

On a Problem Posed by Belcastro and Sherman

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ABSTRACT. In this paper, we compute the number of distinct centralizers and commutativity degree of a class of finite groups. These computations produce a further class of examples of groups answering one question raised by Belcastro and Sherman and another one raised by Lescot.

1. Introduction

Given a group G and $x \in G$, the set $C(x) = \{y \in G : xy = yx\}$ is called the *centralizer* of x in G . The set of all centralizers in G is denoted by $\text{Cent}(G)$. A group G is called an n -centralizer group if $|\text{Cent}(G)| = n$. It is easy to see that one centralizer groups are precisely the abelian groups. It is also not difficult to see that two and three centralizer groups do not exist. In [3], Belcastro and Sherman studied n -centralizer groups for some n and asked the following question:

If n is a positive integer other than two or three, does there exist a group with n centralizers?

By counting the number of distinct centralizers of generalized quaternion groups (Q_{4m}) presented by $\langle a, b : a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1} \rangle$ and dihedral groups (D_{2n}) presented by $\langle a, b : a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$, Ashrafi (in [1, 2]) has answered this question affirmatively.

The commutativity degree of a finite group G , denoted by $d(G)$, is defined by

$$(1.1) \quad d(G) = \frac{1}{|G|^2} \sum_{g \in G} |C(g)|.$$

In [7], Lescot have computed the commutativity degree of dihedral groups and quaternion groups ($Q_{2^{n+1}}$) presented by $\langle a, b : a^{2^n} = 1, b^2 = a^{2^{n-1}}, bab^{-1} = a^{-1} \rangle$, and showed that

$$d(D_{2n}) \rightarrow \frac{1}{4} \quad \text{and} \quad d(Q_{2^{n+1}}) \rightarrow \frac{1}{4}$$

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as the orders of the groups D_{2n} and $Q_{2^{n+1}}$ tend to infinity. He then asked, “whether there are other natural families of groups with the same property”. It may be mentioned here that many authors have answered this problem by computing commutativity degree of different classes of groups (see [4],[5],[6]).

In this paper, we compute the number of centralizers and commutativity degree of a class of metacyclic groups $M_{2m,n}$ presented by $\langle a, b : a^n = b^{2m} = 1, bab^{-1} = a^{-1} \rangle$. These computations produce a further class of examples of groups answering the above mentioned questions.

2. Main Result

In this section, we have the following main result.

Theorem 2.1. *For $n > 2$, $\text{Cent}(M_{2m,n})$ has cardinality $n + 2$ or $\frac{n}{2} + 2$ according as to whether n is odd or even. In particular, for each positive integer $k > 3$, there exists a metacyclic group $M_{2m,n}$ which has exactly k centralizers. Further, the commutativity degree $d(M_{2m,n}) = \frac{n+3}{4n}$ or $\frac{n+6}{4n}$ according as to whether n is odd or even. In particular, $d(M_{2m,n})$ is independent of m and tends to $\frac{1}{4}$ as $n \rightarrow \infty$.*

Proof. From $bab^{-1} = a^{-1}$, we have

$$b^s a^t b^{-s} = a^{(-1)^s t}.$$

Using this, we have

$$\begin{aligned} (a^i b^j)(a^u b^v)(a^i b^j)^{-1}(a^u b^v)^{-1} &= (a^i b^j)(a^u b^v)(b^{-j} a^{-i})(b^{-v} a^{-u}) \\ &= a^i (b^j a^u b^{-j})(b^v a^{-i} b^{-v}) a^{-u} \\ &= a^{((-1)^{(v+1)}+1)i + ((-1)^j - 1)u}. \end{aligned}$$

Therefore, the centralizer of any element is given by:

$$C(a^u b^v) = \{a^i b^j : ((-1)^{(v+1)} + 1)i + ((-1)^j - 1)u \equiv 0 \pmod{n}\}.$$

This breaks up into the following cases.

The m elements in $\langle b^2 \rangle$ have $C(b^{2v}) = M_{2m,n}$ and the m elements in $b\langle b^2 \rangle$ have $C(b^{2v-1}) = \{b^j, a^{\frac{n}{2}} b^j : j \leq 2m\}$; the m elements in $a^{\frac{n}{2}} \langle b^2 \rangle$ have $C(a^{\frac{n}{2}} b^{2v}) = M_{2m,n}$ if n is even; the m elements in $a^u \langle b^2 \rangle$ have

$$C(a^u b^{2v}) = \{a^i b^{2j} : i \leq n, j \leq m\} \text{ for each } \frac{n}{2} \neq u < n;$$

the m elements in $a^u b \langle b^2 \rangle$ have

$$C(a^u b^{2v-1}) = \{b^{2j}, a^u b^{2k-1}, a^{\frac{n}{2}} b^{2j}, a^{u+\frac{n}{2}} b^{2k+1} : j, k \leq m\} \text{ for each } u < n.$$

Hence, for odd n , we have

$$\begin{aligned} \text{Cent}(M_{2m,n}) &= \{M_{2m,n}, \langle b \rangle, \langle a \rangle \langle b^2 \rangle, ab \langle b^2 \rangle \cup \langle b^2 \rangle, a^2 b \langle b^2 \rangle \cup \langle b^2 \rangle, \dots, \\ &\quad a^{n-1} b \langle b^2 \rangle \cup \langle b^2 \rangle\}. \end{aligned}$$

This gives $|\text{Cent}(M_{2m,n})| = n + 2$. Further, we have

$$\sum_{u,v} |C(a^u b^v)| = m(2mn) + m(2m) + m(n-1)mn + m(n-1)2m = m^2 n^2 + 3m^2 n.$$

Similarly, if n is even, then

$$\begin{aligned} \text{Cent}(M_{2m,n}) = \{ & M_{2m,n}, \langle b \rangle \cup a^{\frac{n}{2}} \langle b \rangle, \langle a \rangle \langle b^2 \rangle, \langle b^2 \rangle \cup a^{\frac{n}{2}} \langle b^2 \rangle \cup ab \langle b^2 \rangle \cup a^{1+\frac{n}{2}} b \langle b^2 \rangle, \\ & \langle b^2 \rangle \cup a^{\frac{n}{2}} \langle b^2 \rangle \cup a^2 b \langle b^2 \rangle \cup a^{2+\frac{n}{2}} b \langle b^2 \rangle, \dots, \\ & \langle b^2 \rangle \cup a^{\frac{n}{2}} \langle b^2 \rangle \cup a^{\frac{n}{2}-1} b \langle b^2 \rangle \cup a^{n-1} b \langle b^2 \rangle \}. \end{aligned}$$

This gives $|\text{Cent}(M_{2m,n})| = \frac{n}{2} + 2$ and

$$\begin{aligned} \sum_{u,v} |C(a^u b^v)| &= m(2mn) + m(4m) + m(n-2)mn + m(4m)(n-1) \\ &= m^2 n^2 + 6m^2 n. \end{aligned}$$

Finally, these give the commutativity degree to be $\frac{n+3}{4n}$ or $\frac{n+6}{4n}$ respectively, as n is odd or even (using (1.1)). This completes the proof of the theorem. \square

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