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Algebraic Kripke-style semantics for an extension of HpsUL, CnHpsUL* †

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【Abstract】 This paper deals with Kripke-style semantics for weakening-free non-commutative fuzzy logics. As an example, we consider an algebraic Kripke-style semantics for an extension of the pseudo-uninorm based fuzzy logic **HpsUL**, **CnHpsUL***. For this, first, we recall the system **CnHpsUL***, define its corresponding algebraic structures CnHpsUL*-algebras, and algebraic completeness results for it. We next introduce a Kripke-style semantics for **CnHpsUL***, and connect it with algebraic semantics.

【Key Words】 **HpsUL**, **CnHpsUL***, Kripke-style semantics, algebraic semantics, fuzzy logic, standard completeness.

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1. Introduction

This paper is a contribution to the study of Kripke-style semantics, i.e., semantics with binary accessibility relations, for *weakening-free non-commutative substructural fuzzy* logics. First recall the concepts of substructural fuzzy logics. Substructural logics, in general, mean logics lacking structural rules such as weakening, contraction, and commutativity (or exchange). These logics encompass classical logic, intuitionistic logic, relevance logic, linear logic, many-valued logic, fuzzy logic, etc. (see Galatos et al. (2007), Metcalfe & Montagna (2007)). Among these logics, fuzzy logic is known as dealing with vagueness: According to Cintula (and Behounek), a (weakly implicative) logic **L** is said to be *fuzzy* if it is complete with respect to (w.r.t.) linearly ordered matrices (or algebras) and *core fuzzy* if it is complete w.r.t. *standard* algebras (i.e., algebras on the real unit interval [0,1]) (Behounek & Cintula (2006), Cintula (2006)).

We next recall some historical facts associated with Kripke-style semantics for many-valued logics. A lot of Kripke-style semantics have been provided for three- and four-valued logics. As Yang mentioned in Yang (2014b), Thomason gave a three-valued Kripke-style semantics for the Nelson's system **N** of constructible falsity by allowing partial evaluations (“gaps” (N)) (Thomason (1969)). Dunn provided a three-valued Kripke-style semantics for the **R** of Relevance with mingle (**RM**) by allowing non-functional evaluations (“gluts” (B)) (Dunn (1976; 2000)). Furthermore, Yang provided Kripke-style semantics for three- and four-valued logics, which can be regarded as the three-valued Dummett-Gödel logic $\mathbf{G}_3$ and neighbors of the relevance logics **R**, **E** of Entailment, and **T** of Ticket entailment (Yang (2009; 2012b)).

In particular, several algebraic Kripke-style semantics have been provided for core fuzzy logics. As Yang further mentioned in Yang (2014a), after introducing algebraic semantics for t-norm[1)] (based) logics, their corresponding algebraic Kripke-style semantics have been introduced. More precisely, after Esteva and Godo introducing algebraic semantics for monoidal t-norm (based) logics in Esteva & Godo (2001), their corresponding algebraic Kripke-style semantics were introduced in Montagna & Ono (2002), Montagna & Sacchetti (2003; 2004), and Diaconescu & Georgescu (2007). Furthermore, algebraic semantics and corresponding algebraic Kripke-style semantics for core fuzzy logic systems based on more general structures have been introduced: After Hájek introducing algebraic semantics for *non-commutative* pseudo-t-norm (based) logics in Hájek (2003a; 2003b), one corresponding algebraic Kripke-style semantics for the pseudo-t-norm (based) logic $\mathbf{psMTL}^r$ was introduced in Diaconescu (2010). After Metcalfe and Montagna introducing algebraic semantics for *weakening-free* uninorm (based) logics in Metcalfe & Montagna (2007), their corresponding algebraic Kripke-style semantics were introduced in Yang (2012a; 2014a).

Pseudo-t-norm and uninorm functions are t-norms dropping commutativity and t-norms having the identity lying anywhere in

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1) T-norms are commutative, associative, increasing, binary functions with identity 1 on the real unit interval [0,1].

