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PRINCIPAL FIBRATIONS AND GENERALIZED H-SPACES

Yeon Soo Yoon*

Abstract. For a map $f : A \to X$, there are concepts of H^f-spaces, T^f-spaces, which are generalized ones of H-spaces [17,18]. In general, Any H-space is an H^f-space, any H^f-space is a T^f-space. For a principal fibration $E_k \to X$ induced by $k : X \to X'$ from $\epsilon : PX' \to X'$, we obtain some sufficient conditions to having liftings $H^{\bar{f}}$-structures and $T^{\bar{f}}$-structures on E_k of H^f-structures and T^f-structures on X respectively. We can also obtain some results about H^f-spaces and T^f-spaces in Postnikov systems for spaces, which are generalizations of Kahn's result about H-spaces.

1. Introduction

A map $f : A \to X$ is *cyclic* [14] if there is a map $F : X \times A \to X$ such that $F|_X \sim 1_X$ and $F|_A \sim f$. It is clear that a space X is an H-space if and only if the identity map 1_X of X is cyclic. We called a space X as an *H^f-space for a map $f : A \to X$* [17] if there is a cyclic map $f : A \to X$, that is, there is an H^f-structure $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$, where $j : X \vee A \to X \times A$ is the inclusion. We showed [17] that if a space X is an H-space, then for any space A and any map $f : A \to X$, X is an H^f-space for a map $f : A \to X$, but the converse does not hold. In [1], Aguade introduced a T-space as a space X having the property that the evaluation fibration $\Omega X \to X^{S^1} \to X$ is fibre homotopically trivial. It is easy to show that any H-space is a T-space. However, there are many T-spaces which are not H-spaces in [16]. Let ΣX denotes the reduced suspension of X, and ΩX denotes the based loop space of X. Let τ be the adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$and $\tau(1_{\Sigma X})$

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respectively. It is well known [1] that a space X is a T-space if and only if the evaluating map $e : \Sigma\Omega X \to X$ is cyclic. We called a space X as a *T^f-space for a map* $f : A \to X$ [18] if $e : \Sigma\Omega X \to X$ is f-cyclic, that is, there is a T^f-structure $F : \Sigma\Omega X \times A \to X$ such that $Fj \sim \nabla(e \vee f)$, where $j : \Sigma\Omega X \vee A \to \Sigma\Omega X \times A$ is the inclusion. We also showed [18] that if X is a T-space, then for any space A and any map $f : A \to X$, X is a T^f-space for a map $f : A \to X$, but the converse does not hold. We called a space X as a *G^f-space for a map* $f : A \to X$ [19] if $e : \Sigma\Omega X \to X$ is weakly f-cyclic, that is, $e_{\#}(\pi_n(\Sigma\Omega X)) \subset G_n(A, f, X)$ for all n. For a map $f : A \to X$, there are concepts of H^f-spaces, T^f-spaces and G^f-spaces which are generalized ones of H-spaces. In general, Any H-space is an H^f-space, any H^f-space is a T^f-space and any T^f-space is a G^f-space. In this paper, for a principal fibration $E_k \to X$ induced by $k : X \to X'$ from $\epsilon : PX' \to X'$, we obtain some sufficient conditions to having liftings $H^{\bar{f}}$-structures and $T^{\bar{f}}$-structures on E_k of H^f-structures and T^f-structures on X respectively. We can also obtain some results about H^f-spaces and T^f-spaces in Postnikov systems for spaces, which are generalizations of Kahn's result about H-spaces.

