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ON SYMMETRIC BI-DERIVATIONS OF B-ALGEBRAS

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Abstract. In this paper, we introduce the notion of symmetric bi-derivations of a B-algebra and investigate some related properties. We study the notion of symmetric bi-derivations of a 0-commutative B-algebra and state some related properties.

1. Introduction

The notion of B-algebra was introduced by J. Neggers and H. S. Kim and some of its related properties in [10] were studied. This class of algebras is related to several classes of interest such as $BCH/BCI/BCK$-algebras. Later, the notion of a ranked trigroupoid as a natural followup on the idea of a ranked bigroupoid was given by N. O. Alshehri et al. in [2]. The notion of derivation in ring theory and near ring theory was applied to BCI-algebras by Y. B. Jun and Xin and some of its related properties were given by them [6]. Later, in [11] the notion of a regular derivation in BCI-algebras was applied to BCC-algebras by Prabpayak and Leerawat and also some of its related properties were investigated. In [1] the notion of derivation in B-algebra was given and some related properties were stated by N. O. Alshehri. Also, in [2] the notion of derivation on ranked bigroupoids was introduced and $(X, *, \omega)$-self-(co)derivations were discussed by N. O. Alshehri, H. S. Kim and J. Neggers. The concept of symmetric bi-derivation was introduced by G. Maksa in [8] (see also [9]). J. Vukman proved some results concerning symmetric bi-derivation on prime and semiprime rings in [12, 13]. Later, the notion of left-right (resp. right-left) symmetric bi-derivation of BCI-algebras was introduced by S. Ilbira, A. Firat and Y. B. Jun in [5].

In this paper, we apply the notion of symmetric bi-derivation in rings, near rings and lattices to B-algebras. We introduce the concept of symmetric bi-derivation of a B-algebra. Additionally, this definition in 0-commutative B-algebra is studied and related properties are given.

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2. Preliminaries

Definition 2.1 ([10])**.** A B-*algebra* is a non-empty set X with a constant 0 and with a binary operation $*$ satisfying the following axioms for all $x, y, z \in X$:
(I) $x * x = 0$.
(II) $x * 0 = x$.
(III) $(x * y) * z = x * (z * (0 * y))$.

Proposition 2.2 ([10])**.** *If* $(X, *, 0)$ *is a* B*-algebra, then for all* $x, y, z \in X$:
(1) $(x * y) * (0 * y) = x$.
(2) $x * (y * z) = (x * (0 * z)) * y$.
(3) $x * y = 0$ *implies* $x = y$.
(4) $0 * (0 * x) = x$.

Theorem 2.3 ([10])**.** $(X, *, 0)$ *is a* B*-algebra if and only if it satisfies the following axioms for all for all* $x, y, z \in X$:
(5) $(x * z) * (y * z) = x * y$.
(6) $0 * (x * y) = y * x$.

Theorem 2.4 ([4])**.** *In any* B*-algebra, the left and right cancellation laws hold.*

Definition 2.5 ([7])**.** A B-algebra $(X, *, 0)$ is said to be 0-*commutative* if for all for all $x, y \in X$:

$$x * (0 * y) = y * (0 * x).$$

Proposition 2.6 ([7])**.** *If* $(X, *, 0)$ *is a* 0*-commutative* B*-algebra, then for all* $x, y, z \in X$:
(7) $(0 * x) * (0 * y) = y * x$.
(8) $(z * y) * (z * x) = x * y$.
(9) $(x * y) * z = (x * z) * y$.
(10) $[x * (x * y)] * y = 0$.
(11) $(x * z) * (y * t) = (t * z) * (y * x)$.
From (11) *and* (3) *we get that, if* $(X, *, 0)$ *is a* 0*-commutative* B*-algebra, then*:
(12) $x * (x * y) = y$ *for all* $x, y \in X$.

For a B-algebra X, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$.

Definition 2.7. Let X be a B-algebra. A mapping $D : X \times X \to X$ is called *symmetric* if $D(x, y) = D(y, x)$ holds for all $x, y \in X$.

Definition 2.8. Let X be a B algebra. A mapping $d : X \to X$ is said to be *regular* if $d(0) = 0$.

3. The symmetric bi-derivations of *B*-algebras

The following definition introduces the notion of symmetric bi-derivation for a B-algebra.

