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ON GENERALIZED DERIVATIONS OF BCH-ALGEBRAS

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Abstract. The aim of this paper is to introduce the notion of a generalized derivations of BCH-algebras and some related properties are investigated.

1. Introduction

In 1966, Imai and Iseki introduced two classes of abstract algebras, BCK-algebra and BCI-algebras [6]. It is known that the class of BCI-algebras is a generalization of the class of BCK-algebras In 1983, Hu and Li [3] introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK-algebras and BCI-algebras. They have studied a few properties of these algebras. In this paper, we introduce the notion of generalized derivations of BCH-algebras and investigate some properties of generalized derivations in a BCH-algebra. Moreover, we introduce the notions of fixed set and kernel set of generalized derivations in a BCH-algebra and obtained some interesting properties in medial BCH-algebras. Also, we discuss the relations between ideals in a medial BCH-algebras.

2. Preliminary

By a *BCH-algebra,* we mean an algebra $(X, *, 0)$ with a single binary operation "$*$" that satisfies the following identities for any $x, y, z \in X$:

(BCH1) $x * x = 0$,

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(BCH2) $x \leq y$ and $y \leq x$ imply $x = y$, where $x \leq y$ if and only if $x * y = 0$.
(BCH3) $(x * y) * z = (x * z) * y$.

In a BCH-algebra X, the following identities are true for all $x, y \in X$:

(BCH4) $(x * (x * y)) * y = 0$,
(BCH5) $x * 0 = 0$ implies $x = 0$,
(BCH6) $0 * (x * y) = (0 * x) * (0 * y)$,
(BCH7) $x * 0 = x$,
(BCH8) $(x * y) * x = 0 * y$,
(BCH9) $x * y = 0$ implies $0 * x = 0 * y$,
(BCH10) $x * (x * y) \leq y$.

Definition 2.1. Let I be a nonempty subset of a BCH-algebra X. Then I is called an *ideal* of X if it satisfies:

(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.2. A BCH-algebra X is said to be *medial* if it satisfies

$$(x * y) * (z * w) = (x * z) * (y * w)$$

for all $x, y, z, w \in X$.

In a medial BCH-algebra X, the following identity hold:

(BCH11) $x * (x * y) = y$ for all $x, y \in X$.

Definition 2.3. Let X be a BCH-algebra. Then the set $X_+ = \{x \in X | 0 * x = 0\}$ is called a *BCA-part* of X.

Definition 2.4. Let X be a BCH-algebra. Then the set $G(X) = \{x \in X | 0 * x = x\}$.

Definition 2.5. Let X be a BCH-algebra. If we define an operation "$+$", called *addition,* as $x + y = x * (0 * y)$, for all $x, y \in X$, then $(X, +)$ is an abelian group with identity 0 and the additive inverse $-x = 0 * x$, for all $x \in X$.

Remark 2.6. If we have a BCH-algebra $(X, *, 0)$, it follows from the above definition that $(X, +)$ is an abelian group with $-y = 0 * y$, for all $y \in X$. Then we have $x - y = x * y$, for all $x, y \in X$. On the other hand, if we choose an abelian group $(X, +)$ with an identity 0 and define $x * y = x - y$, we get a BCH-algebra $(X, *, 0)$ where $x + y = x * (0 * y)$, for every $x, y \in X$.

For a BCH-algebra X, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$. A BCH-algebra X is said to be *commutative* if for all $x, y \in X$,

$$y * (y * x) = x * (x * y), \quad \text{i.e.,} \quad x \wedge y = y \wedge x.$$

3. Generalized derivations of BCH-algebras

In what follows, let X denote a BCH-algebra unless otherwise specified.

DEFINITION 3.1. Let X be a BCH-algebra. A map $D : X \to X$ is called a *generalized left-right derivation* (briefly, *generalized* (l, r)-*derivation*) of X if there exists a derivation $d : X \to X$ such that

$$D(x * y) = (D(x) * y) \wedge (x * d(y)),$$

for every $x, y \in X$. If D satisfies the identity $D(x * y) = (x * D(y)) \wedge (d(x) * y)$, for all $x, y \in X$, then it is said that D is a *generalized right-left derivation* (briefly, *generalized* (r, l)*-derivation)* of X.

Moreover, If D is both a generalized (l, r) and (r, l)-derivation of X, it is said that D is a *generalized derivation* of X.