[0,1], respectively. Note that pseudo-uninorm functions as uninorms dropping commutativity were introduced in Metcalfe et al. (2009) and that, although standard completeness proof for **HpsUL***, the logic of pseudo-uninorms and their residua, remains open, such proof for **CnHpsUL***, the **HpsUL*** with n-potency, were provided in Wang (2013). Then, these raise the following interesting question:

- Can we introduce Kripke-style semantics for pseudo-uninorm based logics, in particular **CnHpsUL***?

The answer to the question is positive in the sense that we can provide algebraic Kripke-style semantics for **CnHpsUL***. For this, first, in Section 2 we introduce **CnHpsUL*** and the corresponding algebraic semantics as the necessary notions for treating the question. In Section 3, we introduce an algebraic Kripke-style semantics for **CnHpsUL***, and connect it with algebraic semantics.

For convenience, we shall adopt the notation and terminology similar to those in Cintula (2006), Metcalfe & Montagna (2007), Montagna & Sacchetti (2003; 2004), and Yang (2012a; 2014a), and we assume reader familiarity with them (along with results found therein).

2. Preliminaries: The logic **CnHpsUL*** and its algebraic semantics

We base **CnHpsUL*** on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives →, ⇟, &, ∧, ∨, and constants **T**, **F**, **t**[2], with a defined connective:

df1. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We moreover define ϕ^n_t as ϕ_t & ⋯ & ϕ_t, n factors, where $\phi_t := \phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiomatization of **HpsUL** as the most basic weakening-free non-commutative (substructural) fuzzy logic introduced here.

Definition 2.1 (Metcalfe et al. (2009), Tsinakis & Blount (2003), Wang (2009; 2013)) **HpsUL** consists of the following axiom schemes and rules:

A1. $\phi \rightarrow \phi$ (self-implication, SI)

A2. $(\phi \wedge \psi) \rightarrow \phi$, $(\phi \wedge \psi) \rightarrow \psi$ (∧-elimination, ∧-E)

A3. $((\phi \rightarrow \psi) \wedge (\phi \rightarrow \chi)) \rightarrow (\phi \rightarrow (\psi \wedge \chi))$ (∧-introduction, ∧-I)

A4. $\phi \rightarrow (\phi \vee \psi)$, $\psi \rightarrow (\phi \vee \psi)$ (∨-introduction, ∨-I)

A5. $((\phi \rightarrow \chi) \wedge (\psi \rightarrow \chi)) \rightarrow ((\phi \vee \psi) \rightarrow \chi)$ (∨-elimination, ∨-E)

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2) The constant **t** corresponds to the least designated element.

A6. $\phi \rightarrow \mathbf{T}$ (verum ex quolibet, VE)
A7. $\mathbf{F} \rightarrow \phi$ (ex falso quadlibet, EF)
A8. $\mathbf{t}$
A9. $\phi \rightarrow (\mathbf{t} \rightarrow \phi)$
A10. $(\psi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$ (prefixing, PF)
A11. $\phi \rightarrow ((\phi \rightsquigarrow \psi) \rightarrow \psi)$
A12. $(\phi \rightsquigarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\phi \rightsquigarrow \chi)$
A13. $\psi \rightarrow (\phi \rightarrow (\phi \,\&\, \psi))$
A14. $(\psi \rightarrow (\phi \rightarrow \chi)) \rightarrow ((\phi \,\&\, \psi) \rightarrow \chi)$
A15. $((\psi \rightsquigarrow \psi) \,\&\, (\psi \rightarrow \phi)) \rightarrow (\psi \rightsquigarrow \phi)$
A16. $(\phi_t \,\&\, \psi_t) \rightarrow (\phi \wedge \psi)$
A17. $(\phi \vee \psi)_t \rightarrow (\phi_t \vee \psi_t)$ (prelinearity, PRL1)
A18. $(\chi\rightarrow(((\phi\vee\psi)\rightarrow\phi)\&\chi)) \vee (\chi \rightsquigarrow (\chi\&((\phi\vee\psi)\rightarrow\psi)))$ (PRL2)
$\phi \rightarrow \psi, \phi \vdash \psi$ (mp)
$\phi \vdash \phi_t$ (adj$_t$)
$\phi \vdash \psi \rightarrow (\phi \,\&\, \psi)$ (pn$_\rightarrow$)
$\phi \vdash \psi \rightsquigarrow (\psi \,\&\, \phi)$ (pn$_\rightsquigarrow$).

