JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **29**, No. 1, 2016
http://dx.doi.org/10.14403/jcms.2016.29.1.73

SOME UMBRAL CHARACTERISTICS OF THE ACTUARIAL POLYNOMIALS

Eun Woo Kim* and Yu Seon Jang**

Abstract. The utility of exponential generating functions is that they are relevant for combinatorial problems involving sets and subsets. Sequences of polynomials play a fundamental role in applied mathematics, such sequences can be described using the exponential generating functions. The actuarial polynomials $a_n^{(\beta)}(x)$, $n = 0, 1, 2, \cdots$, which was suggested by Toscano, have the following exponential generating function:

$$\sum_{n=0}^{\infty} \frac{a_n^{(\beta)}(x)}{n!} t^n = \exp(\beta t + x(1-e^t)).$$

A linear functional on polynomial space can be identified with a formal power series. The set of formal power series is usually given the structure of an algebra under formal addition and multiplication. This algebra structure, the additive part of which agree with the vector space structure on the space of linear functionals, which is transferred from the space of the linear functionals. The algebra so obtained is called the umbral algebra, and the umbral calculus is the study of this algebra. In this paper, we investigate some umbral representations in the actuarial polynomials.

1. Introduction

Let V_1 and V_2 be linear spaces over the same field F. A mapping $T : V_1 \to V_2$ from V_1 to V_2 is said to be a linear transformation, sometimes linear operation, if for any $u, v \in V_1$ and $a, b \in F$

$$T(au + bv) = aT(u) + bT(u). \tag{1.1}$$

The collection of all linear transformations from V_1 to V_2 will be denoted by $\mathbb{L}(V_1, V_2)$. When V is a linear space over F, the linear transformations

---

Received December 15, 2015; Accepted February 22, 2016.

2010 Mathematics Subject Classification: Primary 05A15, 05A40.

Key words and phrases: actuarial polynomials, linear functional, formal power series, Sheffer sequence, umbral calculus.

Correspondence should be addressed to Yu Seon Jang, ysjang@kangnam.ac.kr.

from V to F are called the linear functionals on V (see [5]). The linear functional T from V to F is in $\mathbb{L}(V, F)$. Let $F[\omega]$ be the algebra of all polynomials in a single variable ω over F. Then an element $p(\omega)$ in $F[\omega]$ can be written uniquely as a finite sum

$$p(\omega) = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_n\omega^n \tag{1.2}$$

for some nonnegative integer n and $a_0, a_1, \cdots, a_n \in F$ with $a_n \neq 0$. The degree of $p(\omega)$ is defined by n and is denoted $\deg(p(\omega)) = n$. The set $\mathbb{L}(F[\omega], F)$ of all linear functionals on $F[\omega]$ is linear space which is usually thought of as a vector space over F (see [2]).

Consider an operator $\langle\cdot|\cdot\rangle : \mathbb{L}(F[\omega], F) \times F[\omega] \to F$ such that for all $p(\omega) \in F[\omega]$ and for all $T, S \in \mathbb{L}(F[\omega], F)$

$$\langle T + S|p(\omega)\rangle = \langle T|p(\omega)\rangle + \langle S|p(\omega)\rangle, \tag{1.3}$$

$$\langle cT|p(\omega)\rangle = c\langle T|p(\omega)\rangle, \tag{1.4}$$

where $c \in F$. For any nonnegative integer n there exists polynomial $p_n(\omega)$ in $F[\omega]$ with $\deg(p(\omega)) = n$, thus the linear functional T is uniquely determined by the sequence of constants $\langle T|\omega^n\rangle$ (see [4]). Occasionally, the value of its evaluation has $\langle T|p_n(\omega)\rangle = s_n(x)$ for some $s_n(x) \in F[x]$.

Let $F[[t]]$ be the set of all formal power series in the variable t over field F. An element of $F[[t]]$ has the form

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \tag{1.5}$$

for $a_n \in F$. Two formal power series are equal if and only if the coefficients of like powers of t are equal. If addition and multiplication are defined by

$$\sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} (a_n + b_n) t^n, \tag{1.6}$$

$$\left(\sum_{n=0}^{\infty} a_n t^n\right)\left(\sum_{n=0}^{\infty} b_n t^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) t^n, \tag{1.7}$$

then $F[[t]]$ is a ring (see [2]). It is well known that a linear functional on $F[\omega]$ can be identified with a formal power series. In fact, there is a one-to-one correspondence between $\mathbb{L}(F[\omega], F)$ and $F[[t]]$. A formal power series

$$f_T(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \tag{1.8}$$

is defined by a linear functional T on $F[\omega]$ by setting

$$\langle T|\omega^n\rangle = a_n \tag{1.9}$$

for all $n \geq 0$. Thus we have

$$f_T(t) = \sum_{n=0}^{\infty} \frac{\langle T|\omega^n\rangle}{n!} t^n. \tag{1.10}$$