2. Gottlieb sets for maps and generalized H-spaces

Let $f : A \to X$ be a map. A based map $g : B \to X$ is called *f-cyclic* [12] if there is a map $\phi : B \times A \to X$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi} & X \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \nabla} \\ A \vee B & \xrightarrow{(f \vee g)} & X \vee X \end{array}$$

is homotopy commute, where $j : A \vee B \to A \times B$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. We call such a map ϕ an *associated map* of a f-cyclic map g. Clearly, g is f-cyclic iff f is g-cyclic. In the case, $f = 1_X : X \to X$, $g : B \to X$ is called cyclic [14]. We denote the set of all homotopy classes of f-cyclic maps from B to X by $G(B; A, f, X)$ which is called the *Gottlieb set for a map* $f : A \to X$. In the case $f = 1_X : X \to X$, we called such a set $G(B; X, 1, X)$ the *Gottlieb set* denoted $G(B; X)$. In particular, $G(S^n; A, f, X)$ will be denoted by $G_n(A, f, X)$. Gottlieb [3,4] introduced and studied the *evaluation subgroups* $G_n(X) = G_n(X, 1, X)$ of $\pi_n(X)$.

In general, $G(B;X) \subset G(B;A,f,X) \subset [B,X]$ for any map $f : A \to X$ and any space B. However, there is an example [20] such that $G(B,X) \neq G(B;A,f,X) \neq [B,X]$.

The next proposition is an immediate consequence from the definition.

Proposition 2.1.

(1) *For any maps* $f : A \to X, \theta : C \to A$ *and any space* B, $G(B;A,f,X) \subset G(B;C,f\theta,X)$.
(2) $G(B,X) = G(B;X,1_X,X) \subset G(B;A,f,X) \subset G(B;A,*,X) = [B,X]$ *for any spaces* X, A *and* B.
(3) $G(B,X) = \cap\{G(B;A,f,X) | f : A \to X$ *is a map and* A *is a space*$\}$.
(4) *If* $h : C \to A$ *is a homotopy equivalence, then* $G(B;A,f,X) = G(B;C,fh,X)$.
(5) *For any map* $k : X \to Y$, $k_\#(G(B;A,f,X)) \subset G(B;A,kf,Y)$.
(6) *For any map* $k : X \to Y$, $k_\#(G(B,X)) \subset G(B;X,k,Y)$.
(7) *For any map* $s : C \to B, s^\#(G(B;A,f,X)) \subset G(C;A,f,X)$.

Proposition 2.2.

(1) [9] X *is an* H*-space* $\Longleftrightarrow G(B,X) = [B,X]$ *for any space* B.
(2) [16] X *is a* T*-space* $\Longleftrightarrow G(\Sigma C,X) = [\Sigma C,X]$ *for any space* C.
(3) [4] X *is a* G*-space* $\Longleftrightarrow G_n(X) = \pi_n(X)$ *for all* n.

It is clear that any H-space is a T-space and any T-space is a G-space.

Proposition 2.3. *Let* $f : A \to X$ *be a map. Then*

(1) [17] X *is an* H^f*-space* $\Longleftrightarrow G(B;A,f,X) = [B,X]$ *for any space* B.
(2) [18] X *is a* T^f*-space* $\Longleftrightarrow G(\Sigma C;A,f,X) = [\Sigma C,X]$ *for any space* C.
(3) [19] X *is a* G^f*-space* $\Longleftrightarrow G_n(A,f,X) = \pi_n(X)$ *for all* n.

It is clear that any H^f-space is a T^f-space and any T^f-space is a G^f-space.

3. Principal fibrations and generalized H-spaces

Let $f : A \to X$, $f' : A' \to X'$, $l : A \to A'$, $k : X \to X'$ be maps. Then a pair of maps $(k,l) : (X,A) \to (X',A')$ is called *a map from* f *to*

f' if the following diagram is commutative;

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle l}\downarrow & & {\scriptstyle k}\downarrow \\ A' & \xrightarrow{f'} & X'. \end{array}$$

It will be denoted by $(k,l) : f \to f'$.

