathscr{F}}(X)$. And by Lemma 2.3, both $I_{\mathscr{F}}(X)$ and $S_{\mathscr{F}}(X)$ are submonoids of $T_{\mathscr{F}}(X)$, and $B_{\mathscr{F}}(X)$ is a submonoid of both $I_{\mathscr{F}}(X)$ and $S_{\mathscr{F}}(X)$. Notice that $I_{\Lambda_X}(X)$, $S_{\Lambda_X}(X)$ and $B_{\Lambda_X}(X)$ are exactly the sets of elements in $T(X)$ which are injective, surjective and bijective respectively. And that $I_{\Sigma_X}(X) = S_{\Sigma_X}(X) = B_{\Sigma_X}(X) = T_{\Sigma_X}(X) = T(X)$. We end this paper by studying the regularity of the semigroups $B_{\mathscr{F}}(X)$, $S_{\mathscr{F}}(X)$ and $I_{\mathscr{F}}(X)$.

Theorem 3.5. (1) *The semigroup $B_{\mathscr{F}}(X)$ is regular.*
(2) $\mathcal{R}(I_{\mathscr{F}}(X)) = \mathcal{R}(S_{\mathscr{F}}(X)) = B_{\mathscr{F}}(X)$.

Proof. (1) Let $\alpha \in B_{\mathscr{F}}(X)$, and let $\gamma = \chi^{(\alpha)}$. For $i \in I$, let α_i be the restriction of α to Y_i. Then α_i is a function from Y_i into $Y_{i\gamma}$. We now let $i \in I$ be arbitrarily fixed, and for each $x \in Y_i\alpha_i$, let $a_x^{(i)}$ be a fixed element in $x\alpha_i^{-1}$, and let a_i be another point in Y_i which is also fixed. Let $\beta_i : Y_{i\gamma} \to Y_i$ be defined by

$$\beta_i = \begin{pmatrix} x & Y_{i\gamma} \setminus Y_i\alpha_i \\ a_x^{(i)} & a_i \end{pmatrix}.$$

Next, we define $\beta \in T(X)$ by

$$\beta = \begin{pmatrix} Y_i \\ \beta_j \end{pmatrix},$$

where $j \in I$ such that $i = j\gamma$. We now want to show that β is well-defined. Let $x, z \in X$. Then there are $\nu, \mu \in I$ such that $x \in Y_\nu$ and $z \in Y_\mu$. Since γ is surjective, there are $i, j \in I$ such that $\nu = i\gamma$ and $\mu = j\gamma$. Assume that $x = z$. Then $Y_\nu = Y_\mu$, which yields that $i\gamma = \nu = \mu = j\gamma$. Thus, by the injectivity of γ, we have $i = j$. So $x\beta = x\beta_i = x\beta_j = z\beta_j = z\beta$. Hence β is well-defined, that is, $\beta \in T(X)$. It is evident that $\beta \in T_{\mathscr{F}}(X)$, and that the character of

β is exactly γ^{-1}. Therefore $\beta \in B_{\mathscr{F}}(X)$. Finally, we show that $\alpha\beta\alpha = \alpha$. To see this, let $x \in X$. Then $x \in Y_i$ for some $i \in I$, and hence $x\alpha \in Y_i\alpha_i$. Thus $((x\alpha)\beta)\alpha = ((x\alpha_i)\beta_i)\alpha = \left(a^{(i)}_{x\alpha_i}\right)\alpha = \left(a^{(i)}_{x\alpha_i}\right)\alpha_i$. Since $a^{(i)}_{x\alpha_i} \in x\alpha_i\alpha_i^{-1}$, it follows that $((x\alpha)\beta)\alpha = \left(a^{(i)}_{x\alpha_i}\right)\alpha_i = x\alpha_i = x\alpha$. Consequently, α is a regular element of the semigroup $B_{\mathscr{F}}(X)$.