On the other hand, let $f_T(t) \in F[[t]]$ be the formal power series. Taking $\langle f_T(t)|\omega^n\rangle = a_n$, we have

$$f_T(t) = \sum_{n=0}^{\infty} \frac{\langle f_T(t)|\omega^n\rangle}{n!} t^n. \tag{1.11}$$

From equations (1.10) and (1.11), we have

$$\langle T|\omega^n\rangle = \langle f_T(t)|\omega^n\rangle. \tag{1.12}$$

Let x, ω be the indeterminates in F. Then there exists a unique linear functional $T : F[\omega] \to F$ such that

$$T\left(p_n(\omega)\right) = T\left(\sum_{k=0}^{n} a_k \omega^k\right) = \sum_{k=0}^{n} \psi(a_k) x^k, \quad \text{say } s_n(x), \tag{1.13}$$

where ψ is a homomorphism from F to F with $\psi(1_F) = 1_F$, $\psi(\omega) = x$ and 1_F is identity of F (see [2]). Thus the linear functional T can be defined by the operator $\langle\ \cdot|\cdot\rangle$ as following;

$$T\left(p_n(\omega)\right) = \langle T|p_n(\omega)\rangle = s_n(x) \tag{1.14}$$

for any $p_n(\omega) \in F[\omega]$ with $\deg(p_n(x)) = n$. Therefore for any $T \in \mathbb{L}(F[\omega], F)$ there exists a unique sequence of polynomials $s_n(x), n \geq 0$ such that

$$\sum_{n=0}^{\infty} \frac{s_n(x)}{n!} t^n = f_T(t). \tag{1.15}$$

The function $f_T(t)$ is called the exponential generating function of the sequence polynomials $s_n(x)$ (see [1]). The set $F[[t]]$ of all formal power series is usually given the structure of an algebra under formal addition and multiplication. This algebra structure, the additive part of which agree with the vector space structure on $\mathbb{L}(F[\omega],F)$ which is transferred from $\mathbb{L}(F[\omega],F)$ (see [4]). In this algebra, the new variable x is used instead to the original variable ω. In this viewpoint, the variable ω is called the shadow variable or umbra. The algebra so obtained is called the umbral algebra, and the umbral calculus is the study of this algebra.

When $f_{_T}(t)=\exp(\beta t+x(1-e^t))$, the sequence of polynomials $s_n(x)$ satisfying the relation (1.15) was suggested by Toscano (see [8]). Since the polynomials are used as the useful tool in the solving the problem in the actuarial mathematics, it is called the actuarial polynomials and denoted by $a_n^{(\beta)}(x)$ (see [4, 9]). That is, the actuarial polynomials $a_n^{(\beta)}(x)$ can be represented by

$$\sum_{n=0}^{\infty}\frac{a_n^{(\beta)}(x)}{n!}t^n=\exp(\beta t+x(1-e^t)). \tag{1.16}$$

Recently, Jang et al. studied the characteristics of the special polynomials and umbral representation of the moments in the Poisson distribution (see [3, 6, 7]). In this paper, we investigate some umbral representations in the actuarial polynomials.

2. Umbral characteristics

For any $k,n\geq 0$, if $f_{_T}(t)=t^k$, then

$$\langle f_{_T}(t)|\omega^n\rangle=\langle t^k|\omega^n\rangle=n!\delta_{n,k}, \tag{2.1}$$

where $\delta_{n,k}$ is Kronecker delta which is defined by 1 if $n=k$ and 0 otherwise. Let $f_{_T}(t)$ and $f_{_S}(t)$ be the formal power series related to T and S, respectively. If $\langle f_{_T}|\omega^n\rangle=\langle f_{_S}(t)|\omega^n\rangle$ for any nonnegative integer n, then by uniqueness of T we have $T=S$ and $f_{_T}(t)=f_{_S}(t)$. As a similar result, we have the following lemma.

Lemma 2.1. (see [4]) *For any two polynomials $p(\omega)$ and $q(\omega)$ in $F[\omega]$ if*

$$\langle t^k|p(\omega)\rangle=\langle t^k|q(\omega)\rangle$$

for all $k\geq 0$, then $p(\omega)=q(\omega)$.