Given maps $f : A \to X$, $f' : A' \to X'$, let $(k,l) : f \to f'$ be a map from f to f'. Let PX' and PA' be the spaces of paths in X' and A' which begin at $*$ respectively. Let $\epsilon_{X'} : PX' \to X'$ and $\epsilon_{A'} : PA' \to A'$ be the fibrations given by evaluating a path at its end point. Let $p_k : E_k \to X$ be the fibration induced by $k : X \to X'$ from $\epsilon_{X'}$. Let $p_l : E_l \to A$ induced by $l : A \to A'$ from $\epsilon_{A'}$. Then there is a map $\bar{f} : E_l \to E_k$ such that the following diagram is commutative

$$\begin{array}{ccc} E_l & \xrightarrow{\bar{f}} & E_k \\ {\scriptstyle p_l}\downarrow & & {\scriptstyle p_k}\downarrow \\ A & \xrightarrow{f} & X, \end{array}$$

where $E_l = \{(a,\xi) \in A \times PA' | l(a) = \epsilon(\xi)\}$, $E_k = \{(x,\eta) \in X \times PX' | k(x) = \epsilon(\eta)\}$, $\bar{f}(a,\xi) = (f(a), f' \circ \xi)$, $p_k(x,\eta) = x$, $p_l(a,\xi) = a$.

Definition 3.1. *Let X be an H^f-space for a map $f : A \to X$. Then a map $(k,l) : f \to f'$ is called an H^f-primitive if there is an associated map $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$ and $kF(p_k \times p_l) \sim * : E_k \times E_l \to X'$, where $j : X \vee A \to X \times A$ is the inclusion.*

Definition 3.2. *Let X be a T^f-space for a map $f : A \to X$. Then a map $(k,l) : f \to f'$ is called a T^f-primitive if there is an associated map $F : \Sigma\Omega X \times A \to X$ such that $Fj \sim \nabla(e \vee f)$ and $kF(\Sigma\Omega p_k \times p_l) \sim * : \Sigma\Omega E_k \times E_l \to X'$, where $j : \Sigma\Omega X \vee A \to \Sigma\Omega X \times A$ is the inclusion.*

Definition 3.3. [19] *Let X be a G^f-space for a map $f : A \to X$. Then a map $(k,l) : f \to f'$ is called a G^f-primitive if for each m and each map $g : S^m \to X$, there is a map $F : S^m \times A \to X$ such that $Fj \sim \nabla(g \vee f)$, $kF(1 \times p_l) \sim * : S^m \times E_l \to X'$, where $j : S^m \vee A \to S^m \times A$ is the inclusion.*

It is well known that any map $g : S^m \to X$, $g \sim e\Sigma\tau(g) : S^m \to X$. Thus we know the above definition is equivalent to one in [19].

Proposition 3.4.

(1) *If X is an H^f-space for a map $f : A \to X$ and $(k,l) : f \to f'$ is an H^f-primitive, then $(k,l) : f \to f'$ is a T^f-primitive.*

(2) *If X is a T^f-space for a map $f : A \to X$ and $(k,l) : f \to f'$ is an T^f-primitive, then $(k,l) : f \to f'$ is a G^f-primitive.*

Proof. (1) Since $(k,l) : f \to f'$ is an H^f-primitive, there is an associated map $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$ and $kF(p_k \times p_l) \sim * : E_k \times E_l \to X'$. Let $F' = F(e_X \times 1) : \Sigma\Omega X \times A \to X$. Then $F'j' \sim Fj(e_X \vee 1) \sim \nabla(1 \vee f)(e_X \vee 1) = \nabla(e_X \vee f)$, where $j' : \Sigma\Omega X \vee A \to \Sigma\Omega X \times A$ is the inclusion. Moreover, since $(p_k \times p_l)(e_{E_k} \times 1_{E_l}) \sim (e_X \times 1_A)(\Sigma\Omega p_k \times p_l) : \Sigma\Omega E_k \times E_l \to X \times A$, we have that $kF'(\Sigma\Omega p_k \times p_l) \sim kF(e_X \times 1)(\Sigma\Omega p_k \times p_l) \sim kF(p_k \times p_l)(e_{E_k} \times 1_{E_l}) \sim *$. Thus $(k,l) : f \to f'$ is a T^f-primitive.