EXAMPLE 3.2. Let $X = \{0, 1, 2\}$ be a BCH-algebra with Cayley table as follows:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define a self-map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ 2 & \text{if } x = 2 \end{cases}$$

Then it is easy to check that d is both (l, r) and (r, l)-derivation of a BCH-algebra X. Also, define a map $D : X \to X$ by

$$D(x) = \begin{cases} 2 & \text{if } x = 0, 1 \\ 0 & \text{if } x = 2. \end{cases}$$

It is easy to verify that D is a generalized derivation of X.

EXAMPLE 3.3. Let $X = \{0, 1, 2, 3\}$ be a BCH-algebra with Cayley table as follows:

$*$	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	2	1	0

Define a self-map $d : X \to X$ by

$$d(x) = \begin{cases} 2 & \text{if } x = 0, 1 \\ 0 & \text{if } x = 2, 3 \end{cases}$$

Then it is easy to check that d is a derivation of a BCH-algebra X. Also, define a map $D : X \to X$ by

$$D(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ 2 & \text{if } x = 2, 3. \end{cases}$$

It is easy to verify that D is a generalized derivation of X.

DEFINITION 3.4. A self-map D of a BCH-algebra X is said to be *regular* if $D(0) = 0$.

EXAMPLE 3.5. A generalized derivation D in Example 3.3 is regular.

PROPOSITION 3.6. *Let D be a self-map of a medial BCH-algebra X. Then,*

(1) *If D is a generalized (l, r)-derivation of X, then $D(x) = D(x) \wedge x$, for all $x \in X$,*
(2) *If D is a generalized (r, l)-derivation of X, then $D(0) = 0$ if and only if $D(x) = x \wedge d(x)$, for all $x \in X$.*

Proof. (1) Let D be a generalized (r, l)-derivation of X. Then, for all $x, y \in X$,

$$\begin{aligned} D(x) &= D(x * 0) = (D(x) * 0) \wedge (x * d(0)) \\ &= D(x) \wedge (x * d(0)) = (x * d(0)) * ((x * d(0)) * D(x)) \\ &= (x * d(0)) * ((x * D(x)) * d(0)) && (\text{ since } (x * y) * z = (x * z) * y) \\ &= x * (x * D(x)) && (\text{ since } (x * y) * (t * s) = (x * t) * (y * s)) \\ &= D(x) \wedge x. \end{aligned}$$

(2) Let D be a generalized (r, l)-derivation on X such that $D(0) = 0$. Then

$$D(x * y) = (x * D(y)) \wedge (d(x) * y) \qquad (1)$$

for all $x, y \in X$. Putting $y = 0$ in (1), we have $D(x * 0) = (x * D(0)) \wedge (d(x) * 0)$, that is, $D(x) = (x * 0) \wedge d(x) = x \wedge d(x)$, for all $x \in X$. Conversely, if $D(x) = x \wedge d(x)$, then we have

$$D(0) = 0 \wedge d(0) = d(0) * (d(0) * 0) = d(0) * d(0) = 0.$$

□

PROPOSITION 3.7. *Let D be a generalized derivation of X. If $D(x) * x = 0$ for all $x \in X$, then D is regular.*

Proof. Let $D(x) * x = 0$ for all $x \in X$. Then we have

$$\begin{aligned} D(0) = D(x * x) &= (D(x) * x) \wedge (x * d(x)) \\ &= 0 \wedge (x * d(x)) = (x * d(x)) * ((x * d(x)) * 0) \\ &= (x * d(x)) * (x * d(x)) = 0. \end{aligned}$$

Hence D is regular. □

PROPOSITION 3.8. *Let D be a generalized derivation of X. Then we have for all $x, y \in X$,*

(1) $D(x * y) \leq D(x) * y$,
(2) $D(x * D(x)) = 0$.

Proof. Let D be a generalized derivation of X. Then for all $x, y \in X$,

(1)

$$\begin{aligned} D(x * y) &= (D(x) * y) \wedge (x * d(y)) \\ &= (x * d(x)) * ((x * d(x)) * (D(x) * y)) \\ &\leq D(x) * y. \end{aligned}$$

(2) For any $x \in X$, we have

$$\begin{aligned} D(x * D(x)) &= (D(x) * D(x)) \wedge (x * d(D(x))) \\ &= 0 \wedge (x * d(D(x))) = 0. \end{aligned}$$

□

PROPOSITION 3.9. *Let D be a generalized (l, r)-derivation of X. If there exists $a \in X$ such that $D(x) * a = 0$ for all $x \in X$, then D is regular.*

Proof. Let $D(x) * a = 0$ for all $x \in X$. Then

$$\begin{aligned} 0 = D(x * a) * a &= ((D(x) * a) \wedge (x * d(a))) * a \\ &= (0 \wedge (x * d(a))) * a \\ &= 0 * a, \end{aligned}$$

that is, $a \in X_+$ and so

$$\begin{aligned} D(0) &= D(0 * a) \\ &= (D(0) * a) \wedge (0 * d(a)) \\ &= 0 \wedge (0 * d(a)) = 0. \end{aligned}$$

