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# Semi M-Projective and Semi N-Injective Modules

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Abstract. Let $M$ and $N$ be modules over a ring $R$. The purpose of this paper is to study modules $M, N$ for which the bi-module $[M, N]$ is regular or *pi*. It is proved that the bi-module $[M, N]$ is regular if and only if a module $N$ is semi $M-$projective and $\mathrm{Im}(\alpha) \subseteq^{\oplus} N$ for all $\alpha \in [M, N]$, if and only if a module $M$ is semi $N-$injective and $\mathrm{Ker}(\alpha) \subseteq^{\oplus} N$ for all $\alpha \in [M, N]$. Also, it is proved that the bi-module $[M, N]$ is *pi* if and only if a module $N$ is direct $M-$projective and for any $\alpha \in [M, N]$ there exists $\beta \in [N, M]$ such that $\mathrm{Im}(\alpha\beta) \subseteq^{\oplus} N$, if and only if a module $M$ is direct $N-$injective and for any $\alpha \in [M, N]$ there exists $\beta \in [N, M]$ such that $\mathrm{Ker}(\beta\alpha) \subseteq^{\oplus} M$. The relationship between the Jacobson radical and the (co)singular ideal of $[M, N]$ is described.

## 1. Introduction

In this paper rings $R$, are associative with identity unless otherwise indicated. All modules over a ring $R$ are unitary right modules. The category of right $R-$modules is denoted by $mod - R$. Maps are written on the left. A submodule $N$ of a module $M$ is said to be *small* in $M$, if $N + K \neq M$ for any proper submodule $K$ of $M$ [1]. Also, a submodule $Q$ of a module $M$ is said to be *large* (*essential*) in $M$ if $Q \cap K \neq 0$ for every nonzero submodule $K$ of $M$ [1]. For a submodule $N$ of a module, we use $N \subseteq^{\oplus} M$ to mean that $N$ is a direct summand of $M$, and write $N \leq_e M$ and $N \ll M$ to indicate that $N$ is a large, respectively small, submodule of $M$. Also, we write $J(R)$ and $U(R)$ for the Jacobson radical and the group of units of a ring $R$ respectively. If $M_R$ and $N_R$ are modules, We use the notation: $E_M = \mathrm{End}_R(M)$, and $[M, N] = hom_R(M, N)$. Thus, $[M, N]$ is an $(E_N, E_M)$-bimodule.

An important line of research in module theory is to investigate substructures

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such as the radical, the singular and co-singular ideals, and (semi)regularity, semipotency or (*pi*) of $[M,N]$ which are similar to ones in countered in the ring and module theory.

Our main concern is about when the endomorphism ring $E_M$ of some module $M \in \text{mod}-R$ is regular or $pi-$ring. In section 3, 4 it is proved that for any modules $M, N \in \text{mod}-R$, $[M,N]$ is regular if and only if $N$ is semi $M-$projective and $\text{Im}(\alpha) \subseteq^{\oplus} N$ for any $\alpha \in [M,N]$ if and only if $M$ is semi $N-$injective and $\text{Ker}(\alpha) \subseteq^{\oplus} M$ for any $\alpha \in [M,N]$. It is also proved that for a semi $N-$projective module $M$, $J[M,N] = \{\alpha : \alpha \in [M,N];\ \text{Im}(1_N - \alpha\beta) = N \text{ for all } \beta \in [N,M]\}$ and for a semi $M-$injective module $N$, $J[M,N] = \{\alpha : \alpha \in [M,N];\ \text{Ker}(1_M - \beta\alpha) = \{0\} \text{ for all } \beta \in [N,M]\}$. In section 5, it is proved that for any two modules $M, N \in \text{mod}-R$; $[M,N]$ is *pi* if and only if $N$ is direct $M-$projective and for any $\alpha \in [M,N]$ there exists $\beta \in [N,M]$ such that $\text{Im}(\alpha\beta) \subseteq^{\oplus} N$ if and only if $M$ is direct $N-$injective and for any $\alpha \in [M,N]$ there exists $\beta \in [N,M]$ such that $\text{Ker}(\beta\alpha) \subseteq^{\oplus} M$.

## 2. Some Properties of [M,N]

**Lemma 2.1.**([4, Lemma 2.9]) *Let $M_R$, $N_R$ be modules and $\alpha \in [M,N]$, $\beta \in [N,M]$. The following statements hold:*

(1) $Im(\alpha) + Im(1_N - \alpha\beta) = N$.

(2) $Im(\alpha - \alpha\beta\alpha) = Im(\alpha) \cap Im(1_N - \alpha\beta)$.

(3) $Ker(\alpha) \cap Ker(1_M - \beta\alpha) = \{0\}$.

(4) $Ker(\alpha - \alpha\beta\alpha) = Ker(\alpha) + Ker(1_M - \beta\alpha)$.

*Proof.* We have $\alpha\beta \in E_N$ and $\beta\alpha \in E_M$.
(1) It is clear that $N = \text{Im}(\alpha\beta) + \text{Im}(1_N - \alpha\beta) \subseteq \text{Im}(\alpha) + \text{Im}(1_N - \alpha\beta) \subseteq N$. Similarly (3) holds.
(2) It is obvious that $\alpha - \alpha\beta\alpha \in [M,N]$, $\text{Im}(\alpha - \alpha\beta\alpha) = \text{Im}((1_N - \alpha\beta)\alpha) \subseteq \text{Im}(1_N - \alpha\beta)$ and $\text{Im}(\alpha - \alpha\beta\alpha) = \text{Im}(\alpha(1_M - \beta\alpha)) \subseteq \text{Im}(\alpha)$. So $\text{Im}(\alpha - \alpha\beta\alpha) \subseteq \text{Im}(\alpha) \cap \text{Im}(1_N - \alpha\beta)$.
Let $x \in \text{Im}(\alpha) \cap \text{Im}(1_N - \alpha\beta)$; $x \in N$ and $x = \alpha(y) = (1_N - \alpha\beta)(z)$ where $y \in M$, $z \in N$. So $x = z - \alpha\beta(z)$, $z = x + \alpha\beta(z) = \alpha(y) + \alpha\beta(z) = \alpha(y + \beta(z))$. Let $y_0 = y + \beta(z) \in M$. Then $z = \alpha(y_0)$ and $x = (1_N - \alpha\beta)(z) = (1_N - \alpha\beta)\alpha(y_0) = (\alpha - \alpha\beta\alpha)(y_0) \in \text{Im}(\alpha - \alpha\beta\alpha)$. Thus, $\text{Im}(\alpha) \cap \text{Im}(1_N - \alpha\beta) \subseteq \text{Im}(\alpha - \alpha\beta\alpha)$. Similarly (4) holds. (5) and (7) are clear.
(6) It is clear that $\text{Ker}(\alpha) \subseteq \text{Ker}(\alpha - \alpha\beta\alpha)$ and $\text{Ker}(1_M - \beta\alpha) \subseteq \text{Ker}(\alpha - \alpha\beta\alpha)$, so $\text{Ker}(\alpha) + \text{Ker}(1_M - \beta\alpha) \subseteq \text{Ker}(\alpha - \alpha\beta\alpha)$. Let $x \in \text{Ker}(\alpha - \alpha\beta\alpha)$. Then $x \in M$ and $\alpha(x) = \alpha\beta\alpha(x)$. Since $x = \beta\alpha(x) + (1_M - \beta\alpha)(x)$ and $\beta\alpha(x) \in \text{Ker}(1_M - \beta\alpha)$, $(1_M - \beta\alpha)(x) \in \text{Ker}(\alpha)$, hence $(1_M - \beta\alpha)(\beta\alpha(x)) = \beta\alpha(x) - \beta\alpha\beta\alpha(x) = \beta\alpha(x) - \beta\alpha(x) = 0$, $\alpha(1_M - \beta\alpha)(x) = \alpha(x) - \alpha\beta\alpha(x) = \alpha(x) - \alpha(x) = 0$. So $x \in \text{Ker}(1_M - \beta\alpha) + \text{Ker}(\alpha)$. Thus, $\text{Ker}(\alpha - \alpha\beta\alpha) \subseteq \text{Ker}(\alpha) + \text{Ker}(1_M - \beta\alpha)$. Similarly (8) holds. $\square$