(2) Since $(k,l) : f \to f'$ is a T^f-primitive, there is an associated map $F : \Sigma\Omega X \times A \to X$ such that $Fj \sim \nabla(e \vee f)$ and $kF(\Sigma\Omega p_k \times p_l) \sim * : \Sigma\Omega E_k \times E_l \to X'$. For each m and each $g : S^m \to X$, let $F' = F(\Sigma\tau(g) \times 1) : S^m \times A \to X$. Then $F'j' \sim Fj(\Sigma\tau(g) \vee 1) \sim \nabla(e \vee f)(\Sigma\tau(g) \vee 1) \sim \nabla(g \vee f)$, where $j' : S^m \vee A \to S^m \times A$ is the inclusion. Moreover, since $(1 \times p_l)(\Sigma\tau(g) \times 1_{E_l}) \sim (\Sigma\tau(g) \times 1_A)(1_{S^m} \times p_l) : S^m \times E_l \to \Sigma\Omega X \times A$, we have that $kF'(1_{S^m} \times p_l) = kF(\Sigma\tau(g) \times 1)(1_{S^m} \times p_l) \sim (kF(\Sigma\Omega p_k \times p_l)(\Sigma\tau(g) \times 1_{E_l}) \sim *(\Sigma\tau(g) \times 1_{E_l}) \sim *$. Thus $(k,l) : f \to f'$ is a G^f-primitive. □

Lemma 3.5.

(1) *A map $l : C \to X$ can be lifted to a map $C \to E_k$ if and only if $kl \sim *$.*

(2) [5] *Given maps $g_i : A_i \to E_k$, $i = 1, 2$ and $g : A_1 \times A_2 \to E_k$ satisfying $p_k g|_{A_i} \sim p_k g_i$, $i = 1, 2$, then there is a map $h : A_1 \times A_2 \to E_k$ such that $p_k h = p_k g$ and $h|_{A_i} \sim g_i, i = 1, 2$.*

Theorem 3.6.

(1) *If X is an H^f-space for a map $f : A \to X$ and $(k,l) : f \to f'$ is an H^f-primitive, then E_k is an $H^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$.*

(2) *If X is a T^f-space for a map $f : A \to X$ and $(k,l) : f \to f'$ is a T^f-primitive, then E_k is a $T^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$.*

Proof. (1) Since $(k,l) : f \to f'$ is an H^f-primitive, there is a map $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$ and $kF(p_k \times p_l) \sim * : E_k \times E_l \to X'$, where $j : X \vee A \to X \times A$ is the inclusion. From Lemma 3.5(1), there is a lifting $F' : E_k \times E_l \to E_k$ of $F(p_k \times p_l) : E_k \times E_l \to E_k$, that is, $p_k F' = F(p_k \times p_l)$. Then $p_k F'|_{E_k} = F(p_k \times p_l)|_{E_k} \sim F|_X p_k \sim p_k 1_{E_k}$ and $p_k F'|_{E_l} = F(p_k \times p_l)|_{E_l} \sim F|_A p_l \sim f p_l = p_k \bar{f}$. Thus we have, from Lemma 3.5(2), that there is a map $\bar{F} : E_k \times E_l \to E_k$ such that $p_k\bar{F} = p_kF' = F(p_k \times p_l)$ and $\bar{F}|_{E_k} \sim 1_{E_k}$, $\bar{F}|_{E_l} \sim \bar{f}$. Thus E_k is an $H^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$. This proves the theorem.

