

ON HENSTOCK-STIELTJES INTEGRALS OF INTERVAL-VALUED FUNCTIONS ON TIME SCALES

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ABSTRACT. In this paper we introduce the interval-valued Henstock-Stieltjes integral on time scales and investigate some properties of these integrals.

1. Introduction and preliminaries

The Henstock integral for real functions was first defined by Henstock [2] in 1963. The Henstock integral is more powerful and simpler than the Lebesgue, Wiener and Feynman integrals. The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [5] in 2006. In 2000, Congxin Wu and Zengtai Gong introduced the concept of the Henstock integral of interval-valued functions [6].

In this paper we introduce the concept of the Henstock-Stieltjes delta integral of interval-valued functions on time scales and investigate some properties of the integral.

A time scale T is a nonempty closed subset of real number \mathbb{R} with the subspace topology inherited from the standard topology of \mathbb{R} . For $t \in T$ we define the forward jump operator $\sigma(t) = \inf\{s \in T : s > t\}$ where $\inf \phi = \sup T$, while the backward jump operator $\rho(t) = \sup\{s \in T : s < t\}$ where $\sup \phi = \inf T$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ of $t \in T$ is defined by $\mu(t) = \sigma(t) - t$, while the backward graininess function $\nu(t)$ of $t \in T$ is defined by $\nu(t) = t - \rho(t)$. For $a, b \in T$ we denote the closed interval $[a, b]_T = \{t \in T : a \leq t \leq b\}$. $\delta = (\delta_L, \delta_R)$ is a Δ -gauge on $[a, b]_T$ if $\delta_L(t) > 0$ on $(a, b]_T$, $\delta_R(t) > 0$ on $(a, b]_T$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$ and $\delta_R(t) \geq \mu(t)$ for each $t \in [a, b]_T$.

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A collection $P = \{([t_{i-1}, t_i], \xi_i) : 1 \leq i \leq n\}$ of tagged intervals is δ -fine Henstock partition of $[a, b]_T$ if $U_{i=1}^n [t_{i-1}, t_i] = [a, b]_T$, $[t_{i-1}, t_i]_T \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$ and $\xi_i \in [t_{i-1}, t_i]_T$ for each $i = 1, 2, \dots, n$.

DEFINITION 1.1 ([5]). A function $f : [a, b] \rightarrow \mathbb{R}$ is Henstock delta integrable (or H_Δ -integrable) on $[a, b]$ if there exists a number A such that for each $\epsilon > 0$ there exists a Δ -gauge δ on $[a, b]$ such that

$$\left| \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) - A \right| < \epsilon$$

for every δ -fine Henstock partition $P = \{([t_{i-1}, t_i], \xi_i) : 1 \leq i \leq n\}$ of $[a, b]$. The number A is called the Henstock delta integral of f on $[a, b]$ and we write $A = (H_\Delta) \int_a^b f$.

DEFINITION 1.2. Let $I_\mathbb{R} = \{I = [I^-, I^+]$ is the closed bounded interval on the real \mathbb{R} , where $I^- = \min\{x : x \in I\}$, $I^+ = \max\{x : x \in I\}$. For $A, B, C \in I_\mathbb{R}$, we define $A \leq B$ iff $A^- \leq B^-$ and $A^+ \leq B^+$, $A + B = C$ iff $A^- + B^- = C^-$ and $A^+ + B^+ = C^+$, and $AB = \{ab : a \in A, b \in B\}$, where $(AB)^- = \min(A^-B^-, A^-B^+, A^+B^-, A^+B^+)$ and $(AB)^+ = \max(A^-B^-, A^-B^+, A^+B^-, A^+B^+)$. Define $d(A, B) = \max(|A^- - B^-|, |A^+ - B^+|)$ as the distance between A and B .

DEFINITION 1.3 ([6]). An interval-valued function $F : [a, b] \rightarrow I_\mathbb{R}$ is Henstock integrable to $I_0 \in I_\mathbb{R}$ on $[a, b]$ if for every $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$d\left(\sum_{i=1}^n F(\xi_i)(t_i - t_{i-1}), I_0\right) < \epsilon$$

whenever $P = \{([t_{i-1}, t_i], \xi_i) : 1 \leq i \leq n\}$ of $[a, b]$ is a δ -fine Henstock partition of $[a, b]$. We write $(IH) \int_a^b F(x)dx = I_0$ and $F \in IH[a, b]$.