**Lemma 2.2.** *Let $M_R$, $N_R$ be modules and $\alpha \in [M,N]$, $\beta \in [N,M]$. The following* *statements hold:*

(1) $Im(1_N-\alpha\beta)=N$ *if and only if* $Im(1_M-\beta\alpha)=M$.

(2) $Ker(1_N-\alpha\beta)=\{0\}$ *if and only if* $Ker(1_M-\beta\alpha)=\{0\}$.

(3) $1_N-\alpha\beta\in U(E_N)$ *if and only if* $1_M-\beta\alpha\in U(E_M)$.

*Proof.* (1)($\Rightarrow$) Assume that $\mathrm{Im}(1_N-\alpha\beta)=N$. Then by Lemma 2.1(4) $\mathrm{Im}(\beta)\cap\mathrm{Im}(1_M-\beta\alpha)=\mathrm{Im}(\beta-\beta\alpha\beta)=\beta(\mathrm{Im}(1_N-\alpha\beta))=\mathrm{Im}(\beta)$, which shows that $\mathrm{Im}(\beta)\subseteq\mathrm{Im}(1_M-\beta\alpha)$. Lemma 2.1(3) implies that $M=\mathrm{Im}(\beta)+\mathrm{Im}(1_M-\beta\alpha)=\mathrm{Im}(1_M-\beta\alpha)$. Similarly ($\Leftarrow$) holds.
(2)($\Rightarrow$) Assume that $\mathrm{Ker}(1_N-\alpha\beta)=\{0\}$. Let $x\in\mathrm{Ker}(1_M-\beta\alpha)$. Then $(\alpha-\alpha\beta\alpha)(x)=(1_N-\alpha\beta)(\alpha(x))=0$, so $\alpha(x)\in\mathrm{Ker}(1_N-\alpha\beta)=\{0\}$, thus $\alpha(x)=0$ and that $x\in\mathrm{Ker}(\alpha)$, which shows that $\mathrm{Ker}(1_M-\beta\alpha)\subseteq\mathrm{Ker}(\alpha)$. Lemma 2.1(5) implies that $\{0\}=\mathrm{Ker}(\alpha)\cap\mathrm{Ker}(1_M-\beta\alpha)=\mathrm{Ker}(1_M-\beta\alpha)$. Similarly ($\Leftarrow$) holds.
(3)($\Rightarrow$) If $1_N-\alpha\beta\in U(E_N)$. Then $\mu(1_N-\alpha\beta)=1_N$ for some $\mu\in E_N$. So, $(1_M+\beta\mu\alpha)(1_M-\beta\alpha)=(1_M-\beta\alpha)+\beta\mu\alpha(1_M-\beta\alpha)=(1_M-\beta\alpha)+\beta\mu(1_N-\alpha\beta)\alpha=1_M$. The proof for right inverses is similar. Similarly, ($\Leftarrow$) holds. □

**Lemma 2.3.** *Let $M_R$, $N_R$ be modules and $\alpha\in[M,N]$, $\beta\in[N,M]$. The following statements hold*

(1) $E_N=\alpha[N,M]+(1_N-\alpha\beta)E_N$.

(2) $E_N=[M,N]\beta+E_N(1_N-\alpha\beta)$.

(3) $E_M=\beta[M,N]+(1_M-\beta\alpha)E_M$.

(4) $E_M=[N,M]\alpha+E_M(1_M-\beta\alpha)$.

(5) $(\alpha-\alpha\beta\alpha)[N,M]=\alpha[N,M]\cap(1_N-\alpha\beta)E_N$.

(6) $[N,M](\alpha-\alpha\beta\alpha)=[N,M]\alpha\cap E_M(1_M-\beta\alpha)$.

(7) $(\beta-\beta\alpha\beta)[M,N]=\beta[M,N]\cap(1_M-\beta\alpha)E_M$.

(8) $[M,N](\beta-\beta\alpha\beta)=[M,N]\beta\cap E_N(1_N-\alpha\beta)$.

(9) $[M,N]=\alpha E_M+(1_N-\alpha\beta)[M,N]$.

(10) $[M,N]=E_N\alpha+[M,N](1_M-\beta\alpha)$.

*Proof.* (1) Since $1_N=\alpha\beta+(1_N-\alpha\beta)$, for any $\lambda\in E_N$, $\lambda=\alpha\beta\lambda+(1_N-\alpha\beta)\lambda\in\alpha[N,M]+(1_N-\alpha\beta)E_N$, so $E_N\subseteq\alpha[N,M]+(1_N-\alpha\beta)E_N\subseteq E_N$. Similarly (2), (3) and (4) hold.
(5) Since $(\alpha-\alpha\beta\alpha)[N,M]=(1_N-\alpha\beta)\alpha[N,M]\subseteq(1_N-\alpha\beta)E_N$ and $(\alpha-\alpha\beta\alpha)[N,M]=\alpha(1_M-\beta\alpha)[N,M]\subseteq\alpha[N,M]$, we have $(\alpha-\alpha\beta\alpha)[N,M]\subseteq\alpha[N,M]\cap(1_N-\alpha\beta)E_N$.
Let $\lambda\in\alpha[N,M]\cap(1_N-\alpha\beta)E_N$. Then $\lambda=\alpha\gamma=(1_N-\alpha\beta)\mu$ where $\gamma\in[N,M]$, $\mu\in E_N$, so $\mu=\lambda+\alpha\beta\mu=\alpha\gamma+\alpha\beta\mu=\alpha(\gamma+\beta\mu)\in\alpha[N,M]$. Suppose that $\mu=\alpha\theta$ where $\theta=\gamma+\beta\mu\in[N,M]$. Then $\lambda=(1_N-\alpha\beta)\mu=(1_N-\alpha\beta)\alpha\theta=(\alpha-\alpha\beta\alpha)\theta\in(\alpha-\alpha\beta\alpha)[N,M]$, which shows that $\alpha[N,M]\cap(1_N-\alpha\beta)E_N\subseteq(\alpha-\alpha\beta\alpha)[N,M]$.
Similarly (6), (7), (8), (9) and (10) hold. □

**Lemma 2.4.** *Let $M_R,N_R$ be modules and $\alpha\in[M,N]$, $\beta\in[N,M]$. The following are equivalent:*

(1) $Im(1_N - \alpha\beta) = N$.