Hence D is regular. □

PROPOSITION 3.10. *Let D be a generalized (r, l)-derivation of X. If there exists $a \in X$ such that $a * D(x) = 0$ for all $x \in X$, then D is regular.*

Proof. Let $a * D(x) = 0$ for all $x \in X$. Then

$$\begin{aligned} 0 = a * D(a * x) &= a * ((a * D(x)) \wedge (d(a) * x)) \\ &= a * (0 \wedge (d(a) * x)) \\ &= a * 0 = a \end{aligned}$$

that is, $a \in X_+$ and so

$$\begin{aligned} D(0) &= D(a) = D(a * 0) \\ &= (a * D(0)) \wedge (a * d(0)) \\ &= 0 \wedge (a * d(0)) = 0. \end{aligned}$$

Hence D is regular. □

PROPOSITION 3.11. *Let D be a generalized left derivation of X and let D is regular. Then $D : X \to X$ is an identity map if it satisfies $D(x) * y = x * D(y)$ for all $x, y \in X$.*

Proof. Since D is regular, we have $D(0) = 0$. Let $x * D(y) = D(x) * y$ for all $x, y \in X$. Then $D(x) = D(x) * 0 = x * D(0) = x * 0 = x$. Thus D is an identity map. □

DEFINITION 3.12. Let X be a BCH-algebra. A self-map D on X is said to be *isotone* if $x \leq y$ implies $D(x) \leq D(y)$ for $x, y \in X$.

PROPOSITION 3.13. *Let D be a generalized left derivation of X and let D be regular. Then $D(x * y) = D(x) * D(y)$ implies $D(x \wedge y) = D(x) \wedge D(y)$.*

Proof. Let $D(x * y) = D(x) * D(y)$ for all $x, y \in X$. Then we have for all $x, y \in X$,

$$\begin{aligned} D(x \wedge y) &= D(y * (y * x)) \\ &= D(y) * D(y * x) \\ &= D(y) * [D(y) * D(x)] \\ &= D(x) \wedge D(y) \end{aligned}$$

□

Proposition 3.14. *Let D be a generalized derivation of X. If $D(x \wedge y) = D(x) \wedge D(y)$ for all $x, y \in X$, then D is isotone.*

Proof. Let $D(x \wedge y) = D(x) \wedge D(y)$ and $x \leq y$ for all $x, y \in X$. Then $x * y = 0$. Thus, we have

$$\begin{aligned} D(x) &= D(x * 0) \\ &= D(x * (x * y)) \\ &= D(y \wedge x) \\ &= D(y) \wedge D(x) \\ &= D(x) * [D(x) * D(y)] \\ &\leq D(y). \end{aligned}$$

Hence we get $D(x) \leq D(y)$, and so D is isotone. □

Proposition 3.15. *Let D be a generalized derivation of a medial BCH-algebra X. Then $D(x * y) = D(x) * y$ for all $x, y \in X$.*

Proof. Let $x, y \in X$. Then we have

$$D(x*y) = (D(x)*y) \wedge (x*d(y)) = (x*d(y))*((x*d(y))*(D(x)*y)) = D(x)*y.$$

□

Proposition 3.16. *Let D be a generalized (l, r)-derivation of a medial BCH-algebra X. Then, the following conditions hold,*

(1) $D(a) = D(0) + a$, for all $a \in X$,

(2) $D(a + x) = D(a) + x$, for all $a, x \in X$,

(3) $D(a + b) = D(a) + b = a + D(b)$, for all $a, b \in X$.

Proof. (1) Let D be a generalized (l, r)-derivation of a medial BCH-algebra X. Then we have

$$D(a) = D(0 * (0 * x)) = (D(0) * (0 * a)) \wedge (0 * d(0 * a)) = D(0) * (0 * a),$$

which implies $D(a) = D(0) + a$, for all $a \in X$.

(2) For all $a, x \in X$, we have

$$D(a+x) = D(a*(0*x)) = (D(a)*(0*x)) \wedge (a*d(0*x)) \\ = D(a)*(0*x) = D(a)+x.$$

(3) Since $(X, +)$ is an abelian group, we get

$$D(a)+b = D(a+b) = D(b+a) = D(b)+a,$$

for all $a, b \in X$. $\square$

PROPOSITION 3.17. *Let D be a generalized (r,l)-derivation of a medial BCH-algebra X. Then, the following conditions hold,*
(1) $D(a) \in G(X)$, for all $a \in G(X)$,
(2) $D(a) = a*D(0) = a+D(0)$, for all $a \in X$,
(3) $D(a+b) = D(a)+D(b)-D(0)$, for all $a, b \in X$,
(4) D is an identity map on X if and only if $D(0)=0$.