Definition 2.2 (HpsULs) A logic is a schematic extension of **L** if and only if (iff) it results from **L** by adding axiom schemes. **L** is an HpsUL iff **L** is a schematic extension of **HpsUL**. In particular, the following are weakening-free non-commutative extensions of **HpsUL** introduced in Wang (2013).

- **CnHpsUL** is **HpsUL** plus $\phi^n \leftrightarrow \phi^{n-1}$, for $2 \leq n$ (n-potency, nP)
- **CnHpsUL*** is **CnHpsUL** plus $(\phi \,\&\, \psi) \rightarrow \mathbf{t} \vdash (\psi \,\&\, \phi) \rightarrow \mathbf{t}$[3] (weak commutativity, WCM).

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3) Note that we may instead take $(\phi \rightsquigarrow \mathbf{t}) \rightarrow (\phi \rightarrow \mathbf{t})$

For easy reference, we group the weakening-free non-commutative fuzzy logics introduced in Definitions 2.1 and 2.2 as a set.

Definition 2.3 Ls = {**HpsUL**, **CnHpsUL**, **CnHpsUL***}

Proposition 2.4 L (∈ Ls}) proves:

(1) $\phi \rightarrow \psi \vdash \phi \rightsquigarrow \psi$, $\phi \rightsquigarrow \psi \vdash \phi \rightarrow \psi$

(2) $\phi \rightarrow (\psi \rightarrow \chi) \vdash (\psi \& \phi) \rightarrow \chi$ (residuation1, Res1)

(3) $\phi \rightarrow (\psi \rightsquigarrow \chi) \vdash (\phi \& \psi) \rightarrow \chi$ (Res1$_{\rightsquigarrow}$)

(4) $(\psi \& \phi) \rightarrow \chi \vdash \phi \rightarrow (\psi \rightarrow \chi)$ (Res2)

(5) $(\phi \& \psi) \rightarrow \chi \vdash \phi \rightarrow (\psi \rightsquigarrow \chi)$ (Res2$_{\rightsquigarrow}$)

(6) $\phi \rightarrow \psi \vdash (\chi \& \phi) \rightarrow (\chi \& \psi)$, $\phi \rightarrow \psi \vdash (\phi \& \chi) \rightarrow (\psi \& \chi)$

(7) $(\mathbf{t} \& \phi) \leftrightarrow \phi \leftrightarrow (\phi \& \mathbf{t})$

Proof: For (1) to (6), see Cintula et al. (2013; 2015). We prove (7).

(1) $(\mathbf{t} \& \phi) \leftrightarrow \phi$

(⇒) 1. $(\mathbf{t} \& \phi) \rightarrow (\mathbf{t} \& \phi)$ (A1)

2. $\mathbf{t} \rightarrow (\phi \rightsquigarrow (\mathbf{t} \& \phi))$ (1, Res2$_{\rightsquigarrow}$)

3. $\phi \rightarrow (\mathbf{t} \& \phi)$ (2, A8, mp)

(⇐) 1. $(\mathbf{t} \& \phi) \rightarrow \phi$ (A9, Res1)

(2) $\phi \leftrightarrow (\phi \& \mathbf{t})$

(⇒) 1. $(\phi \& \mathbf{t}) \rightarrow (\phi \& \mathbf{t})$ (A1)

2. $\mathbf{t} \rightarrow (\phi \rightarrow (\phi \& \mathbf{t}))$ (1, Res2)

3. $\phi \rightarrow (\phi \& \mathbf{t})$ (2, A8, mp)

(⇐) 1. $\mathbf{t} \rightarrow (\phi \rightarrow \phi)$ (A1, A8, mp)

2. $(\phi \,\&\, \mathbf{t}) \rightarrow \phi$ (1, Res1). □

A *theory* over L ($\in$ Ls) is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using the two rules in Definition 2.1. T $\vdash \phi$, more exactly T $\vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T.