The order of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k is not vanish and denoted by *order*$(f(t))$. We take *order*$(f(t))=\infty$ if $f(t)=0$. It is easily to see that

$$order(f(t)g(t))=order(f(t))+order(g(t)). \tag{2.2}$$

LEMMA 2.2. *Let $f(t)$ be a formal power series in $F[\omega]$ with* order$(f(t)) = 1$. *Then there exists a unique sequence $A_n(\omega)$ of polynomials satisfying the conditions*

$$\langle f(t)^k | A_n(\omega)\rangle = n!\delta_{n,k}$$

for all $n, k \geq 0$, where $\delta_{n,k}$ is the Kronecker delta which is defined by 1 if $n = k$ and 0 otherwise.

Proof. Since $\langle f(t)|1\rangle = 0$ and $\langle f(t)|\omega\rangle \neq 0$, for all $k \geq 0$ there exist constants $b_{k,i}$ for which $f(t)^k = \sum_{i=k}^{\infty} b_{k,i}t^i$ with $b_{k,k} \neq 0$. To show that the existence of polynomials $A_n(\omega)$ satisfying the orthogonal conditions, let $A_n(\omega) = \sum_{j=0}^{n} a_{n,j}\omega^j$ for all $n \geq 0$. Then

$$n!\delta_{n,k} = \left\langle \sum_{i=k}^{\infty} b_{k,i}t^i \,\middle|\, \sum_{j=0}^{n} a_{n,j}\omega^j \right\rangle = \sum_{i=k}^{\infty}\sum_{j=0}^{\infty} b_{k,i}a_{n,j}\langle t^i|\omega^j\rangle = \sum_{i=k}^{n} b_{k,i}a_{n,i}i!.$$

Taking $k = n$, we obtain

$$a_{n,n} = \frac{1}{b_{n,n}}.$$

By successively taking $k = n, n-1, \cdots, 0$, we have the coefficients $a_{n,j}$ $(j = 0, 1, \cdots, n)$. For the uniqueness, suppose that there exist $A_n(\omega)$ and $B_n(\omega)$ such that

$$\langle f(t)^k|A_n(\omega)\rangle = \langle f(t)^k|B_n(\omega)\rangle$$

for all $n, k \geq 0$. These conditions imply

$$\langle t^k|A_n(\omega)\rangle = \langle t^k|B_n(\omega)\rangle$$

for all $n, k \geq 0$. By Lemma 2.1 we have the desired result for the uniqueness. □

From (1.11), we get $\langle e^{xt}|\omega^n\rangle$. Thus we have

$$\langle e^{xt}|p(\omega)\rangle = p(x) \tag{2.3}$$

for any $p(\omega) \in F[\omega]$. Therefore

$$\langle e^{xt}|A_n(\omega)\rangle = A_n(x). \tag{2.4}$$

We say that the sequence $A_n(x)$ satisfying equation (2.4) for polynomials $A_n(\omega)$ in Lemma 2.2 is the associated sequence for $f(t)$ (see [4]).

THEOREM 2.3. *If the sequence $A_n(x)$ is associated for $f(t) = \ln(1-t)$ $(|t| < 1)$, then the exponential generating function of $A_n(x)$ is*

$$\sum_{n=0}^{\infty} \frac{A_n(x)}{n!}t^n = e^{x(1-e^t)} \ (|t| < 1).$$

Proof. Since

$$\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n} t^n \quad (|t|<1),$$

we know that

$$\langle f(t)|1\rangle = 0, \quad \text{and} \quad \langle f(t)|\omega\rangle \neq 0.$$

For any formal series $h(t) \in F[[t]]$ we have

$$\begin{aligned}\left\langle \sum_{k=0}^{\infty} \frac{\langle h(t)|A_k(\omega)\rangle}{k!} f(t)^k \,\middle|\, A_n(\omega) \right\rangle =& \sum_{k=0}^{\infty} \frac{\langle h(t)|A_k(\omega)\rangle}{k!} \langle f(t)^k | A_n(\omega)\rangle \\ =& \langle h(t)|A_n(\omega)\rangle.\end{aligned}$$

Then

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|A_k(\omega)\rangle}{k!} f(t)^k.$$

Substituting e^{xt} to $h(t)$, we have

$$e^{xt} = \sum_{n=0}^{\infty} \frac{\langle e^{xt}|A_n(\omega)\rangle}{n!} f(t)^n = \sum_{n=0}^{\infty} \frac{A_n(x)}{n!} \left(\ln(1-t)\right)^n.$$

Thus we have

$$\sum_{n=0}^{\infty} \frac{A_n(x)}{n!} t^n = e^{x(1-e^t)}.$$

This is the completion of the proof. □

From equation (1.16) and Theorem 2.3 we have the following corollary.