(2) Since $(k, l) : f \to f'$ is a T^f-primitive, there is a map $F : \Sigma\Omega X \times A \to X$ such that $Fj \sim \nabla(e \vee f)$ and $kF(\Sigma\Omega p_k \times p_l) \sim * : \Sigma\Omega E_k \times E_l \to X'$, where $j : X \vee A \to X \times A$ is the inclusion. From Lemma 3.5(1), there is a lifting $F' : \Sigma\Omega E_k \times E_l \to E_k$ of $F(\Sigma\Omega p_k \times p_l) : \Sigma\Omega E_k \times E_l \to E_k$, that is, $p_kF' = F(\Sigma\Omega p_k \times p_l)$. Then $p_kF'|_{\Sigma\Omega E_k} = F(\Sigma\Omega p_k \times p_l)|_{\Sigma\Omega E_k} \sim F|_{\Sigma\Omega X}\Sigma\Omega p_k \sim e\Sigma\Omega p_k \sim p_k e_{E_k}$ and $p_kF'|_{E_l} = F(\Sigma\Omega p_k \times p_l)|_{E_l} \sim F|_A p_l \sim fp_l = p_k\bar{f}$. Thus we have, from Lemma 3.5(2), that there is a map $\bar{F} : \Sigma\Omega E_k \times E_l \to E_k$ such that $p_k\bar{F} = p_kF' = F(\Sigma\Omega p_k \times p_l)$ and $\bar{F}|_{\Sigma\Omega E_k} \sim e_{E_k}$, $\bar{F}|_{E_l} \sim \bar{f}$. Thus E_k is a $T^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$. This proves the theorem. □

Proposition 3.7. [19] *If X is a G^f-space for a map $f : A \to X$ and $(k, l) : f \to f'$ is a G^f-primitive, then E_k is a $G^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$.*

In 1951, Postnikov [13] introduced the notion of the Postnikov system as follows; *A Postnikov system for X(or homotopy decomposition of X)* $\{X_n, i_n, p_n\}$ consists of a sequence of spaces and maps satisfying (1) $i_n : X \to X_n$ induces an isomorphism $(i_n)_\# : \pi_i(X) \to \pi_i(X_n)$ for $i \leq n$. (2) $p_n : X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$. (3) $p_n i_n \sim i_{n+1}$. It is well known fact [11] that if X is a 1-connected space having a homotopy type of CW-complex, then there is a Postnikov system $\{X_n, i_n, p_n\}$ for X such that $p_{n+1} : X_{n+1} \to X_n$ is the fibration induced from the path space fibration over $K(\pi_{n+1}(X), n+2)$ by a map $k^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$. It is well known [7] that if A and X are spaces having the homotopy type of 1-connected countable CW-complexes and $f; A \to X$ is a map, then there exist Postnikov systems $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ for A and X respectively and induced maps $\{f_n : A_n \to X_n\}$ satisfying (1) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & X_n \\ {\scriptstyle k_A^{n+2}}\downarrow & & \downarrow{\scriptstyle k_X^{n+2}} \\ K(\pi_{n+1}(A), n+2) & \xrightarrow{\tilde{f}_\#} & K(\pi_{n+1}(X), n+2), \end{array}$$

that is, $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$. (2) $f_{n+1} : A_{n+1} \to X_{n+1}$ given by $f_{n+1} = \bar{f}_n$ satisfying commute diagram

$$\begin{array}{ccc} A_{n+1}(=E_{k_A^{n+2}}) & \xrightarrow{f_{n+1}=\bar{f}_n} & X_{n+1}=(E_{k_X^{n+2}}) \\ {\scriptstyle p'_n(=p_{k_A^{n+2}})}\downarrow & & {\scriptstyle p_n(=p_{k_X^{n+2}})}\downarrow \\ A_n & \xrightarrow{f_n} & X_n. \end{array}$$

(3) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i'_n}\downarrow & & {\scriptstyle i_n}\downarrow \\ A_n & \xrightarrow{f_n} & X_n. \end{array}$$

THEOREM 3.8. *Let A and X be spaces having the homotopy type of 1-connected countable CW-complexes and $f; A \to X$ a map, and $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ Postnikov systems for A and X respectively.*

(1) *If X is an H^f-space for a map $f : A \to X$, then each X_n is H^{f_n}-space and the all pair of k invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are H^{f_n}-primitive.*

(2) *If X_{n-1} is an $H^{f_{n-1}}$-space and the pair of k-invariants $(k_X^{n+1}, k_A^{n+1}) : f_{n-1} \to \tilde{f}_\#$ is $H^{f_{n-1}}$-primitive, then X_n is an H^{f_n}-space, where f_n is an induced map from f.*