Proof. (1) For $a \in G(X)$, we have

$$D(a) = D(0*a) = (0*D(a)) \wedge (d(0)*a) = 0*D(a),$$

which implies $D(a) \in G(X)$.

(2) Now, since $D(a) = D(a*0) = (a*D(0)) \wedge (d(a)*0)$, for all $a \in X$, we have

$$D(a) = a*D(0) = a*D(0*0) = a*(0*D(0)) = a+D(0).$$

(3) By (2) we get $D(a+b) = (a+b)+D(0)$ and $D(b) = b+D(0)$. Since $(X,+)$ is an abelian group, we have

$$D(a+b) = (a+b)+D(0) = (a+D(0))+b \\ = D(a)+b = D(a)+(D(b)-D(0)) \\ = D(a)+D(b)-D(0).$$

(4) If $D(0)=0$, then we have, for every $a \in X$,

$$D(a) = D(a*0) = a*D(0) = a*0 = a,$$

which implies D is an identity map on X. Conversely, if D is an identity map on X, then $D(a)=a$ for all $a \in X$, and so $D(0)=0$. $\square$

DEFINITION 3.18. A BCH-algebra X is said to be *Torsion free* if it satisfies

$$x+x=0 \Rightarrow x=0,$$

for all $x \in X$.

If there exists a nonzero element $x \in X$ such that $x+x=0$, then X is not Torsion free.

EXAMPLE 3.19. Let $X = \{0, a, b, c\}$ be a BCH-algebra with Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	c	0	c
c	c	0	0	0

Then X is a Torsion free since $0+0 = 0*(0*0) = 0, a+a = a*(0*a) = a*0 = a, b+b = b*(0*b) = b*0 = b, c+c = c*(0*c) = c*0 = c$. But in Example 3.2, X is not a Torsion free since $2+2 = 2*(0*2) = 2*2 = 0$.

THEOREM 3.20. *Let X be a Torsion free BCH-algebra and let D_1 and D_2 be generalized derivations of X. If $D_1 D_2 = 0$ on X, then $D_2 = 0$ on X.*

Proof. Let $x \in X$. Then $x + x \in X$, and so we have

$$\begin{aligned}
0 &= (D_1 D_2)(x + x) \\
&= D_1(D_2(x + x)) \\
&= D_1(0) + D_2(x + x) \quad \text{(since } D(a) = D(0) + a) \\
&= D_1(0) + D_2(x) + D_2(x) - D_2(0) \quad \text{(by proposition 3.17 (3))} \\
&= D_1(0) - D_2(0) + D_2(x) + D_2(x) \\
&= (D_1(0) * D_2(0)) + D_2(x) + D_2(x) \\
&= (D_1(0) * (0 * D_2(0)) + D_2(x) + D_2(x) \\
&= D_1(D_2(0)) + D_2(x) + D_2(x) \\
&= (D_1 D_2(0)) + D_2(x) + D_2(x) \\
&= 0 + D_2(x) + D_2(x) \\
&= D_2(x) + D_2(x).
\end{aligned}$$

Since X is Torsion free, we have $D_2(x) = 0$, for all $x \in X$, and so $D_2 = 0$ on X. $\square$

In the above theorem, if we replace both the generalized derivations D_1 and D_2 by a generalized derivation D itself, we get the following corollary.

COROLLARY 3.21. *Let X be a Torsion free BCH-algebra and let D be a generalized derivation. If $D^2 = 0$, then $D = 0$ on X.*

Proof. Let $D^2 = 0$ on X. Then $D^2(x) = 0$, for all $x \in X$. Now, for any $x \in X$,

$$\begin{aligned} 0 = D^2(x+x) &= D(D(x+x) \\ &= D(0) + D(x+x) \qquad \text{(since } D(a) = D(0) + a) \\ &= D(0) + D(x) + D(x) - D(0) \\ &= D(x) + D(x). \end{aligned}$$

Since X is Torsion free, we have $D(x) = 0$, for all $x \in X$, proving $D = 0$, for all $x \in X$. □

Let D be a generalized derivation of X. Define a set $Fix_D(X)$ by

$$Fix_D(X) = \{x \in X \mid D(x) = x\}.$$

PROPOSITION 3.22. *Let D be a generalized derivation of a medial BCH-algebra X. If $x \in Fix_D(X)$ and for any $y \in X$, then $x * y \in Fix_D(x)$.*