COROLLARY 2.4. *The actuarial polynomials $a_n^{(\beta)}(x)$ are represented by*

$$a_n^{(\beta)}(x) = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(x) \beta^k,$$

where $A_n(x)$ is the associated sequence for $\ln(1-t)$ $(|t|<1)$.

Proof. Since

$$e^{x(1-e^t)} = \sum_{n=0}^{\infty} \frac{A_n(x)}{n!} t^n$$

and

$$e^{\beta t} = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} t^n,$$

from Theorem 2.3 we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{a_n^{(\beta)}(x)}{n!}t^n&=\left(\sum_{n=0}^{\infty}\frac{A_n(x)}{n!}t^n\right)\left(\sum_{n=0}^{\infty}\frac{\beta^n}{n!}t^n\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{A_{n-k}(x)\beta^k}{n!}\right)t^n.\end{aligned}$$

Comparing the coefficients in the both sides, we have the desired result. □

Let $f_k(t)$ be a formal power series having the order of k ($k \geq 0$). Then there exists a sequence of polynomials $A_n^*(\omega)$ in $F[\omega]$ such that

$$\langle f_k(t)|A_n^*(\omega)\rangle = n!\delta_{n,k}, \tag{2.5}$$

where $\delta_{n,k}$ is the Kronecker delta. Since $f_k(t) = g(t)t^k$ for some $g(t) \in F[[t]]$ with $order(g(t)) = 0$, thus we have

$$\langle g(t)t^k|A_n^*(\omega)\rangle = n!\delta_{n,k}. \tag{2.6}$$

We say that the sequence $A_n^*(x)$ satisfying equation (2.4) for polynomials $A_n^*(\omega)$ in equation (2.6) is the Appell sequence for $g(t)$ (see [4]).

THEOREM 2.5. *Let $A_n^*(x)$ be Appell sequence for $g(t) = (1-t)^{-\beta}$. Then the exponential generating function of $A_n^*(x)$ is*

$$\sum_{n=0}^{\infty}\frac{A_n^*(x)}{n!}t^n = (1-t)^{\beta}e^{xt}.$$

Proof. For any formal series $h(t) \in F[[t]]$ we have

$$\begin{aligned}\left\langle \sum_{k=0}^{\infty}\frac{\langle h(t)|A_k^*(\omega)\rangle}{k!}g(t)t^k \,\middle|\, A_n^*(\omega)\right\rangle&=\sum_{k=0}^{\infty}\frac{\langle h(t)|A_k^*(\omega)\rangle}{k!}\langle g(t)t^k|A_n^*(\omega)\rangle\\&=\langle h(t)|A_n^*(\omega)\rangle.\end{aligned}$$

Then

$$h(t)=\sum_{k=0}^{\infty}\frac{\langle h(t)|A_k^*(\omega)\rangle}{k!}g(t)t^k.$$

Substituting e^{xt} to $h(t)$, we have

$$e^{xt}=\sum_{n=0}^{\infty}\frac{\langle e^{xt}|A_n^*(\omega)\rangle}{n!}g(t)t^n=\sum_{n=0}^{\infty}\frac{A_n^*(x)}{n!}g(t)t^n.$$

Thus

$$\sum_{n=0}^{\infty}\frac{A_n^*(x)}{n!}t^n = \frac{1}{g(t)}e^{xt}.$$

and finally

$$\sum_{n=0}^{\infty}\frac{A_k^*(x)}{n!}t^n = (1-t)^{\beta}e^{xt}.$$

This is the completion of the proof. □

Let $f(t)$ and $g(t)$ be any formal power series with $order(f(x)) = 1$ and $order(g(x)) = 0$. Then there exists a sequence of polynomials $s_n(\omega)$ in $F[\omega]$ such that

$$\langle g(t)f(t)^k | s_n(\omega)\rangle = n!\delta_{n,k}, \tag{2.7}$$

where $\delta_{n,k}$ is the Kronecker delta. We say that the sequence $s_n(x)$ satisfying equation (2.4) for polynomials $s_n(\omega)$ in equation (2.7) is Sheffer sequence for $(f(t), g(t))$ (see [4]). Since

$$\langle g(t)f(t)^k | s_n(\omega)\rangle = n!\langle g(t)f(t)^k | s_n(\omega)\rangle, \tag{2.8}$$

thus the sequence $s_n(x)$ is Sheffer for $(f(t), g(t))$ if and only if $g(t)s_n(x)$ is associated for $f(t)$. And also since

$$\langle g(t)f(t)^k | s_n(\omega)\rangle = n!\left\langle g(t)t^k \;\middle|\; \left(\frac{f(t)}{t}\right)^k s_n(\omega)\right\rangle, \tag{2.9}$$

we know that the sequence $s_n(x)$ is Sheffer for $(f(t), g(t))$ if and only if $(f(t)/t)^k s_n(x)$ is Appell for $g(t)$.