(2) $Im(1_M - \beta\alpha) = M$.

(3) $Im(\alpha - \alpha\beta\alpha) = Im(\alpha)$.

(4) $Im(\beta - \beta\alpha\beta) = Im(\beta)$.

*Proof.* (1) $\Leftrightarrow$ (2) By Lemma 2.2(1) and (1) $\Rightarrow$ (4) by Lemma 2.1(2).
(4) $\Rightarrow$ (1) Assume (4) hold. Then $\mathrm{Im}(\beta) = \mathrm{Im}(\beta - \beta\alpha\beta) = \mathrm{Im}(\beta) \cap \mathrm{Im}(1_M - \beta\alpha)$, which shows that $\mathrm{Im}(\beta) \subseteq \mathrm{Im}(1_M - \beta\alpha)$, so $M = \mathrm{Im}(\beta) + \mathrm{Im}(1_M - \beta\alpha) = \mathrm{Im}(1_M - \beta\alpha)$, proving (1). Similarly, the equivalence (2) $\Leftrightarrow$ (3) holds. □

**Lemma 2.5.** *Let $M_R, N_R$ be modules and $\alpha \in [M,N]$, $\beta \in [N,M]$. The following are equivalent:*

(1) $Ker(1_N - \alpha\beta) = \{0\}$.

(2) $Ker(1_M - \beta\alpha) = \{0\}$.

(3) $Ker(\alpha - \alpha\beta\alpha) = Ker(\alpha)$.

(4) $Ker(\beta - \beta\alpha\beta) = Ker(\beta)$.

*Proof.* (1) $\Leftrightarrow$ (2) By Lemma 2.2(2) and (1) $\Rightarrow$ (4) by Lemma 2.1(8).
(4) $\Rightarrow$ (1) Assume (4) hold. Then $\mathrm{Ker}(\beta) = \mathrm{Ker}(\beta - \beta\alpha\beta) = \mathrm{Ker}(\beta) + \mathrm{Ker}(1_N - \alpha\beta)$, which shows that $\mathrm{Ker}(1_N - \alpha\beta) \subseteq \mathrm{Ker}(\beta)$, so $\mathrm{Ker}(1_N - \alpha\beta) = \mathrm{Ker}(\beta) \cap \mathrm{Ker}(1_N - \alpha\beta) = \{0\}$, proving (1). Similarly, the equivalence (2) $\Leftrightarrow$ (3) holds. □

Let $M_R$ and $N_R$ be modules. Write:

$$\widehat{\nabla}[M,N] = \{\alpha : \alpha \in [M,N]; \mathrm{Im}(1_N - \alpha\beta) = N \text{ for all } \beta \in [N,M]\}.$$

It is clear that $\widehat{\nabla}[M,N]$ is a non empty subset in $[M,N]$, ($0 \in \widehat{\nabla}[M,N]$). By using Lemma 2.2(1) it is easy to see that

$$\widehat{\nabla}[M,N] = \{\alpha : \alpha \in [M,N]; \mathrm{Im}(1_N - \alpha\beta) = N \text{ for all } \beta \in [N,M]\}.$$

$$= \{\alpha : \alpha \in [M,N]; \mathrm{Im}(1_M - \beta\alpha) = M \text{ for all } \beta \in [N,M]\}.$$

In addition to, $\widehat{\nabla}[M,N]$ is an ideal in $mod - R$, which means that it is closed under arbitrary multiplication from either side, by the following Lemma:

**Lemma 2.6.** *For arbitrary $M, N, X, Y \in mod - R$ the following statements hold:*

(1) $\widehat{\nabla}[M,N][X,M] \subseteq \widehat{\nabla}[X,N]$.

(2) $[N,Y]\widehat{\nabla}[M,N] \subseteq \widehat{\nabla}[M,Y]$.

(3) $[N,Y]\widehat{\nabla}[M,N][X,M] \subseteq \widehat{\nabla}[X,Y]$.

*Proof.* (1)Let $\alpha \in \widehat{\nabla}[M,N]$ and $\lambda \in [X,M]$. Then $\alpha\lambda \in [X,N]$ and for all $\beta \in [N,X]$, $\mathrm{Im}(1_N - (\alpha\lambda)\beta) = \mathrm{Im}(1_N - \alpha(\lambda\beta)) = N$, hence $\lambda\beta \in [N,M]$. Thus, $\alpha\lambda \in \widehat{\nabla}[X,N]$. (2) is analogous. (3) by (1) and (2). □

Again, let $M_R$ and $N_R$ be modules. Write:

$$\widehat{\Delta}[M,N] = \{\alpha : \alpha \in [M,N]; \mathrm{Ker}(1_N - \alpha\beta) = \{0\} \text{ for all } \beta \in [N,M]\}.$$

It is clear that $\widehat{\Delta}[M,N]$ is a non empty subset in $[M,N]$, $(0 \in \widehat{\Delta}[M,N])$. By using Lemma 2.2(2) it is easy to see that

$$\widehat{\Delta}[M,N] = \{\alpha : \alpha \in [M,N]; \mathrm{Ker}(1_N - \alpha\beta) = \{0\} \text{ for all } \beta \in [N,M]\}.$$

$$= \{\alpha : \alpha \in [M,N]; \mathrm{Ker}(1_M - \beta\alpha) = \{0\} \text{ for all } \beta \in [N,M]\}.$$

In addition to, $\widehat{\Delta}[M,N]$ is an ideal in $mod - R$, which means that it is closed under arbitrary multiplication from either side, by the following Lemma:

**Lemma 2.7.** *For arbitrary $M, N, X, Y \in mod - R$ the following statements hold:*

(1) $\widehat{\Delta}[M,N][X,M] \subseteq \widehat{\Delta}[X,N]$.

(2) $[N,Y]\widehat{\Delta}[M,N] \subseteq \widehat{\Delta}[M,Y]$.