2. The interval-valued Henstock-Stieltjes delta integral on time scales

In this section, we will define the Henstock-Stieltjes integral of interval-valued function on time scales and investigate some properties of the integral.

DEFINITION 2.1. Let α be an increasing function on $[a, b]_T$. An interval-valued function $F : [a, b]_T \rightarrow I_\mathbb{R}$ is Henstock delta integrable

with respect to α to $I_0 \in I_{\mathbb{R}}$ on $[a, b]_T$ if for every $\epsilon > 0$ there exists a Δ -gauge δ on $[a, b]_T$ such that

$$d\left(\sum_{i=1}^n F(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})), I_0\right) < \epsilon$$

whenever $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \leq i \leq n\}$ is a δ -fine Henstock partition of $[a, b]_T$. We write $(IHS_{\Delta}) \int_a^b F(x) d\alpha = I_0$ and $F \in IHS_{\Delta}^{\alpha}[a, b]_T$.

REMARK 2.2. If $F(x) = F^{-}(x) = F^{+}(x)$ for all $x \in [a, b]_T$. It is clear that Definition 2.1 implies the real-valued Henstock-Stieltjes integral on $[a, b]_T$.

REMARK 2.3. If $F \in IHS_{\Delta}^{\alpha}[a, b]_T$, then the integral is unique.

THEOREM 2.4. An interval-valued function $F : [a, b]_T \rightarrow I_{\mathbb{R}}$ is Henstock-Stieltjes delta integrable with respect to α on $[a, b]_T$ if and only if $F^{-}, F^{+} \in HS_{\Delta}^{\alpha}[a, b]_T$ and

$$(IHS_{\Delta}) \int_a^b F(x) d\alpha = \left[(HS_{\Delta}) \int_a^b F^{-}(x) d\alpha, (HS_{\Delta}) \int_a^b F^{+}(x) d\alpha \right],$$

where $F(x) = [F^{-}(x), F^{+}(x)]$.

Proof. Let $F \in IHS_{\Delta}^{\alpha}[a, b]_T$. Then there exists an interval $I_0 = [I_0^{-}, I_0^{+}]$ with the property that for each $\epsilon > 0$ there exists a Δ -gauge δ with respect to α on $[a, b]_T$ such that

$$d\left(\sum_{i=1}^n F(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})), I_0\right) < \epsilon$$

whenever $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \leq i \leq n\}$ is a δ -fine Henstock partition of $[a, b]_T$. Let $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \leq i \leq n\}$ be a δ -fine Henstock partition of $[a, b]_T$. Since

$$\begin{aligned} & d\left(\sum_{i=1}^n F(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})), I_0\right) \\ &= \max\left(\left|\sum_{i=1}^n F^{-}(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})) - I_0^{-}\right|, \left|\sum_{i=1}^n F^{+}(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})) - I_0^{+}\right|\right). \end{aligned}$$

Hence,

$$\left|\sum_{i=1}^n F^{-}(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})) - I_0^{-}\right| < \epsilon, \left|\sum_{i=1}^n F^{+}(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})) - I_0^{+}\right| < \epsilon.$$

Conversely, let $F^-, F^+ \in HS_\Delta^\alpha[a, b]_T$. then there exist $H_1, H_2 \in \mathbb{R}$ with the property that given Δ -gauge δ on $[a, b]_T$ such that

$$\left| \sum_{i=1}^n F^-(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})) - H_1 \right| < \epsilon, \left| \sum_{i=1}^n F^+(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})) - H_2 \right| < \epsilon.$$

whenever $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \leq i \leq n\}$ is a δ -fine Henstock partition of $[a, b]_T$. Let $I_0 = [H_1, H_2]$. For any δ -fine Henstock partition $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \leq i \leq n\}$ of $[a, b]_P$, we have

$$d\left(\sum_{i=1}^n F(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})), I_0\right) < \epsilon.$$

Hence $F : [a, b]_T \rightarrow I_{\mathbb{R}}$ is Henstock-Stieltjes delta integrable with respect to α on $[a, b]_T$. \square

From Theorem 2.4 and the properties of Henstock-Stieltjes delta integral, we can easily obtain the following theorems.