THEOREM 2.6. *Let* $s_n(x)$ *be Sheffer sequence for* $(\ln(1-t), (1-t)^{-\beta})$ $(|t| < 1)$. *Then the exponential generating function of* $s_n^{(\beta)}(x)$ *is*

$$\sum_{n=0}^{\infty}\frac{s_n(x)}{n!}t^n = \exp(\beta t + x(1-e^t)).$$

Proof. For any formal series $h(t) \in F[[t]]$ we have

$$\begin{aligned}\left\langle \sum_{k=0}^{\infty}\frac{\langle h(t)|s_k(\omega)\rangle}{k!}g(t)f(t)^k \;\middle|\; s_n(\omega)\right\rangle &= \sum_{k=0}^{\infty}\frac{\langle h(t)|s_k(\omega)\rangle}{k!}\langle g(t)f(t)^k|s_n(\omega)\rangle \\ &= \langle h(t)|s_n(\omega)\rangle.\end{aligned}$$

Then

$$h(t) = \sum_{k=0}^{\infty}\frac{\langle h(t)|s_k(\omega)\rangle}{k!}g(t)f(t)^k.$$

Substituting e^{xt} to $h(t)$, we have

$$e^{xt} = \sum_{n=0}^{\infty} \frac{\langle e^{xt} | s_n(\omega) \rangle}{n!} g(t) f(t)^n = \sum_{n=0}^{\infty} \frac{s_n(x)}{n!} (1-t)^{-\beta} \left(\ln(1-t)\right)^n .$$

Thus

$$\sum_{n=0}^{\infty} \frac{s_n(x)}{n!} \left(\ln(1-t)\right)^n = (1-t)^{-\beta} e^{xt}.$$

and finally

$$\sum_{n=0}^{\infty} \frac{s_n(x)}{n!} t^n = e^{\beta t} e^{x(1-e^t)}.$$

This is the completion of the proof. □

From equation (1.16) and Theorem 2.6 we have the following corollary.

Corollary 2.7. *Let* $f(t) = \ln(1-t)$ $(|t| < 1)$ *and* $g(t) = (1-t)^{-\beta}$. *Then the sequence of the actuarial polynomials* $a_n^{(\beta)}(x)$ *is Sheffer sequence for* $(f(t), g(t))$.

References

[1] M. Aigner, *A course in Enumeration*, Springer-Verlag, Berlin Heidellberg, 2007.

[2] T. W. Hungerford, *Algebra*, Springer, 1980.

[3] Y. S. Jang, Combinatorial representations of moments in Poisson distribution, *Far East J. Math.* **96** (2015), no. 7, 855-867.

[4] S. Roman, *The Umbral Calculus*, Dover Publications, 1984.

[5] W. Rudin, *Functional Analysis*, 2nd Edition, McGraw-Hill, 1991.

[6] C. S. Ryoo, D. V. Dolgy, H. I. Kwon, and Y. S. Jang, Functional Equations associated with generalized Bernoulli numbers and polynomials, *Kyungpook Math. J.*, **55** (2015), 29-39.

[7] C. S. Ryoo, H. I. Kwon, J. Yoon, and Y. S. Jang, Representation of higher-order Euler numbers using the solution of Bernoulli equation, *J. Comput. Anal. Appl.* **19** (2015), no. 3, 570-577.

[8] L. Toscano, *Una classe di polinomi della matematica attuariale*, Rivista di Matematica della Università di Parma (in Italian) **1** (1950), 459-470.

[9] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics*, Second Edition, 2002.

[10] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, 4th Edition, Cambridge University Press, 1990.

*
Department of Liberal Education
Kangnam University
Yongin 446-702, Republic of Korea
E-mail: `ewkim72@kangnam.ac.kr`

**
Department of Applied Mathematics
Kangnam University
Yongin 446-702, Republic of Korea
E-mail: `ysjang@kangnam.ac.kr`