(3) $[N,Y]\widehat{\Delta}[M,N][X,M] \subseteq \widehat{\Delta}[X,Y]$.

*Proof.* As in Lemma 2.6. □

Following [2, 7], the Jacobson radical of the bimodule $[M,N]$ defined by Kasch as follows:

$$J[M,N] = \{\alpha : \alpha \in [M,N]; (1_N - \alpha\beta) \in U(E_N) \text{ for all } \beta \in [N,M]\}.$$

$$= \{\alpha : \alpha \in [M,N]; (1_M - \beta\alpha) \in U(E_M) \text{ for all } \beta \in [N,M]\}.$$

**Lemma 2.8.** *Let $M_R$ and $N_R$ be modules. The following statements hold:*

(1) $J[M,N] \subseteq \widehat{\nabla}[M,N]$.

(2) $J[M,N] \subseteq \widehat{\Delta}[M,N]$.

*Proof.* This is obvious. □

Let $M_R$ and $N_R$ be modules. Recall that a morphism $\alpha \in [M,N]$ is *regular* [2], if there exists $\beta \in [N,M]$ such that $\alpha = \alpha\beta\alpha$. Also, $[M,N]$ is called regular if and only if every $\alpha \in [M,N]$ is *regular.*

**Lemma 2.9.**([7]) *Let $M_R$ and $N_R$ be modules. The following are equivalent:*

(1) *$[M,N]$ is regular.*

(2) *For every $\alpha \in [M,N]$, Im$(\alpha) \subseteq^{\oplus} N$ and Ker$(\alpha) \subseteq^{\oplus} M$.*

*In particular, for a module $M$, $E_M$ is regular if and only if Im$(\alpha) \subseteq^{\oplus} M$ and Ker$(\alpha) \subseteq^{\oplus} M$ for all $\alpha \in E_M$,* [5, *Lemma* 3.1].

**Proposition 2.10.** *Let $M$ and $N$ be modules and $\alpha, \beta \in [M,N]$. If $[M,N]$ is regular. Then the following statements hold:*

(1) *Im$(\alpha) \subseteq$ Im$(\beta)$ if and only if $\alpha[N,M] \subseteq \beta[N,M]$.*

(2) $Im(\alpha) = Im(\beta)$ *if and only if* $\alpha[N, M] = \beta[N, M]$.

(3) $\alpha[N, M] = \{\mu : \mu \in E_N;\ Im(\mu) \subseteq Im(\alpha)\}$.

(4) $Ker(\alpha) \subseteq Ker(\beta)$ *if and only if* $[N, M]\beta \subseteq [N, M]\alpha$.

(5) $Ker(\alpha) = Ker(\beta)$ *if and only if* $[N, M]\beta = [N, M]\alpha$.

(6) $[N, M]\alpha = \{\mu : \mu \in E_M;\ Ker(\alpha) \subseteq Ker(\mu)\}$.

*Proof.* (1) ($\Rightarrow$) Suppose that $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(\beta)$. Since $[M, N]$ is regular, there exists $\mu \in [N, M]$ such that $\beta = \beta\mu\beta$. For $e = \beta\mu$; $e^2 = e \in E_N$ and $\mathrm{Im}(e) = \mathrm{Im}(\beta)$, so $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(e)$. Thus, for all $x \in M$, $e(\alpha(x)) = \alpha(x)$, so $\alpha = e\alpha = \beta\mu\alpha \in \beta E_M$. Therefore, $\alpha[N, M] \subseteq \beta E_M[N, M] \subseteq \beta[N, M]$.
($\Leftarrow$) Suppose that $\alpha[N, M] \subseteq \beta[N, M]$. Since $[M, N]$ is regular, $\alpha = \alpha\lambda\alpha$ for some $\lambda \in [N, M]$. Since $\alpha\lambda \in \alpha[N, M] \subseteq \beta[N, M]$, $\alpha\lambda = \beta\delta$ for some $\delta \in [N, M]$. Thus, $\mathrm{Im}(\alpha) = \mathrm{Im}(\alpha\lambda\alpha) = \mathrm{Im}(\beta\delta\alpha) \subseteq \mathrm{Im}(\beta)$.
(2) and (3) are clear by (1).
(4) ($\Rightarrow$) Suppose that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$. Then $\beta(\mathrm{Ker}(\alpha)) = 0$. Since $[M, N]$ is regular, there exists $\mu \in [N, M]$ such that $\alpha = \alpha\mu\alpha$. For $e = \mu\alpha \in E_M$; $e^2 = e$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(e)$, so $\beta(\mathrm{Ker}(\alpha)) = \beta(\mathrm{Ker}(e)) = \beta(\mathrm{Im}(1_M - e)) = \mathrm{Im}(\beta(1_M - e)) = 0$. Thus, $\beta(1_M - e) = 0$ and that $\beta = \beta e = \beta\mu\alpha \in (E_N)\alpha$. So $[N, M]\beta \subseteq [N, M](E_N)\alpha \subseteq [N, M]\alpha$.
($\Leftarrow$) Suppose that $[N, M]\beta \subseteq [N, M]\alpha$. Since $[M, N]$ is regular, $\beta = \beta\delta\beta$ for some $\delta \in [N, M]$ and $\delta\beta \in [N, M]\beta \subseteq [N, M]\alpha$. So $\delta\beta = \lambda\alpha$ for some $\lambda \in [N, M]$. Thus, $\beta = \beta\lambda\alpha$ and $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$.
(5) and (6) are clear by (4). □

### 3. Semi M-Projective Modules.

**Theorem 3.1.** *Let $M_R$ and $N_R$ be modules. The following are equivalent:*

(1) *For every submodule $K \subseteq N$ and every epimorphism $\alpha : M \to K$, homomorphism $\lambda : N \to K$ there exists $\beta : N \to M$ such that $\alpha\beta = \lambda$.*

(2) *For every $\alpha \in [M, N]$, $\alpha[N, M] = [N, Im(\alpha)]$.*

(3) *For every $\alpha \in [M, N]$, $\alpha[N, M] = \{\lambda : \lambda \in E_N; Im(\lambda) \subseteq Im(\alpha)\}$.*

*Proof.* (1) $\Rightarrow$ (2) Let $\alpha \in [M, N]$. It is clear that $\alpha[N, M] \subseteq [N, \mathrm{Im}(\alpha)]$. Let $\lambda \in [N, \mathrm{Im}(\alpha)]$. Then by assumption there exists $\beta \in [N, M]$ such that $\alpha\beta = \lambda$, so $\lambda \in \alpha[N, M]$.
(2) $\Rightarrow$ (1) Let $K$ be a submodule of $N$, $\alpha : M \to K$ be an epimorphism and $\lambda : N \to K$ be a homomorphism. Since $\mathrm{Im}(\lambda) \subseteq K = \mathrm{Im}(\alpha)$, we have $\lambda \in [N, \mathrm{Im}(\alpha)] = \alpha[N, M]$ by assumption. So there exists $\beta \in [N, M]$ such that $\alpha\beta = \lambda$, which proves (1). The equivalence (2) $\Leftrightarrow$ (3) is clear. □

Let $M_R$ and $N_R$ be modules. Now a module $N$ is called semi $M-$projective if $M, N$ are satisfy the conditions of Theorem 3.1.
We remark that a module $M_R$ is semi projective [6], if and only if $M$ is a semi $M-$projective module.