(2) We obtain immediately from (1) that $B_{\mathscr{F}}(X) \subseteq \mathcal{R}(I_{\mathscr{F}}(X))$ and that $B_{\mathscr{F}}(X) \subseteq \mathcal{R}(S_{\mathscr{F}}(X))$. To see that $B_{\mathscr{F}}(X) = \mathcal{R}(I_{\mathscr{F}}(X))$, let α be a regular element of $I_{\mathscr{F}}(X)$. We want to show that $\alpha \in B_{\mathscr{F}}(X)$. Suppose to the contrary that α is not a member of $B_{\mathscr{F}}(X)$, which means $\chi^{(\alpha)}$ is not surjective. Since α is regular, there is $\beta \in I_{\mathscr{F}}(X)$ such that $\alpha\beta\alpha = \alpha$, which yields from Lemma 2.3 that $\chi^{(\alpha)}\chi^{(\beta)}\chi^{(\alpha)} = \chi^{(\alpha)}$. Thus, by the injectivity of $\chi^{(\alpha)}$ and $\chi^{(\beta)}$, we have that $\chi^{(\beta)}|_{I\chi^{(\alpha)}} = \left(\chi^{(\alpha)}\right)^{-1}$. Since $\chi^{(\alpha)}$ is not surjective, it follows that $I\chi^{(\alpha)} \neq I$, which implies that $\chi^{(\beta)}$ is not injective. This is a contradiction. Hence $\alpha \in B_{\mathscr{F}}(X)$, and so $\mathcal{R}(I_{\mathscr{F}}(X)) = B_{\mathscr{F}}(X)$. We now turn our attention to showing that $B_{\mathscr{F}}(X) = \mathcal{R}(S_{\mathscr{F}}(X))$. Let $\alpha \in \mathcal{R}(S_{\mathscr{F}}(X))$. Then there is $\beta \in S_{\mathscr{F}}(X)$ such that $\alpha\beta\alpha = \alpha$. And by the surjectivity of $\chi^{(\alpha)}$, we have for each $i \in I$ that $i\left(\chi^{(\alpha)}\right)^{-1} \neq \emptyset$. We claim that for each $i \in I$, there is a unique $j_i \in i\left(\chi^{(\alpha)}\right)^{-1}$ such that $Y_i\beta \subseteq Y_{j_i}$. Let $i \in I$, and fix $x \in Y_i\alpha^{-1}$. Then $x\alpha \in Y_i$. To get what we claim, we will show that there is a $j_i \in i\left(\chi^{(\alpha)}\right)^{-1}$ such that $x\alpha\beta \in Y_{j_i}$. If $x\alpha\beta$ were not in Y_j for all $j \in i\left(\chi^{(\alpha)}\right)^{-1}$, there would be a $k \in I \setminus \{i\}$ such that $x\alpha\beta \in Y_\mu$ for some $\mu \in k\left(\chi^{(\alpha)}\right)^{-1}$, which yields that $x\alpha = x\alpha\beta\alpha \in Y_\mu\alpha \subseteq Y_k$. Since $x\alpha \in Y_i$, it follows that $Y_i \cap Y_k \neq \emptyset$, which is a contradiction. Hence there is a $j_i \in i\left(\chi^{(\alpha)}\right)^{-1}$ such that $x\alpha\beta \in Y_{j_i}$, which implies by the definition of $T_{\mathscr{F}}(X)$ that $Y_i\beta \subseteq Y_{j_i}$. It is clear that j_i is unique. Therefore $i\chi^{(\beta)} = j_i$ for all $i \in I$. To see that $\alpha \in B_{\mathscr{F}}(X)$, suppose to the contrary that $\alpha \notin B_{\mathscr{F}}(X)$, that is, $\chi^{(\alpha)}$ is not injective. Then there is a $\nu \in I$ such that $\left|\nu\left(\chi^{(\alpha)}\right)^{-1}\right| > 1$, which yields that $\chi^{(\beta)}$ is not surjective. This is a contradiction. Therefore, $\alpha \in B_{\mathscr{F}}(X)$, and thus we obtain $B_{\mathscr{F}}(X) = \mathcal{R}(S_{\mathscr{F}}(X))$ as asserted. □

From Theorem 3.5, we obtain the following result on the regularity of $I_{\mathscr{F}}(X)$.

Corollary 3.6. *$I_{\mathscr{F}}(X)$ is regular if and only if I is finite.*

Proof. Suppose that $I_{\mathscr{F}}(X)$ is regular. Then by Theorem 3.5, $I_{\mathscr{F}}(X) = B_{\mathscr{F}}(X)$. Thus, by Theorem 2.4, we have that for every $\gamma \in T(I)$, if γ is injective, then γ is bijective. This occurs only when I is finite. Conversely, suppose that I is finite. Then $I_{\mathscr{F}}(X) = B_{\mathscr{F}}(X)$. Hence, by Theorem 3.5 again, $I_{\mathscr{F}}(X)$ is regular. □

Similarly, the following result on the regularity of the semigroup $S_{\mathscr{F}}(X)$ is obtained.

Corollary 3.7. *$S_{\mathscr{F}}(X)$ is regular if and only if I is finite.*

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