Definition 3.1. Let X be a B-algebra. A map $D : X \times X \to X$ is said to be *a left-right symmetric bi-derivation* (briefly, an (l, r) symmetric bi-derivation) of X, if it satisfies the identity $D(x * y, z) = (D(x, z) * y) \wedge (x * D(y, z))$ for all $x, y, z \in X$.

If D satisfies the identity $D(x * y, z) = (x * D(y, z)) \wedge (D(x, z) * y)$ for all $x, y, z \in X$, then D is said to be *a right-left derivation* (briefly, an (r, l) symmetric bi-derivation) of X. Moreover, if D is both (l, r) and (r, l) symmetric bi-derivations, then it is said that D is *a symmetric bi-derivation*.

Example 3.1. Let $X = \{0, 1, 2\}$ be a 0-commutative B-algebra with Cayley table as follows.

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Define a mapping $D : X \times X \to X$

$$D(x,y) = \begin{cases} 0, & (x,y) = (2,2) \text{ and } (x,y) = (0,1) \text{ and } (x,y) = (1,0) \\ 1, & (x,y) = (0,2) \text{ and } (x,y) = (2,0) \text{ and } (x,y) = (1,1) \\ 2, & (x,y) = (2,1) \text{ and } (x,y) = (1,2) \text{ and } (x,y) = (0,0). \end{cases}$$

Then it can be checked that D is an (l, r) symmetric bi-derivation on X.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a B-algebra with Cayley table as follows.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define a mapping $D : X \times X \to X$

$$D(x,y) = \begin{cases} 0, & (x,y) = (0,0) \text{ and } (x,y) = (1,1) \text{ and } (x,y) = (2,2) \text{ and } (x,y) = (3,3) \\ 1, & (x,y) = (3,2) \text{ and } (x,y) = (2,3) \text{ and } (x,y) = (0,1) \text{ and } (x,y) = (1,0) \\ 2, & (x,y) = (2,0) \text{ and } (x,y) = (0,2) \text{ and } (x,y) = (1,3) \text{ and } (x,y) = (3,1) \\ 3, & (x,y) = (3,0) \text{ and } (x,y) = (0,3) \text{ and } (x,y) = (1,2) \text{ and } (x,y) = (2,1). \end{cases}$$

Then it can be checked that D is a symmetric bi-derivation on X.

Definition 3.2. Let X be a B-algebra. A mapping $d : X \to X$ defined by $d(x) = D(x, x)$ for all $x \in X$ is called *a trace of* D, where $D : X \times X \to X$ is a symmetric mapping.

Proposition 3.3. *Let D be an (l,r) symmetric bi-derivation on a B-algebra X. Then the followings hold:*
(i) $D(x,y) = D(x,y) \wedge (x * D(0,y))$ *for all* $x, y \in X$.
(ii) $D(0,y) = D(x,y) * x$ *for all* $x, y \in X$.
(iii) $D(0,x) = d(x) * x$ *for all* $x \in X$ *where* d *is the trace of* D.

Proof. (i) Let $x, y \in X$. By using (II) and the definition of an (l,r) symmetric bi-derivation we get $D(x,y) = D(x*0,y) = (D(x,y)*0) \wedge (x * D(0,y)) = D(x,y) \wedge (x * D(0,y))$. Hence we find that $D(x,y) = D(x,y) \wedge (x * D(0,y))$.
(ii) Let $x, y \in X$.

$$\begin{aligned} D(0,y) &= D(x*x,y) \\ &= (D(x,y)*x) \wedge (x*D(x,y)) \\ &= (x*D(x,y)) * [(x*D(x,y)) * (D(x,y)*x)] \\ &= [(x*D(x,y)) * (0*(D(x,y)*x))] * (x*D(x,y)) \qquad \text{by (2)} \\ &= [(x*D(x,y)) * (x*D(x,y))] * (x*D(x,y)) \qquad \text{by (I)} \\ &= 0*(x*D(x,y)) \\ &= D(x,y)*x \qquad \text{by (6)}. \end{aligned}$$

Therefore, $D(0,y) = D(x,y) * x$.
(iii) Let $x \in X$. By using (I) and the definition of an (l,r) symmetric bi-derivation we get

$$\begin{aligned} D(0,x) = D(x*x,x) &= (D(x,x)*x) \wedge (x*D(x,x)) \\ &= (d(x)*x) \wedge (x*d(x)) \\ &= (x*d(x)) * [(x*d(x)) * (d(x)*x)]. \end{aligned}$$