**Theorem 3.2.** *Let $M_R$ and $N_R$ be modules. The following are equivalent:*

(1) $[M, N]$ *is regular.*

(2) *For every $\alpha \in [M, N]$, $Im(\alpha) \subseteq^{\oplus} N$, and $N$ is a semi $M-$projective module.*

(3) *For every finite set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subseteq [M, N]$, $\Sigma_{i=1}^{n} Im(\alpha_i) \subseteq^{\oplus} N$, and $N$ is a semi $M-$projective module.*

*Proof.* (1) $\Rightarrow$ (2) If $\alpha \in [M, N]$. Then $\mathrm{Im}(\alpha) \subseteq^{\oplus} N$ by Lemma 2.9. On the other hand, since $[M, N]$ is regular, $\alpha[N, M] = \{\mu : \mu \in E_N;\ \mathrm{Im}(\mu) \subseteq \mathrm{Im}(\alpha)\}$ for every $\alpha \in [M, N]$ by Proposition 2.10(3). So Theorem 3.1(3) implies that $N$ is semi $M-$projective.

(2) $\Rightarrow$ (3) We prove (3) by induction on $n$. The case $n = 1$ holds by (2). Assume that $n > 1$ and $\Sigma_{i=1}^{n-1}\mathrm{Im}(\alpha_i) = \mathrm{Im}(e)$, where $e^2 = e \in E_N$, hence $\Sigma_{i=1}^{n-1}\mathrm{Im}(\alpha_i) \subseteq^{\oplus} N$. Since $1_N - e \in E_N$ and $\alpha_n \in [M, N]$, $(1_N - e)\alpha_n \in [M, N]$, so $\mathrm{Im}((1 - e)\alpha_n) \subseteq^{\oplus} N$ and by assumption $\mathrm{Im}((1 - e)\alpha_n) = \mathrm{Im}(f)$ where $f^2 = f \in E_N$. Then $ef = 0$, and for $\mu = e + f - fe$, we have $\mu^2 = \mu \in E_N$. Since $f\mu = f$ and $\mu e = e$, $\mathrm{Im}(\mu) = \mathrm{Im}(e) + \mathrm{Im}(f)$. Therefore $\mathrm{Im}(\alpha) = \mathrm{Im}(e) + \mathrm{Im}(f) = \mathrm{Im}(e) + \mathrm{Im}((1-e)\alpha_n)$. Thus $\Sigma_{i=1}^{n}\mathrm{Im}(\alpha_i) = \Sigma_{i=1}^{n-1}\mathrm{Im}(\alpha_i) + \mathrm{Im}(\alpha_n) = \mathrm{Im}(e) + \mathrm{Im}(\alpha_n) = \mathrm{Im}(e) + \mathrm{Im}((1 - e)\alpha_n) = \mathrm{Im}(e) + \mathrm{Im}(f) = \mathrm{Im}(\mu) \subseteq^{\oplus} N$, proving (3).

(3) $\Rightarrow$ (1) Let $\alpha \in [M, N]$. Then $\mathrm{Im}(\alpha) \subseteq^{\oplus} N$ by assumption. Denote by $\pi : N \to \mathrm{Im}(\alpha)$ the projection. Then $\pi \in E_N$ and $\mathrm{Im}(\pi) = \mathrm{Im}(\alpha)$, so $\pi \in [N, \mathrm{Im}(\alpha)] = \alpha[N, M]$ by Theorem 3.1, hence $N$ is semi $M-$projective. Thus $\pi = \alpha\beta$ for some $\beta \in [N, M]$. Since $\alpha(x) \in \mathrm{Im}(\alpha) = \mathrm{Im}(\pi)$ for all $x \in M$; $\pi\alpha(x) = \alpha(x)$ and $\alpha(x) = \alpha\beta\alpha(x)$, so $\alpha = \alpha\beta\alpha$, which shows that $\alpha$ is regular, proving (1). □

Following [4, Lemma 2.1], for any module $M$, $\mathrm{Ker}(\alpha) \subseteq^{\oplus} M$ for every $\alpha \in E_M$ if and only if $\mathrm{Im}(\alpha)$ is semi $M-$projective for every $\alpha \in E_M$. From [4, Lemma 2.1] and Theorem 3.2 in case $N = M$, we derive the following:

**Corollary 3.3.** *Let $M_R$ be a module. The following are equivalent:*

(1) *$E_M$ is a regular rings.*

(2) *For every $\alpha \in E_M$; $Im(\alpha) \subseteq^{\oplus} M$ and $M$ is a semi projective module.*

(3) *For every $\alpha \in E_M$, $Ker(\alpha) \subseteq^{\oplus}$ and $Im(\alpha)$ is semi $M-$projective for every $\alpha \in E_M$.*

(4) *For every finite set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subseteq E_M$; $\Sigma_{i=1}^{n} Im(\alpha_i) \subseteq^{\oplus} M$, and $M$ is a semi projective module.*

Following [2], let $M_R, N_R$ be modules, the co-singular ideal of $[M, N]$ is

$$\nabla[M, N] = \{\alpha : \alpha \in [M, N];\ \mathrm{Im}(\alpha) \ll N\}.$$

**Corollary 3.4.** *Let $M_R$ and $N_R$ be modules. If $N$ is semi $M-$projective, then:*

(1) $J[M, N] = \widehat{\nabla}[M, N]$.

(2) $\nabla[M, N] \subseteq J[M, N]$.

*Proof.* (1) By Lemma 2.8 we have $J[M, N] \subseteq \widehat{\nabla}[M, N]$.