For the deduction theorem for L, we first introduce the notion of conjugate.

Definition 2.5 (Left, right and iterated conjugates, Cintula & Niguera (2011)) Given a formula α, we define left and right conjugates w.r.t. α as $\lambda_\alpha(\bigstar) = (\alpha \rightarrow (\bigstar \,\&\, \alpha))_{\mathbf{t}}$ and $\rho_\alpha(\bigstar) = (\alpha \rightsquigarrow (\alpha \,\&\, \bigstar))_{\mathbf{t}}$. An iterated conjugate is a formula of the form $\gamma(\bigstar) = \gamma_{\alpha 1}(\gamma_{\alpha 2}(\cdots(\gamma_{\alpha n}(\bigstar))\cdots)$, where each $\gamma_{\alpha i}$ is either $\lambda_{\alpha i}$ or $\rho_{\alpha i}$.

A formula of the form $\gamma(\phi)$, where γ is a left, right, or iterated conjugate is called left, right, or iterated (resp.) conjugate of ϕ.

Theorem 2.6 (Cintula & Noguera (2011)) Let T be a theory over L ($\in$ Ls), and ϕ, ψ formulas. L is almost (MP)-based with the set of basic deduction terms $\{\lambda_\alpha(\bigstar), \rho_\alpha(\bigstar) : \alpha \in Fm\}$. Therefore, the following holds:

T, $\phi \vdash_L \psi$ iff T $\vdash \chi(\phi) \rightarrow \psi$ for some conjunction χ of iterated conjugates.

Proof: Since the system **FL** satisfies this theorem and L ($\in$ Ls) is its extension, L also satisfies this theorem. □

A theory T is *inconsistent* if T $\vdash$ **F**; otherwise it is *consistent*.

For convenience, "$\wedge$", "$\vee$", "$\rightarrow$", and "$\rightsquigarrow$" are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meanings.

The algebraic counterpart of L is the class of the so-called *L-algebras*. Let $x_t := x \wedge t$. They are defined as follows.

Definition 2.7 (Metcalfe et al. (2009), Wang (& Zhao) (2009; 2013))

(i) (HpsUL-algebra) An *HpsUL algebra* is a bounded residuated lattice **A** = (A, $\top$, $\bot$, t, $\wedge$, $\vee$, *, $\rightarrow$, $\rightsquigarrow$) such that:

(I) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded lattice with top element $\top$ and bottom element $\bot$.

(II) (A, *, t) is a monoid.

(III) $y \leq x \rightarrow z$ iff $x * y \leq z$ iff $x \leq y \rightsquigarrow z$, for all x, y, z $\in$ A (residuation).

(IV) $(x \vee y)_t = (x_t \vee y_t)$, for all x, y $\in$ A.

(V) $t \leq (u \rightarrow (((x \vee y) \rightarrow x) * u)) \vee (u \rightsquigarrow (u * ((x \vee y) \rightarrow y)))$, for all x, y, u $\in$ A.

(ii) (CnHpsUL-algebra) A *CnHpsUL-algebra* is an HpsUL-algebra satisfying the n-potency condition: $x^n = x^{n-1}$, $2 \leq n$, for all x $\in$ A.

(iii) (CnHpsUL*-algebra) A *CnHpsUL*-algebra* is a CnHpsUL-algebra satisfying the WCM condition: $x \rightsquigarrow t \leq x \rightarrow t$, for all x $\in$ A.