THEOREM 2.5. *Let $F, G \in IHS_\Delta^\alpha[a, b]_T$ and $\beta, \gamma \in \mathbb{R}$. Then*

- (1) $\beta F + \gamma G \in IHS_\Delta^\alpha[a, b]_T$ and $(IHS_\Delta[a, b]_T) \int_a^b (\beta F + \gamma G) d\alpha = \beta(IHS_\Delta[a, b]_T) \int_a^b F d\alpha + \gamma(IHS_\Delta[a, b]_T) \int_a^b G d\alpha$
- (2) If $F(x) \leq G(x)$ a.e. in $[a, b]_T$, then $(IHS_\Delta[a, b]_T) \int_a^b F d\alpha \leq (IHS_\Delta[a, b]_T) \int_a^b G d\alpha$

THEOREM 2.6. *Let $F \in IHS_\Delta^\alpha[a, c]_T$ and $F \in IHS_\Delta^\alpha[c, b]_T$. Then $F \in IHS_\Delta^\alpha[a, b]_T$ and*

$$(IHS_\Delta) \int_a^b F d\alpha = (IHS_\Delta) \int_a^c F d\alpha + (IHS_\Delta) \int_c^b F d\alpha$$

THEOREM 2.7. *Let $F, G \in IHS_\Delta^\alpha[a, b]_T$ and $d(F, G)$ be a Henstock-Stieltjes delta integrable on $[a, b]_T$. Then*

$$d\left((IHS_\Delta) \int_a^b F d\alpha, (IHS_\Delta) \int_a^b G d\alpha\right) \leq (H_\Delta) \int_a^b d(F, G) d\alpha$$

Proof. By definition of distance, we have

$$\begin{aligned}
 & d\left((IHS_{\Delta}) \int_a^b F d\alpha, (IHS_{\Delta}) \int_a^b G d\alpha\right) \\
 &= \max \left(\left| \left((IHS_{\Delta}) \int_a^b F d\alpha \right)^{-} - \left((IHS_{\Delta}) \int_a^b G d\alpha \right)^{-} \right|, \right. \\
 &\quad \left. \left| \left((IHS_{\Delta}) \int_a^b F d\alpha \right)^{+} - \left((IHS_{\Delta}) \int_a^b G d\alpha \right)^{+} \right| \right) \\
 &= \max \left(\left| (HS_{\Delta}) \int_a^b (F^{-} - G^{-}) d\alpha \right|, \left| (HS_{\Delta}) \int_a^b (F^{+} - G^{+}) d\alpha \right| \right) \\
 &\leq \max \left((HS_{\Delta}) \int_a^b |F^{-} - G^{-}| d\alpha, (HS_{\Delta}) \int_a^b |F^{+} - G^{+}| d\alpha \right) \\
 &\leq (HS_{\Delta}) \int_a^b d(F, G) d\alpha.
 \end{aligned}$$

□

3. The Henstock-Stieltjes delta integral of fuzzy number valued functions on time scales

DEFINITION 3.1 ([1]). Let $\tilde{A} \in F(\mathbb{R})$ be a fuzzy subset on \mathbb{R} . If for any $\lambda \in [0, 1]$, $A_{\lambda} = [A_{\lambda}^{-}, A_{\lambda}^{+}]$ and $A_1 \neq \emptyset$, where $A_{\lambda} = \{x : \tilde{A}(x) \geq \lambda\}$, then \tilde{A} is called a fuzzy number.

Let $\tilde{\mathbb{R}}$ denote the set of all fuzzy numbers.

DEFINITION 3.2 ([3]). Let $\tilde{A}, \tilde{B} \in \tilde{\mathbb{R}}$, we define $\tilde{A} \leq \tilde{B}$ iff $A_{\lambda} \leq B_{\lambda}$ for all $\lambda \in (0, 1]$, $\tilde{A} + \tilde{B} = \tilde{C}$ iff $A_{\lambda} + B_{\lambda} = C_{\lambda}$ for any $\lambda \in (0, 1]$, $\tilde{A} \cdot \tilde{B} = \tilde{D}$ iff $A_{\lambda} \cdot B_{\lambda} = D_{\lambda}$ for any $\lambda \in (0, 1]$. For $D(\tilde{A}, \tilde{B}) = \sup_{\lambda \in [0, 1]} d(A_{\lambda}, B_{\lambda})$ is called the distance between \tilde{A}, \tilde{B} .