Proof. (1) Clearly $\{X_n \times A_n, i_n \times i'_n, p_n \times p'_n\}$ is a Postnikov system for $X \times A$. Then we have, by Kahn's result [7,Theorem 2.2], that there are families of maps $f_n : A_n \to X_n$ and $F_n : X_n \times A_n \to X_n$ such that $p_n f_n = f_{n-1} p'_n$ and $i_n f \sim f_n i'_n$, and $p_n F_n = F_{n-1}(p_n \times p'_n)$ and $i_n F \sim F_n(i_n \times i'_n)$ for $n = 2, 3, \cdots$ respectively, and $k_X^{n+2} f_n \sim \tilde{f} k_A^{n+2}$, $k_X^{n+2} F_n \sim \tilde{F}_\#(k_X^{n+2} \times k_A^{n+2})$, where $k_A^{n+2} : A_n \to K(\pi_{n+1}(A), n+2)$ and $k_X^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$ are k-invariants of A and X respectively, $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \to K(\pi_{n+1}(X), n+2)$ and $\tilde{F}_\# : K(\pi_{n+1}(X), n+2) \times K(\pi_{n+1}(A), n+2) \approx K(\pi_{n+1}(X \times A), n+2) \to K(\pi_{n+1}(X), n+2)$ are the induced maps by $f : A \to X$ and $F : X \times A \to X$ respectively. Since $F|_X \sim 1$ and $F|_A \sim f$, we know, from Kahn's another result [8, Theorem 1.2], that $F_{n|X_n} = (F|_X)_n \sim 1$ and $F_{n|A_n} = (F|_A)_n \sim f_n$. Thus for each n, there exists an H^{f_n}-structure $F_n : X_n \times A_n \to X_n$ on X_n such that $F_n j_n \sim \nabla(1 \vee f_n)$, where $j_n : X_n \vee A_n \to X_n \times A_n$ is the inclusion and f_n is an induced map from f, and X_n is an H^{f_n}-space. Moreover, since there is a lifting $F_{n+1} : X_{n+1} \times A_{n+1} \to X_{n+1}$ of F_n such that $p_{n+1} F_{n+1} \sim F_n(p_{n+1} \times p'_{n+1})$, we know, from Lemma

3.5(1), that $k_X^{n+2}F_n(p_{n+1}\times p'_{n+1})\sim *$ and all the pair of k-invariants $(k_X^{n+2},k_A^{n+2}):f_n\to\tilde{f}_\#$ are H^{f_n}-primitive, where $\tilde{f}_\#:K(\pi_{n+1}(A),n+2)\to K(\pi_{n+1}(X),n+2)$ is the induced map by $f:A\to X$.

(2) It follows from Theorem 3.6(1). □

Taking $f=1_X,\ f'=1_{K(\pi_{n+1}(X),n+2)},\ l=k=k_X^{n+2}$, we can obtain, from the fact [15] $p_{n+1}:X_{n+1}\to X_n$ is an H-map if and only if k_X^{n+1} is primitive and the above theorem, the following corollary given by Kahn [8].

COROLLARY 3.9. [8, Theorem 1.3] *Let X be space having the homotopy type of 1-connected countable CW-complexes and $\{X_n,i_n,p_n\}$ Postnikov systems for X.*

(1) *If X is an H-space, then each X_n is H-space and all the k invariants k_X^{n+2} is primitive.*
(2) *If X_{n-1} is an H-space and the k-invariants k_X^{n+1} is primitive, then X_n is an H-space, where f_n is an induced map from f.*

THEOREM 3.10. *Let A and X be spaces having the homotopy type of 1-connected countable CW-complexes and $f;A\to X$ a map, and $\{A_n,i'_n,p'_n\}$ and $\{X_n,i_n,p_n\}$ Postnikov systems for A and X respectively.*