Additional binary equivalence operation is defined as follows: $x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x)$, for all $x, y \in A$.

The class of all L-algebras is a variety which will be denoted by $\mathbb{L}$.

L-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 2.8 (Evaluation) Let $\mathcal{A}$ be an L algebra. An *$\mathcal{A}$-evaluation* is a function $v : FOR \rightarrow \mathcal{A}$ satisfying: $v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$, $v(\phi \nrightarrow \psi) = v(\phi) \nrightarrow v(\psi)$, $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$, $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$, $v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$, $v(\mathbf{F}) = \bot$, $v(\mathbf{t}) = t$, (and hence $v(\mathbf{T}) = \top$).

Definition 2.9 (Cintula (2006)) Let $\mathcal{A}$ be an L-algebra, T a theory, ϕ a formula, and $\mathbb{K}$ a class of L-algebras.

(i) (Tautology) ϕ is a *t-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq t$ for each $\phi \in T$. We denote the class of $\mathcal{A}$-models of T, by *Mod(T, $\mathcal{A}$).*

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. $\mathbb{K}$, denoting by $T \vDash_{\mathbb{K}} \phi$, if $Mod(T, \mathcal{A}) = Mod(T \cup \{\phi\}, \mathcal{A})$ for each $\mathcal{A} \in \mathbb{K}$.

Definition 2.10 (**L**-algebra, Cintula (2006)) Let $\mathcal{A}$, T, and ϕ be as in Definition 2.8. $\mathcal{A}$ is an ***L**-algebra* iff, whenever ϕ is

L-provable in T (i.e. T $\vdash_L \phi$, L an L logic), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. T$\vDash_{\{A\}}$ ϕ), $\mathcal{A}$ a corresponding L-algebra). By *MOD$^{(l)}$(L)*, we denote the class of (linearly ordered) **L**-algebras. Finally, we write T $\vDash^{(l)}_L \phi$ in place of T $\vDash_{MOD^{(l)}(L)} \phi$.

Theorem 2.11 (Strong completeness, Wang (2009; 2013)) Let T be a theory, and ϕ a formula. T $\vdash_L \phi$ iff T $\vDash_L \phi$ iff T $\vDash^l_L \phi$.

Proof: We obtain this theorem as a corollary of Theorem 3.1.8 in Cintula & Noguera (2011). □

We define standard L-algebras and pseudo-uninorms on [0,1].

Definition 2.12 An L-algebra is standard iff its lattice reduct is [0,1].

Definition 2.13 (Metcalfe et al. (2009)) A *pseudo-uninorm* is a function $\circ : [0,1]^2 \rightarrow [0,1]$ such that, for some t $\in$ [0,1] and for all x, y, z $\in$ [0,1]:

(a) $(x \circ y) \circ z = x \circ (y \circ z)$ (associativity)

(b) $t \circ x = x = x \circ t$ (identity), and

(c) $x \leq y$ implies $(x \circ z) \leq (y \circ z)$ (monotonicity).

A pseudo-uninorm $\circ$ is called conjunctive if $0 \circ 1 = 1 \circ 0 = 0$, disjunctive if $0 \circ 1 = 1 \circ 0 = 1$, and residuated if there are binary functions $\rightarrow$ and $\rightsquigarrow$ satisfying (residuation). $\circ$ is further called n-potent if it satisfies the n-potency condition.

Theorem 2.14 (Wang (& Zhao) (2009; 2013))

(i) There is a linearly ordered HpsUL-algebra, which is not satisfying the WCM condition.

(ii) The systems **HpsUL** and **CnHpsUL** are not standard complete.

Theorem 2.15 (Wang (2013)) The system **CnHpsUL*** is standard complete, i.e., complete w.r.t. the class of standard CnHpsUL*-algebras.

3. Kripke-style semantics for CnHpsUL*

We consider here algebraic Kripke-style semantics for **CnHpsUL***.