LEMMA 3.3 ([1]). If a mapping $H : [0, 1] \rightarrow I_{\mathbb{R}}, \lambda \mapsto H(\lambda) = [m_{\lambda}, n_{\lambda}]$, satisfies $[m_{\lambda_1}, n_{\lambda_1}] \supset [m_{\lambda_2}, n_{\lambda_2}]$ when $\lambda_1 < \lambda_2$, then

$$\tilde{A} := \bigcup_{\lambda \in (0, 1]} \lambda H(\lambda) \in \tilde{\mathbb{R}}$$

and

$$A_{\lambda} = \bigcap_{n=1}^{\infty} H(\lambda_n),$$

where $\lambda_n = [1 - 1/(n + 1)]\lambda$.

DEFINITION 3.4. Let $\tilde{F} : [a, b]_T \rightarrow \tilde{\mathbb{R}}$ and let α be an increasing on $[a, b]_T$. If the interval-valued function $F_\lambda(x) = [F_\lambda^-(x), F_\lambda^+(x)]$ is Henstock-Stieltjes delta integrable with respect to α on $[a, b]_T$ for any $\lambda \in (0, 1]$, then we say that $\tilde{F}(x)$ is Henstock-Stieltjes delta integrable with respect to α on $[a, b]_T$ and the integral is defined by Henstock-Stieltjes delta integral as follow:

$$\begin{aligned} (FHS_\Delta) \int_a^b \tilde{F}(x) d\alpha &:= \bigcup_{\lambda \in (0, 1]} \lambda (IHS_\Delta) \int_a^b F_\lambda(x) d\alpha \\ &= \bigcup_{\lambda \in (0, 1]} \lambda \left[(HS_\Delta) \int_a^b F_\lambda^- d\alpha, (HS_\Delta) \int_a^b F_\lambda^+ d\alpha \right]. \end{aligned}$$

We will write $\tilde{F} \in FHS_\Delta^\alpha[a, b]_T$.

THEOREM 3.5. $\tilde{F} \in FHS_\Delta^\alpha[a, b]_T$, then $(FHS_\Delta) \int_a^b \tilde{F}(x) d\alpha \in \tilde{\mathbb{R}}$ and

$$\left[(FHS_\Delta) \int_a^b \tilde{F}(x) d\alpha \right]_\lambda = \bigcap_{n=1}^{\infty} (IHS_\Delta) \int_a^b F_{\lambda_n}(x) d\alpha,$$

where $\lambda_n = [1 - 1/(n + 1)]\lambda$.

Proof. Let $H : (0, 1] \rightarrow I_{\mathbb{R}}$ be defined by

$$H(\lambda) = \left[(FH_\Delta) \int_a^b F_\lambda^-(x) d\alpha, (FH_\Delta) \int_a^b F_\lambda^+(x) d\alpha \right].$$

Since $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are increasing and decreasing on λ , respectively, therefore, when $0 < \lambda_1 \leq \lambda_2 \leq 1$ we have $F_{\lambda_1}^-(x) \leq F_{\lambda_2}^-(x)$, $F_{\lambda_1}^+(x) \geq F_{\lambda_2}^+(x)$ on $[a, b]_T$. Thus from Theorem 2.5, we have

$$\begin{aligned} &\left[(HS_\Delta) \int_a^b F_{\lambda_1}^-(x) d\alpha, (HS_\Delta) \int_a^b F_{\lambda_1}^+(x) d\alpha \right] \\ &\supset \left[(HS_\Delta) \int_a^b F_{\lambda_2}^-(x) d\alpha, (HS_\Delta) \int_a^b F_{\lambda_2}^+(x) d\alpha \right]. \end{aligned}$$

Using Theorem 2.5 and Lemma 3.3 we obtain

$$(FHS_\Delta) \int_a^b \tilde{F}(x) d\alpha := \bigcup_{\lambda \in (0, 1]} \lambda \left[(HS_\Delta) \int_a^b F_\lambda^-(x) d\alpha, (HS_\Delta) \int_a^b F_\lambda^+(x) d\alpha \right] \in \tilde{\mathbb{R}}$$

and for all $\lambda \in (0, 1]$,

$$\left[(FHS_{\Delta}) \int_a^b \tilde{F}(x) d\alpha \right]_{\lambda} = \bigcap_{n=1}^{\infty} (IHS_{\Delta}) \int_a^b F_{\lambda_n}(x) d\alpha,$$

where $\lambda_n = [1 - 1/(n + 1)]\lambda$. \square

Using Theorem 3.5 and the properties of (IHS) integral, we can obtain the properties of (FHS_{Δ}) integral. For examples, we get the linearity, monotonicity and interval additivity properties of (FHS_{Δ}) integral.

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