By using (3) and (6),

$$\begin{aligned} &= [(x*d(x)) * (0*(d(x)*x))] * (x*d(x)) \\ &= [(x*d(x)) * (x*d(x))] * (x*d(x)) \\ &= 0*(x*d(x)) \\ &= d(x)*x. \end{aligned}$$

Hence $D(0,x) = d(x) * x$. □

Proposition 3.4. *Let d be the trace of the (l,r) symmetric bi-derivation D on a B-algebra X. Then the followings hold:*
(i) $d(0) = D(x,0) * x$ *for all* $x \in X$.
(ii) *If* $D(x,0) = D(y,0)$ *for all* $x, y \in X$, *then* d *is* $1-1$.
(iii) *If* d *is regular, then* $D(x,0) = x$.
(iv) *If there is an element* $x \in X$ *such that* $D(x,0) = x$, *then* d *is regular.*
(v) *If there exists* $x \in X$ *such that* $d(y)*x = 0$ *or* $x*d(y) = 0$ *for all* $y \in X$, *then* $d(y) = x$.

Proof. (i) Let x be an element in X. Since $x * x = 0$ we have

$$\begin{aligned} d(0) = D(0,0) = D(x * x, 0) \\ &= (D(x,0) * x) \wedge (x * D(x,0)) \\ &= (x * D(x,0)) * [(x * D(x,0)) * (D(x,0) * x)]. \end{aligned}$$

By using (I) and (6),

$$\begin{aligned} &= ((x * D(x,0)) * (0 * (D(x,0) * x)) * (x * D(x,0)) \\ &= ((x * D(x,0)) * (x * D(x,0))) * (x * D(x,0)) \\ &= 0 * (x * D(x,0)) \\ &= D(x,0) * x. \end{aligned}$$

Therefore, $d(0) = D(x,0) * x$ for all $x \in X$.

(ii) Let $x, y \in X$ such that $d(x) = d(y)$. Then by (i), we have $d(0) = D(x,0) * x$ and $d(0) = D(y,0) * y$. Thus $D(x,0) * x = D(y,0) * y$. Using Theorem 2.4 we have $x = y$. Hence we get d is $1 - 1$.

(iii) Let d be a regular. By part (i) we have $d(0) = D(x,0) * x$. Since d is regular we have $d(0) = D(x,0) * x = 0$ and by (3) we get $D(x,0) = x$.

(iv) Let $D(x,0) = x$ for some $x \in X$. By (1) we have $D(x,0) * x = 0$ then we can write by part (i) $d(0) = D(x,0) * x = 0$ therefore $d(0) = 0$.

Hence we get that d is regular.

(v) Let x be an element in X such that $d(y) * x = 0$ or $x * d(y) = 0$ for all $y \in X$ then by (3) we get $d(y) = x$. □

Proposition 3.5. *Let d be the trace of an (r, l) symmetric bi-derivation of a B-algebra X. Then the followings hold*:

(i) $d(0) = x * D(x,0)$ *for all* $x \in X$.

(ii) $d(x) = (x * D(0,x)) \wedge d(x)$ *for all* $x \in X$.

(iii) *If* $D(x,0) = D(y,0)$ *for all* $x, y \in X$*, then* d *is* $1 - 1$.

(iv) *If* d *is regular, then* $D(x,0) = x$.

(v) *If there is an element* $x \in X$ *such that* $D(x,0) = x$*, then* d *is regular.*

(vi) *If there exists* $x \in X$ *such that* $d(y) * x = 0$ *or* $x * d(y) = 0$ *for all* $y \in X$*, then* $d(y) = x$.

Proof. (i) Let x be an element in X. Since $x * x = 0$ we have

$$\begin{aligned} d(0) = D(0,0) = D(x * x, 0) \\ &= (x * D(x,0)) \wedge (D(x,0) * x) \\ &= (D(x,0) * x) * [(D(x,0) * x) * (x * D(x,0))]. \end{aligned}$$

By using (I) and (6),

$$\begin{aligned} &= [(D(x,0) * x) * (0 * (x * D(x,0))] * (D(x,0) * x) \\ &= [(D(x,0) * x) * (D(x,0) * x)] * (D(x,0) * x) \\ &= 0 * (D(x,0) * x)) \end{aligned}$$

$$= x * D(x,0).$$

Therefore, $d(0) = x * D(x,0)$ for all $x \in X$.