Definition 3.1 (i) (Operational Kripke frame) An *operational Kripke frame* is a structure **X** = (X, ⊤, ⊥, t, f, ≤, ∗) such that (X, ⊤, ⊥, t, f, ≤, ∗) is a linearly ordered pointed bounded monoid. The elements of **X** are called *nodes*.

(ii) (Residuated operational Kripke frame) An operational Kripke frame is said to be *residuated* if it has suprema w.r.t. ∗, i.e., for every x, y ∈ X, the sets {z: x ∗ z ≤ y} and {z: z ∗ x ≤ y} have suprema.

Definition 3.2 (CnHpsUL* frame) A *CnHpsUL* frame* is a residuated operational Kripke frame, where $*$ is conjunctive (i.e., $\bot * \top = \bot$) and left-continuous (i.e., whenever $\sup\{x_i : i \in I\}$ exists, $x * \sup\{x_i : i \in I\} = \sup\{x * x_i : i \in I\}$) and $\sup\{x_i : i \in I\} * x = \sup\{x_i * x : i \in I\}$).

Definition 3.2 ensures that a CnHpsUL* frame has suprema w.r.t. $*$, i.e., for every $x, y \in X$, the sets $\{z: x * z \leq y\}$ and $\{z: z * x \leq y\}$ have the suprema. **X** is said to be *complete* if $\leq$ is a complete order.

An *evaluation* or *forcing* on an algebraic Kripke frame is a relation $\Vdash$ between nodes and propositional variables, and arbitrary formulas subject to the conditions below: for every propositional variable p,

(AHC) if $x \Vdash p$ and $y \leq x$, then $y \Vdash p$;
(min) $\bot \Vdash p$; and

for arbitrary formulas,

(t) $x \Vdash \mathbf{t}$ iff $x \leq t$;
(f) $x \Vdash \mathbf{f}$ iff $x \leq f$;
($\bot$) $x \Vdash \mathbf{F}$ iff $x = \bot$;
($\wedge$) $x \Vdash \phi \wedge \psi$ = iff $x \Vdash \phi$ and $x \Vdash \psi$;
($\vee$) $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$;
(&) $x \Vdash \phi \,\&\, \psi$ iff there are $y, z \in X$ such that $y \Vdash \phi$, $z \Vdash \psi$, and $x \leq y * z$;
($\rightarrow$) $x \Vdash \phi \rightarrow \psi$ iff for all $y \in X$, if $y \Vdash \phi$, then $y * x \Vdash \psi$;

($\ddagger$) $x \Vdash \phi \ddagger \psi$ iff for all $y \in X$, if $y \Vdash \phi$, then $x * y \Vdash \psi$.

An evaluation or forcing on a CnHpsUL* frame is an evaluation or forcing further satisfying that (max) for every atomic sentence p, $\{x : x \Vdash p\}$ has a maximum.

Definition 3.3 (i) (Algebraic Kripke model) An *algebraic Kripke model* is a pair ($\mathbf{X}$, $\Vdash$), where $\mathbf{X}$ is an algebraic Kripke frame and $\Vdash$ is a forcing on $\mathbf{X}$.

(ii) (CnHpsUL* model) A *CnHpsUL* model* is a pair ($\mathbf{X}$, $\Vdash$), where $\mathbf{X}$ is a CnHpsUL* frame and $\Vdash$ is a forcing on $\mathbf{X}$. A CnHpsUL* model ($\mathbf{X}$, $\Vdash$) is said to be *complete* if $\mathbf{X}$ is a complete frame and $\Vdash$ is a forcing on $\mathbf{X}$.