Let $\alpha \in \widehat{\nabla}[M,N]$. Then $\mathrm{Im}(1_N - \alpha\beta) = N$ for all $\beta \in [N,M]$, by Lemma 2.1(2), $\mathrm{Im}(\alpha - \alpha\beta\alpha) = \mathrm{Im}(\alpha)$. Since $N$ is semi $M-$projective and that $\alpha, \alpha - \alpha\beta\alpha \in [M,N]$, $[N, \mathrm{Im}(\alpha - \alpha\beta\alpha)] = (\alpha - \alpha\beta\alpha)[N,M]$, $[N, \mathrm{Im}(\alpha)] = \alpha[N,M]$, so $\alpha[N,M] = (\alpha - \alpha\beta\alpha)[N,M] = \alpha[N,M] \cap (1_N - \alpha\beta)E_N$, by Lemma 2.3(5). This shows that $\alpha[N,M] \subseteq (1_N - \alpha\beta)E_N$, again by Lemma 2.3(1) it follows that $E_N = (1_N - \alpha\beta)E_N$, thus $\alpha \in J[M,N]$.
(2) Let $\alpha \in \nabla[M,N]$. Then by Lemma 2.1(1), $N = \mathrm{Im}(\alpha) + \mathrm{Im}(1_N - \alpha\beta) = \mathrm{Im}(1_N - \alpha\beta)$ for all $\beta \in [N,M]$, hence $\mathrm{Im}(\alpha) \ll N$. So $\alpha \in \widehat{\nabla}[M,N]$, by (1) $\alpha \in J[M,N]$. □

## 4. Semi N-Injective Modules.

**Theorem 4.1.** *Let $M_R$ and $N_R$ be modules. The following are equivalent:*

(1) *For every factor module $K$ of $M$ and every monomorphism $\alpha : K \to N$, homomorphism $\lambda : K \to M$, there exists $\beta : N \to M$ such that $\beta\alpha = \lambda$.*

(2) *For every $\alpha \in [M,N]$; $[N,M]\alpha = \{\beta : \beta \in E_M; Ker(\alpha) \subseteq Ker(\beta)\}$ $= \{\beta : \beta \in E_M; \beta(Ker(\alpha)) = 0\}$.*

*Proof.* (1) $\Rightarrow$ (2) Let $\alpha \in [M,N]$. It is clear that $[N,M]\alpha \subseteq \{\beta : \beta \in E_M; \mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)\}$.
Let $\beta \in E_M$ such that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$. Then the map $\alpha' : M/Ker(\alpha) \to N$ defined by $\alpha'(\overline{x}) = \alpha(x)$ for all $\overline{x} \in M/\mathrm{Ker}(\alpha)$ is a monomorphism. Also, since $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$, the map $\beta' : M/\mathrm{Ker}(\alpha) \to M$ defined by $\beta'(\overline{x}) = \beta(x)$ for all $\overline{x} \in M/\mathrm{Ker}(\alpha)$ is a homomorphism. By assumption, there exists $\lambda : N \to M$ such that $\lambda\alpha' = \beta'$. Thus, $\lambda\alpha(x) = \lambda\alpha'(\overline{x}) = \beta'(\overline{x}) = \beta(x)$ for all $x \in M$, so $\lambda\alpha = \beta$ and $\beta \in [N,M]\alpha$, proving (2).
(2) $\Rightarrow$ (1) Let $K$ be a factor module of $M$, $\alpha : K \to N$ be a monomorphism and $\lambda : K \to M$ be a homomorphism. Denote by $\pi : M \to K$ the canonical homomorphism of a module $M$ onto factor module $K$. Then $\lambda\pi \in E_M$, $\alpha\pi \in [M,N]$ and $\mathrm{Ker}(\alpha\pi) \subseteq \mathrm{Ker}(\lambda\pi)$. By assumption $\lambda\pi \in [N,M](\alpha\pi)$, so there exists $\beta \in [N,M]$ such that $\lambda\pi = \beta(\alpha\pi)$. Let $y \in K$. Then $y = \pi(x)$ for some $x \in M$ and $\lambda(y) = \lambda\pi(x) = \beta\alpha\pi(x) = \beta\alpha(y)$. Thus, $\lambda = \beta\alpha$, this proves (1). □

Let $M_R$ and $N_R$ be modules. Now a module $M$ is called semi $N-$injective if $M, N$ are satisfy the conditions of Theorem 4.1.
We remark that a module $N_R$ is semi injective [6], if and only if $N$ is a semi $N-$injective module.

**Theorem 4.2.** *Let $M_R$ and $N_R$ be modules. The following are equivalent:*

(1) *$[M,N]$ is regular.*

(2) *For every $\alpha \in [M,N]$, $Ker(\alpha) \subseteq^{\oplus} M$, and $M$ is a semi $N-$injective module.*

(3) *For every finite set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subseteq [M,N]$; $\cap_{i=1}^{n} Ker(\alpha_i) \subseteq^{\oplus} M$, and $M$ is a semi $N-$injective module.*

*Proof.* (1) $\Rightarrow$ (2) If $\alpha \in [M,N]$. Then $\mathrm{Ker}(\alpha) \subseteq^{\oplus} M$ by Lemma 2.9. On the other hand, since $[M, N]$ is regular, $[N, M]\alpha = \{\mu : \mu \in E_M;\ \mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\mu)\}$ for all $\alpha \in [M, N]$ by Proposition 2.10(6). So Theorem 4.1 implies that $M$ is semi $N-$injective.

(2) $\Rightarrow$ (3) We prove (3) by induction on $n$. The case $n = 1$ holds by (2). Assume $n > 1$ and that $X = \cap_{i=1}^{n-1}\mathrm{Ker}(\alpha_i) \subseteq^{\oplus} M$, say $M = X \oplus Y$ where $Y$ is a submodule of $M$. Denote by $\pi : M \to X$ the projection. Then $\alpha_n\pi \in [M, N]$ and $\mathrm{Ker}(\alpha_n\pi) = [X \cap \mathrm{Ker}(\alpha_n)] \oplus Y$. Since $\mathrm{Ker}(\alpha_n\pi) \subseteq^{\oplus} M$ by assumption, $[X \cap \mathrm{Ker}(\alpha_n)] \subseteq^{\oplus} M$. Thus, $\cap_{i=1}^{n}\mathrm{Ker}(\alpha_i) = X \cap \mathrm{Ker}(\alpha_n) \subseteq^{\oplus} M$ which proves (3).

(3) $\Rightarrow$ (1) Let $\alpha \in [M, N]$. Then $\mathrm{Ker}(\alpha) \subseteq^{\oplus} M$ by assumption, say $M = \mathrm{Ker}(\alpha)\oplus P$ for some submodule $P$ of $M$. Denote by $\pi : M \to P$ the projection. Then $\pi \in E_M$ and $\mathrm{Ker}(\pi) = \mathrm{Ker}(\alpha)$. Also, since $\alpha(\mathrm{Ker}(\pi)) = \alpha(\mathrm{Im}(1 - \pi)) = 0$, $\alpha = \alpha\pi$. On the other hand, since $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\pi)$ and $M$ is semi $N-$injective, by assumption $\pi \in [N, M]\alpha$ by Theorem 4.1, so $\pi = \beta\alpha$ for some $\beta \in [N, M]$, which gives $\alpha = \alpha\beta\alpha$, proving (1). □

Taking $N = M$ in Theorem 4.2 gives

**Corollary 4.3.** *Let $M_R$ be a module. The following are equivalent:*

(1) *$E_M$ is a regular ring.*

(2) *For every $\alpha \in E_M$, Ker$(\alpha) \subseteq^{\oplus} M$ and $M$ is a semi injective module.*