(1) *If X is a T^f-space for a map $f:A\to X$, then each X_n is T^{f_n}-space and the all pair of k invariants $(k_X^{n+2},k_A^{n+2}):f_n\to\tilde{f}_\#$ are T^{f_n}-primitive.*
(2) *If X_{n-1} is a $T^{f_{n-1}}$-space and the pair of k-invariants $(k_X^{n+1},k_A^{n+1}):f_{n-1}\to\tilde{f}_\#$ is $T^{f_{n-1}}$-primitive, then X_n is a T^{f_n}-space, where f_n is an induced map from f.*

Proof. (1) Clearly $\{\Sigma\Omega X_n\times A_n,\Sigma\Omega i_n\times i'_n,\Sigma\Omega p_n\times p'_n\}$ is a Postnikov system for $\Sigma\Omega X\times A$. Then we have, by Kahn's result [7,Theorem 2.2], that there are families of maps $f_n:A_n\to X_n$ and $F_n:\Sigma\Omega X_n\times A_n\to X_n$ such that $p_nf_n=f_{n-1}p'_n$ and $i_nf\sim f_ni'_n$, and $p_nF_n=F_{n-1}(\Sigma\Omega p_n\times p'_n)$ and $i_nF\sim F_n(\Sigma\Omega i_n\times i'_n)$ for $n=2,3,\cdots$ respectively, and $k_X^{n+2}f_n\sim\tilde{f}k_A^{n+2}$, $k_X^{n+2}F_n\sim\tilde{F}_\#(k_{\Sigma\Omega X}^{n+2}\times k_A^{n+2})$, where $k_A^{n+2}:A_n\to K(\pi_{n+1}(A),n+2)$ and $k_X^{n+2}:X_n\to K(\pi_{n+1}(X),n+2)$ and $k_{\Sigma\Omega X}^{n+2}:\Sigma\Omega X_n\to K(\pi_{n+1}(\Sigma\Omega X),n+2)$ are k-invariants of A, X and $\Sigma\Omega X$ respectively, $\tilde{f}_\#:K(\pi_{n+1}(A),n+2)\to K(\pi_{n+1}(X),n+2)$ and $\tilde{F}_\#:K(\pi_{n+1}(\Sigma\Omega X),n+2)\times K(\pi_{n+1}(A),n+2)\approx K(\pi_{n+1}(\Sigma\Omega X\times A),n+2)\to K(\pi_{n+1}(X),n+2)$ are the induced maps by $f:A\to X$ and $F:\Sigma\Omega X\times A\to X$ respectively. Since $F|_{\Sigma\Omega X}\sim e$ and $F|_A\sim f$, we know, from Kahn's another result [8, Theorem 1.2], that $F_{n|\Sigma\Omega X_n} = (F|_{\Sigma\Omega X})_n \sim 1$ and $F_{n|A_n} = (F|_A)_n \sim f_n$. Thus for each n, there exists a T^{f_n}-structure $F_n : \Sigma\Omega X_n \times A_n \to X_n$ on X_n such that $F_n j_n \sim \nabla(e \vee f_n)$, where $j_n : \Sigma\Omega X_n \vee A_n \to \Sigma\Omega X_n \times A_n$ is the inclusion and f_n is an induced map from f, and X_n is a T^{f_n}-space. Moreover, since there is a lifting $F_{n+1} : \Sigma\Omega X_{n+1} \times A_{n+1} \to X_{n+1}$ of F_n such that $p_{n+1}F_{n+1} \sim F_n(\Sigma\Omega p_{n+1} \times p'_{n+1})$, we know, from Lemma 3.5(1), that $k_X^{n+2} F_n(\Sigma\Omega p_{n+1} \times p'_{n+1}) \sim *$ and all the pair of k-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are T^{f_n}-primitive, where $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \to K(\pi_{n+1}(X), n+2)$ is the induced map by $f : A \to X$.

(2) It follows from Theorem 3.6(2). □

In [19], the similar result with the above is known as follows.