Definition 3.4 (Cf. Montagna & Sacchetti (2004)) Given an algebraic Kripke model ($\mathbf{X}$, $\Vdash$), a node x of $\mathbf{X}$ and a formula ϕ, we say that *x forces ϕ* to express $x \Vdash \phi$. We say that ϕ is *true* in ($\mathbf{X}$, $\Vdash$) if $t \Vdash \phi$, and that ϕ is *valid* in the frame $\mathbf{X}$ (expressed by $\mathbf{X}$ models ϕ) if ϕ is true in ($\mathbf{X}$, $\Vdash$) for every forcing $\Vdash$ on $\mathbf{X}$.

For soundness and completeness for **CnHpsUL***, let $\vdash_{\mathbf{CnHpsUL^*}} \phi$ be the theoremhood of ϕ in **CnHpsUL***. First we note the following lemma.

Lemma 3.5 (i) (Hereditary Lemma, HL) Let $\mathbf{X}$ be an algebraic Kripke frame. For any sentence ϕ and for all nodes x, y $\in \mathbf{X}$, if $x \Vdash \phi$ and $y \leq x$, then $y \Vdash \phi$.

(ii) Let $\Vdash$ be a forcing on a CnHpsUL* frame, and ϕ a sentence. Then the set $\{x \in X : x \Vdash \phi\}$ has a maximum.

Proof: (i) Easy. (ii) See Lemma 2.11 in Montagna & Sacchetti (2003) and Proposition 3.3 in Diaconescu (2010). □

Proposition 3.6 (Soundness) If $\vdash_{\mathbf{CnHpsUL^*}} \phi$, then ϕ is valid in every CnHpsUL* frame.

Proof: We prove the validity of A18 as an example: It suffices to show that either $t \Vdash \chi \to (((\phi \vee \psi) \to \phi) \& \chi)$ or $t \Vdash \chi \ddagger (\chi \& ((\phi \vee \psi) \to \psi))$. As mentioned in proof of Lemma 2.11 in Montagna & Sacchetti (2003), for every α, the set $\alpha^\circ = \{x : x \Vdash \alpha\}$ is downwards closed, therefore either $\phi^\circ \subseteq \psi^\circ$ or $\psi^\circ \subseteq \phi^\circ$. Thus $t \Vdash \phi \to \psi$ or $t \Vdash \psi \to \phi$. Let $t \Vdash \phi \to \psi$. Then, since $t \Vdash \psi \to \psi$, we can obtain that $t \Vdash (\phi \vee \psi) \to \psi$ and thus $t \Vdash \mathbf{t} \to ((\phi \vee \psi) \to \psi)$ by A9 and mp. Then, using Proposition 2.4 (6), we obtain $t \Vdash (\chi \& \mathbf{t}) \to (\chi \& ((\phi \vee \psi) \to \psi))$; therefore, $t \Vdash \chi \ddagger (\chi \& ((\phi \vee \psi) \to \psi))$ by Proposition 2.4 (6) and (1). Let $t \Vdash \psi \to \phi$. Analogously we can obtain that $t \Vdash \chi \to (((\phi \vee \psi) \to \phi) \& \chi)$, as wished.

The proof for the other cases is left to the interested reader. □

By a *chain*, we mean a linearly ordered algebra. The next proposition connects algebraic Kripke semantics and algebraic semantics for **CnHpsUL*** (cf. see Montagna & Sacchetti (2004)).

Proposition 3.7 (i) The {⊤, ⊥, t, ≤, ∗} reduct of a CnHpsUL*-chain A is a CnHpsUL* frame, which is complete iff A is complete.

(ii) Let **X** = (X, ⊤, ⊥, t, ≤, ∗) be a CnHpsUL* frame. Then the structure **A** = (X, ⊤, ⊥, t, max, min, ∗, →, ⇟) is a CnHpsUL*-algebra (where *max* and *min* are meant w.r.t. ≤).

(iii) Let **X** be the {⊤, ⊥, t, ≤, ∗} reduct of a CnHpsUL*-chain A, and let v be an evaluation in A. Let for every atomic formula p and for every x ∈ A, x ⊩ p iff x ≤ v(p). Then (**X**, ⊩) is a CnHpsUL* model, and for every formula ϕ and for every x ∈ A, we obtain that: x ⊩ ϕ iff x ≤ v(ϕ).