(ii) Let x be an element in X then we have $x * 0 = x$ and

$$\begin{aligned} d(x) = D(x,x) &= D(x*0,x) \\ &= (x * D(0,x)) \wedge (D(x,x) * 0). \end{aligned}$$

By using (II) and (i),

$$\begin{aligned} &= (x * D(0,x)) \wedge D(x,x) \\ &= (x * D(0,x)) \wedge d(x). \end{aligned}$$

Hence we get $d(x) = (x * D(0,x)) \wedge d(x)$.

(iii) Let $x, y \in X$ such that $d(x) = d(y)$. Then by (i), we have $d(0) = x * D(x,0)$ and $d(0) = y * D(y,0)$. Thus $x * D(x,0) = y * D(y,0)$. Using Theorem 2.4 we have $x = y$. Hence we get d is $1-1$.

(iv) Let d be a regular. By part (i) we have $d(0) = x * D(x,0)$. Since d is regular we have $d(0) = x * D(x,0) = 0$ and by (3) we get $D(x,0) = x$.

(v) Let $D(x,0) = x$ for some $x \in X$. By (1) we have $x * D(x,0) = 0$ then we can write by part (i) $d(0) = x * D(x,0) = 0$ therefore $d(0) = 0$.

Hence we get that d is regular.

(vi) Let x be an element in X such that $d(y) * x = 0$ or $x * d(y) = 0$ for all $y \in X$ then by (3) we get $d(y) = x$. □

Proposition 3.6. *Let $(X, *, 0)$ be a 0-commutative B-algebra and d be the trace of an (l,r) symmetric bi-derivation D of X. Then the followings hold for all $x, y \in X$:*

(i) $d(x * y) = (d(x) * y) * y$.

(ii) $D(x,0) * D(y,0) = x * y$.

Proof. (i) Let $x, y \in X$, then we have

$$\begin{aligned} d(x*y) &= D(x*y, x*y) \\ &= (D(x, x*y) * y) \wedge (x * D(y, x*y)) \\ &= (x * D(y, x*y)) * ((x * D(y, x*y)) * (D(x, x*y) * y)). \end{aligned}$$

Then by (12) we have,

$$\begin{aligned} &= D(x, x*y) * y \\ &= [(D(x,x) * y) \wedge (x * D(x,y))] * y. \end{aligned}$$

Then by (12) we have,

$$= (D(x,x) * y) * y.$$

Hence we get $d(x * y) = (d(x) * y) * y$.

(ii) If $x, y \in X$, then from Proposition 3.4(i) we can write $d(0) = D(x,0) * x$ and $d(0) = D(y,0) * y$. From here we get $D(y,0) * y = D(x,0) * x$ and we have

$(D(y,0)*y)*(D(x,0)*x) = 0$ and we can also write by (11) $(x*y)*(D(x,0)*D(y,0)) = 0$. Hence, by (3) we have $D(x,0)*D(y,0) = x*y$. □

Proposition 3.7. *Let $(X,*,0)$ be a 0-commutative B-algebra and d be the trace of an (r,l) symmetric bi-derivation of X. Then the followings hold for all $x,y \in X$:*
(i) *$d(x*y) = d(y)$.*
(ii) *d is constant.*
(iii) *$D(y,0)*D(x,0) = y*x$.*

Proof. (i)

$$\begin{aligned} d(x*y) &= D(x*y, x*y) \\ &= (x*D(y,x*y)) \wedge (D(x,x*y)*y) \\ &= (D(x,x*y)*y)*((D(x,x*y)*y)*(x*D(y,x*y))) \\ &= (x*D(y,x*y)) \\ &= x*[(x*D(y,y)) \wedge (D(y,x)*y)] \\ &= x*(x*D(y,y)) = x*(x*d(y)) \\ &= d(y) \quad \text{by (12)}. \end{aligned}$$

Hence we get $d(x*y) = d(y)$.

(ii) Let $x \in X$. By using (II) and (ii) we have $d(x) = d(x*0) = d(0)$. Hence d is constant.

(iii) Let $x,y \in X$, then from Proposition 3.5(i) we can write $d(0) = x*D(x,0)$ and $d(0) = y*D(y,0)$. From here we get $x*D(x,0) = y*D(y,0)$ and we have $(x*D(x,0))*(y*D(y,0)) = 0$ and we can also write by (13) $(D(y,0)*D(x,0))*(y*x) = 0$. Hence, by (3) we have $D(y,0)*D(x,0) = y*x$. □

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