PROPOSITION 3.11. [19] *Let A and X be spaces having the homotopy type of 1-connected countable CW-complexes and $f; A \to X$ a map, and $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ Postnikov systems for A and X respectively.*

(1) *If X is a G^f-space for a map $f : A \to X$, then each X_n is G^{f_n}-space and the all pair of k invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are G^{f_n}-primitive.*
(2) *If X_{n-1} is a $G^{f_{n-1}}$-space and the pair of k-invariants $(k_X^{n+1}, k_A^{n+1}) : f_{n-1} \to \tilde{f}_\#$ is $G^{f_{n-1}}$-primitive, then X_n is a G^{f_n}-space, where f_n is an induced map from f.*

Taking $f = 1_X$, $f' = 1_{K(\pi_{n+1}(X), n+2)}$, $l = k = k_X^{n+2}$, we can obtain the following corollary given by Haslam[5].

COROLLARY 3.12. [5] *Let X be space having the homotopy type of 1-connected countable CW-complexes and $\{X_n, i_n, p_n\}$ Postnikov systems for X.*

(1) *If X is a G-space, then each X_n is G-space and all the k invariants k_X^{n+2} are G-primitive.*
(2) *If X_{n-1} is a G-space and the k-invariants k_X^{n+1} is G-primitive, then X_n is a G-space, where f_n is an induced map from f.*

References

[1] J. Aguade, *Decomposable free loop spaces*, Canad. J. Math. **39** (1987), 938-955.
[2] B. Eckmann and P. Hilton, *Decomposition homologique d'un polyedre simplement connexe*, ibid **248** (1959), 2054-2558.

[3] D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840-856.
[4] D. H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729-756.
[5] H. B. Haslam, *G-spaces and H-spaces*, Ph. D. Thesis, Univ. of California, Irvine, 1969.
[6] P. Hilton, *Homotopy Theory and Duality*, Gordon and Breach Science Pub., 1965.
[7] D. W. Kahn, *Induced maps for Postnikov systems*, Trans. Amer. Math. Soc. **107** (1963), 432-450.
[8] D. W. Kahn, *A note on H-spaces and Postnikov systems of spheres*, Proc. Amer. Math. Soc. **15** (1964), 300-307.
[9] K. L. Lim, *On cyclic maps*, J. Austral. Math. Soc. (Series A) **32** (1982), 349-357.
[10] K. L. Lim, *Cocyclic maps and coevaluation subgroups*, Canad. Math. Bull. **30** (1987), 63-71.
[11] R. E. Mosher and M. C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row, New York, 1968.
[12] N. Oda, *The homotopy of the axes of pairings* , Canad. J. Math. **17** (1990), 856-868.
[13] M. Postnikov, *On the homotopy type of polyhedra*, Dokl. Akad. Nauk. SSSR **76** (1951), no. 6, 789-791.
[14] K. Varadarajan, *Generalized Gottlieb groups*, J. Indian Math. Soc. **33** (1969), 141-164.
[15] G. W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, New York Inc., 1978.
[16] M. H. Woo and Y. S. Yoon, *T-spaces by the Gottlieb groups and duality*, J. Austral. Math. Soc. (Series A) **59** (1995), 193-203.
[17] Y. S. Yoon, *H^f-spaces for maps and their duals*, J. Korea Soc. Math. Edu. Series B **14**, no. 4, 287-306.
[18] Y. S. Yoon, *Lifting T-structures and their duals*, J. Chungcheong Math. Soc. **20** (2007), no. 3, 245-259.
[19] Y. S. Yoon, *G^f-spaces for maps and Postnikov systems*, J. Chungcheong Math. Soc. **22** (2009), no. 4, 831-841.
[20] Y. S. Yoon, *Generalized Gottlieb groups and generalized Wang homomorphisms*, Sci. Math. Japon. **55** (2002), no. 1, 139-148.

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Department of Mathematics Education
Hannam University
Daejeon 306-791, Republic of Korea
E-mail: yoon@hannam.ac.kr