(iv) Let (**X**, ⊩) be a CnHpsUL* model, and let A be the CnHpsUL*-algebra defined as in (ii). Define for every atomic formula p, v(p) = max{x ∈ X : x ⊩ p}. Then for every formula ϕ, v(ϕ) = max{x ∈ X : x ⊩ ϕ}.

Proof: The proof for (i) and (ii) is easy. Since (iv) follows almost directly from (iii) and Lemma 3.1.5 (ii), we prove (iii). As regards to claim (iii), we consider the induction steps corresponding to the cases where ϕ = ψ & χ, ϕ = ψ → χ and ϕ = ψ ⇟ χ. (The proof for the other cases is trivial.)

For the cases ϕ = ψ & χ, see Proposition 3.8 in Yang (2012a). We prove the case ϕ = ψ → χ and ϕ = ψ ⇟ χ.

Suppose ϕ = ψ → χ. By the condition (→), x ⊩ ψ → χ iff for all y ∈ X, if y ⊩ ψ, then y ∗ x ⊩ χ, hence by the induction hypothesis, y ⊩ ψ only if y ∗ x ⊩ χ iff y ≤ v(ψ) only if y ∗ x ≤ v(χ), therefore iff v(ψ) ∗ x ≤ v(χ), therefore by residuation, iff $x \leq v(\psi) \rightarrow v(\chi) = v(\psi \rightarrow \chi)$, as desired.

Suppose $\phi = \psi \rightsquigarrow \chi$. By the condition ($\rightsquigarrow$), $x \Vdash \psi \rightsquigarrow \chi$ iff for all $y \in X$, if $y \Vdash \psi$, then $x * y \Vdash \chi$, hence by the induction hypothesis, $y \Vdash \psi$ only if $x * y \Vdash \chi$ iff $y \leq v(\psi)$ only if $x * y \leq v(\chi)$, therefore iff $x * v(\psi) \leq v(\chi)$, therefore by residuation, iff $x \leq v(\psi) \rightsquigarrow v(\chi) = v(\psi \rightsquigarrow \chi)$, as desired. □

Theorem 3.8 (Strong completeness)

(i) **CnHpsUL*** is strongly complete w.r.t. the class of all CnHpsUL*-frames.

(ii) **CnHpsUL*** is strongly complete w.r.t. the class of complete CnHpsUL*-frames.

Proof: (i) and (ii) follow from Proposition 3.7 and Theorem 2.11, and from Proposition 3.7 and Theorem 2.14, respectively. □

4. Concluding remark

We investigated algebraic Kripke-style semantics for weakening-free non-commutative substructural fuzzy logics. As an example we introduced an algebraic Kripke-style semantics for **CnHpsUL***. We proved soundness and completeness theorems. But we did not provide algebraic Kripke-style semantics for **HpsUL*** since its standard completeness proof has not yet provided. This is an open problem left in this paper.

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CnHpsUL*을 위한 대수적 크립키형 의미론

양 은 석

이 글에서 우리는 약화 없는 비교환적인 퍼지 논리의 크립키형 의미론을 다룬다. 이의 한 예로, 우리는 가-유니놈에 기반한 퍼지 논리 **HpsUL**의 한 확장 체계인 **CnHpsUL***을 위한 대수적 크립키형 의미론을 고려한다. 이를 위하여 먼저 **CnHpsUL*** 체계를 소개하고 그에 상응하는 CnHpsUL*-대수를 정의한 후 **CnHpsUL***이 대수적으로 완전하다는 것을 보인다. 다음으로 **CnHpsUL***을 위한 크립키형 의미론을 소개하고 이를 대수적 의미론과 연관 짓는다.

주요어: **HpsUL**, **CnHpsUL***, 크립키형 의미론, 대수적 의미론, 퍼지 논리, 표준 완전성