(3) *For every finite set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subseteq E_M$; $\cap_{i=1}^{n}$Ker$(\alpha_i) \subseteq^{\oplus} M$, and $M$ is a semi injective module.*

Following [2], let $M_R, N_R$ be modules, the singular ideal of $[M, N]$ is

$$\Delta[M, N] = \{\alpha : \alpha \in [M, N];\ \mathrm{Ker}(\alpha) \leq_e M\}.$$

**Corollary 4.4.** *Let $M_R$ and $N_R$ be modules. If $M$ is semi $N-$injective, then:*

(1) *For any $\alpha, \theta \in [M, N]$ such that Ker$(\alpha) =$ Ker$(\theta)$, then $[N, M]\alpha = [N, M]\theta$.*

(2) *$J[M, N] = \widehat{\Delta}[M, N]$.*

(3) *$\Delta[M, N] \subseteq J[M, N]$.*

*Proof.* (1) Assume $\alpha,\ \theta \in [M, N]$ with $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\theta)$. Let $\beta \in [N, M]\alpha$. Then $\beta \in E_M$ and by Theorem 4.1, $\beta(\mathrm{Ker}(\alpha)) = \{0\}$, so $\beta(\mathrm{Ker}(\theta)) = \{0\}$, thus $\beta \in [N, M]\theta$, therefore $[N, M]\alpha \subseteq [N, M]\theta$. The converse is analogous.

(2) By Lemma 2.8 we have $J[M, N] \subseteq \widehat{\Delta}[M, N]$.

Let $\alpha \in \widehat{\Delta}[M, N]$. Then for all $\beta \in [N, M]$; $\mathrm{Ker}(1_M - \beta\alpha) = \{0\}$, so by Lemma 2.1(6) $\mathrm{Ker}(\alpha - \alpha\beta\alpha) = \mathrm{Ker}(\alpha)$ and by (1), $[N, M](\alpha - \alpha\beta\alpha) = [N, M]\alpha$, hence $\alpha - \alpha\beta\alpha, \alpha \in [M, N]$. Thus by Lemma 2.3(6), $[N, M]\alpha = [N, M](\alpha - \alpha\beta\alpha) = [N, M]\alpha \cap E_M(1_M - \beta\alpha)$, which shows that $[N, M]\alpha \subseteq E_M(1_M - \beta\alpha)$. By Lemma 2.3(4), $E_M = [N, M]\alpha + E_M(1_M - \beta\alpha) = E_M(1_M - \beta\alpha)$, so $\alpha \in J[M, N]$.

(3) Let $\alpha \in \Delta[M, N]$. Then $\mathrm{Ker}(\alpha) \leq_e M$. Since for all $\beta \in [N, M]$, $\mathrm{Ker}(\alpha) \cap \mathrm{Ker}(1_M - \beta\alpha) = \{0\}$ implies that $\mathrm{Ker}(1_M - \beta\alpha) = \{0\}$, so $\alpha \in \widehat{\Delta}[M, N]$, by (2) $\alpha \in J[M, N]$. □

## 5. Direct M-Projective (N-Injective) Modules.

**Lemma 5.1.**([6]) *Let $M_R$ and $N_R$ be modules. The following are equivalent:*

(1) *For any submodule $K$ of $M$ and any direct summand $P$ of $N$ such that $M/K \cong P$ we have $K \subseteq^{\oplus} M$.*

(2) *For any direct summand $P$ of $N$, every epimorphism $\alpha : M \to P$ splits.*

(3) *For every direct summand $K$ of $N$ and every epimorphism $\alpha : M \to K$, there exists $\beta : N \to M$ such that $\alpha\beta = \pi$ where $\pi : N \to K$ is the projection.*

Let $M_R$ and $N_R$ be modules. Recall a module $N$ is direct $M-$projective if $M, N$ are satisfy the conditions of Lemma 5.1. From Lemma 5.1 we derive the following:

**Corollary 5.2.** *Let $M_R$ and $N_R$ be modules. The following are equivalent:*

(1) *A module $N$ is direct $M-$projective.*

(2) *For every direct summand $K$ of $N$ and every epimorphism $\alpha : M \to K$, $\alpha[N, M] = [N, K]$.*

(3) *For every direct summand $K$ of $N$ and every epimorphism $\alpha : M \to K$, $\alpha[N, M] = \{\beta : \beta \in E_N; Im(\beta) \subseteq K\}$.*

*Proof.* (1) $\Rightarrow$ (2) Let $K$ be a direct summand of $N$ and $\alpha : M \to K$ be an epimorphism. It is clear that $\alpha[N, M] \subseteq [N, K]$. Let $\lambda \in [N, K]$. Since $N$ is direct $M-$projective, there exists $\beta \in [N, M]$ such that $\alpha\beta = \pi$. Since $\mathrm{Im}(\lambda) \subseteq K = \mathrm{Im}(\pi)$, for every $x \in N$, $\lambda(x) \in K$, $\pi(\lambda(x)) = \lambda(x)$, so $\lambda = \pi\lambda = \alpha\beta\lambda \in \alpha[N, M]$, proving (2).
(2) $\Rightarrow$ (1) Let $K$ be a direct summand of $N$ and $\alpha : M \to K$ be an epimorphism. Denote by $\pi : N \to K$ the projection. Since $\pi \in [N, K] = \alpha[M, N]$, by assumption, there exists $\beta \in [N, M]$ such that $\alpha\beta = \pi$, proving (1). The equivalence (2) $\Leftrightarrow$ (3) is clear. □

Let $M_R$ and $N_R$ be modules. Recall that $[M, N]$ is semi-potent [7], if for any $\alpha \in [M, N]$, $\alpha \notin J[M, N]$ there exists $\beta \in [N, M]$ such that $0 \neq (\alpha\beta)^2 = \alpha\beta \in E_N$, [6]. In particular, a ring $R$ is called semi-potent (or $I_0-$ring [3]), if every principal right ideal not contained in $J(R)$ contains a nonzero idempotent. $[M, N]$ is called *partial invertible* or *pi* [2], if $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in [N, M]$ which is equivalent to that $[M, N]$ is semi-potent and $J[M, N] = 0$.

**Lemma 5.3.** *Let $M_R$ and $N$ be a modules, $\alpha \in [M, N]$. The following are equivalent:*

(1) *An element $\alpha$ is partial invertible.*

(2) *There exists $0 \neq \beta \in [N, M]$ such that $Im(\alpha\beta)$ and $Ker(\alpha\beta)$ are direct summands of $N$.*

(3) *There exists $0 \neq \beta \in [N, M]$ such that $Im(\beta\alpha)$ and $Ker(\beta\alpha)$ are direct summands of $M$.*

*Proof.* Is obvious. □

**Proposition 5.4.** *Let $M_R$ and $N$ be a modules. The following are equivalent:*

(1) *For every* $\alpha \in [M, N]$, $\alpha$ *is partial invertible.*

(2) *A module* $N$ *is direct* $M-$*projective and for any* $\alpha \in [M, N]$, *there exists* $0 \neq \beta \in [N, M]$ *such that* $Im(\alpha\beta) \subseteq^{\oplus} N$.

*Proof.* (1) $\Rightarrow$ (2) Let $\alpha \in [M, N]$. By assumption $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in [N, M]$. Since $(\alpha\beta)^2 = \alpha\beta \in E_N$, $\mathrm{Im}(\alpha\beta) \subseteq^{\oplus} N$.
Let $K$ be a direct summand of $N$ and $\alpha : M \to K$ be an epimorphism. Denote by $\pi : N \to K$ the projection. By assumption $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in [N, N]$. Then for $e = \alpha\beta$; $0 \neq e^2 = e \in E_N$ and $\mathrm{Im}(e) = \mathrm{Im}(\alpha\beta) \subseteq \mathrm{Im}(\alpha) = K$. Since for any $x \in N$, $x = e(x) + (1_N - e)(x)$ and $e(x) \in K$ implies that $\pi(x) = e(x)$. This shows that $\alpha\beta = \pi$, by Lemma 5.1 which implies that a module $N$ is direct $M-$projective.
(2) $\Rightarrow$ (1) Let $\alpha \in [M, N]$. By assumption there exists $0 \neq \beta \in [N, M]$ such that $\mathrm{Im}(\alpha\beta) \subseteq^{\oplus} N$. Since $N$ is direct $M-$projective, the epimorphism $\alpha\beta : M \to \mathrm{Im}(\alpha\beta)$ splits, so $\mathrm{Ker}(\alpha\beta) \subseteq^{\oplus} N$, by Lemma 5.2 which implies that $\alpha$ is pi. □

**Lemma 5.5.**([6]) *Let* $M_R$ *and* $N_R$ *be modules. The following are equivalent:*

(1) *For any submodule* $K$ *of* $N$ *and any direct summand* $P$ *of* $M$ *such that* $K \cong P$, *we have* $K \subseteq^{\oplus} N$.

(2) *For any direct summand* $P$ *of* $M$, *every monomorphism* $\alpha : P \to N$ *splits.*

(3) *For every direct summand* $K$ *of* $M$ *and every monomorphism* $\alpha : K \to N$, *there exists* $\beta : N \to M$ *such that* $\beta\alpha = \tau$ *where* $\tau : K \to M$ *the inclusion.*

Let $M_R$ and $N_R$ be modules. Recall a module $M$ is direct $N-$injective if $M, N$ are satisfy the conditions of Lemma 5.4. From Lemma 5.5 we derive the following:

**Corollary 5.6.** *Let* $M_R$ *and* $N_R$ *be modules. The following are equivalent:*

(1) *A module* $M$ *is direct* $N-$*injective.*

(2) *For every direct summand* $K$ *of* $M$ *and every monomorphism* $\alpha : K \to N$, $[N, M]\alpha = [K, M]$.

*Proof.* (1) $\Rightarrow$ (2) Assume (1) holds. It is clear that $[N, M]\alpha \subseteq [K, M]$. Let $\lambda \in [K, M]$. By assumption there exists $\beta \in [N, M]$ such that $\beta\alpha = \tau$ where $\tau : K \to M$ is the inclusion, so $\lambda = \lambda\tau = \lambda\beta\alpha \in [N, M]\alpha$, proving (2).
(2) $\Rightarrow$ (1) Assume (2) hold. Let $K$ be a direct summand of $M$, $\alpha : K \to M$ be a monomorphism and $\tau : K \to M$ be the inclusion. By assumption $\tau \in [K, M] = [N, M]\alpha$, so there exists $\beta \in [N, M]$ such that $\beta\alpha = \tau$, proving (1). □

**Proposition 5.7.** *Let* $M_R$ *and* $N$ *be modules. The following are equivalent:*

(1) *For every* $\alpha \in [M, N]$; $\alpha$ *is partial invertible.*

(2) *A module* $M$ *is direct* $N-$*injective and for any* $\alpha \in [M, N]$ *there exists* $0 \neq \beta \in [N, M]$ *such that* $Ker(\beta\alpha) \subseteq^{\oplus} M$.

*Proof.* (1) $\Rightarrow$ (2) Let $\psi \in [M, N]$. By assumption $\beta = \beta\psi\beta$ for some $0 \neq \beta \in [N, M]$. Since $(\beta\psi)^2 = \beta\psi \in E_M$, $\mathrm{Ker}(\beta\psi) \subseteq^{\oplus} M$.
Let $K$ be a direct summand of $M$ and $\alpha : K \to N$ be a monomorphism. Denote by $\pi : M \to K$ the projection. Then $\alpha\pi \in [M, N]$. By assumption $\mu = \mu\alpha\pi\mu$ for

some $0 \neq \mu \in [N, M]$ and $\pi\mu = \pi\mu\alpha\pi\mu$. Then for $e = \pi\mu\alpha\pi$, $0 \neq e^2 = e \in E_M$ and $\mathrm{Im}(e) \subseteq K$. Since for any $x \in M$, $x = e(x) + (1_M - e)(x)$ and $e(x) \in K$ we have that $\pi(x) = e(x)$. So for any $x \in K$, $x = \pi(x) = e(x) = \pi\mu\alpha(x)$. For $\beta = \pi\mu \in [N, M]$, $\beta\alpha = \tau$ where $\tau : K \to M$ the inclusion. By Lemma 5.4 it follows that a module $M$ is direct $N-$injective.

(2) $\Rightarrow$ (1) Let $\alpha \in [M, N]$. By assumption there exists $0 \neq \lambda \in [N, M]$ such that $\mathrm{Ker}(\lambda\alpha) \subseteq^{\oplus} M$. Then $M = \mathrm{Ker}(\lambda\alpha) \oplus K$ for some submodule $K$ of $M$. Since $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\lambda\alpha)$, $\alpha : K \to M$ is a monomorphism. Hence $M$ is direct $N-$injective and so there exists $\varphi \in [N, M]$ such that $\varphi\alpha = \tau$ where $\tau : K \to M$ is the inclusion. Let $\pi : M \to K$ the projection. Note for any $m \in M$, $\pi(m) \in K$ implies that $\varphi\alpha\pi = \pi$ and that $(\pi\varphi)\alpha(\pi\varphi) = \pi\varphi$. Then, for $\beta = \pi\varphi$, $\beta\alpha\beta = \beta$ where $0 \neq \beta \in [N, M]$. This shows that $\alpha$ is *pi*. $\square$

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