Proof. Let $x \in Fix_D(X)$ and $y \in X$. Then $D(x) = x$, and so we have

$$\begin{aligned} D(x*y) &= (D(x)*y) \wedge (x*d(x)) \\ &= (x*y) \wedge (x*d(y)) \\ &= (x*d(y))*[(x*d(y))*(x*y)] \\ &= x*y \end{aligned}$$

which implies $x * y \in Fix_D(X)$. □

PROPOSITION 3.23. *Let D be a generalized derivation of a medial BCH-algebra X. If $x \in Fix_D(X)$ and $y \in X$, then $x \wedge y \in Fix_D(X)$.*

Proof. Let $x \in Fix_D(X)$ and $y \in X$. Then $D(x) = x$, and so we have

$$\begin{aligned} D(x \wedge y) &= D(x*(x*y)) \\ &= (D(x)*(x*y)) \wedge (x*d(x*y)) \\ &= (x*(x*y)) \wedge (x*d(x*y)) \\ &= (x*d(x*y))*[(x*d(x*y))*(x*(x*y))] \\ &= x*(x*y) = x \wedge y, \end{aligned}$$

which implies $x \wedge y \in Fix_D(X)$. □

PROPOSITION 3.24. *Let D be a generalized derivation of X. If $x \in Fix_D(X)$, then we have $(D \circ D)(x) = x$.*

Proof. Let $x \in Fix_D(X)$. Then we have

$$(D \circ D)(x) = D(D(x)) = D(x) = x.$$

This completes the proof. $\square$

THEOREM 3.25. *Let D be a generalized derivation of a medial BCH-algebra of X. If $Fix_D(X) \neq \phi$, then D is regular.*

Proof. Let $y \in Fix_D(X)$. Then we get $D(y) = y$ and

$$\begin{aligned} D(0) &= D(0 \wedge y) \\ &= D(y * (y * 0)) \\ &= (D(y) * (y * 0)) \wedge (y * d(y * 0)) \\ &= (y * (y * 0)) \wedge (y * d(y)) \\ &= (y * y) \wedge (y * d(y)) \\ &= 0 \wedge (y * d(y)) = 0. \end{aligned}$$

Hence D is regular. $\square$

THEOREM 3.26. *Let D be a generalized derivation of a medial BCH-algebra X. Then $Fix_D(X)$ is an ideal of X.*

Proof. Let X be a medial BCH-algebra and let D be a generalized derivation of X. Then by Theorem 3.25, D is regular, and so $0 \in Fix_D(X)$. Let $x * y \in Fix_D(X)$ and $y \in Fix_D(X)$. Then we get $D(x * y) = x * y$ and $D(y) = y$. Thus we have

$$\begin{aligned} D(x) = D(x \wedge y) &= D(y * (y * x)) \\ &= (D(y) * (y * x)) \wedge (y * d(y * x)) \\ &= (y * (y * x)) \wedge (y * d(y * x)) \\ &= (y * d(y * x)) * [(y * d(y * x)) * (y * (y * x))] \\ &= y * (y * x) = x, \end{aligned}$$

which implies $x \in Fix_D(X)$. This implies that $Fix_D(X)$ is an ideal of X. $\square$

THEOREM 3.27. *Let D is a generalized derivation of X and let D is regular. Then the following identities are equivalent:*

(1) *D is an isotone generalized derivation of X.*
(2) *$x \leq y$ implies $D(x * y) = D(x) * D(y)$.*

Proof. (1) $\Rightarrow$ (2). Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$. Hence $D(x * y) = D(0) = 0 = D(x) * D(y)$ since $D(x) \leq D(y)$.

(2) $\Rightarrow$ (1). Let $x \leq y$. Then $0 = D(0) = D(x * y) = D(x) * D(y)$, which implies $D(x) \leq D(y)$. □

Let D be a generalized derivation of X. Define a $KerD$ by

$$KerD = \{x \mid D(x) = 0\}$$

for all $x \in X$.

Proposition 3.28. *Let D be a generalized (r, l)-derivation of a medial BCH-algebra X and let D is regular. Then $KerD$ is an ideal of X.*

Proof. Clearly, $0 \in KerD$. Let $x * y \in KerD$ and $y \in KerD$. Then we have $0 = D(x * y) = (x * D(y)) \wedge (d(x) * y) = x * D(y) = x * 0 = x$, and so $D(x) = D(0) = 0$. This implies $x \in KerD$. Hence $KerD$ is an ideal of X. □

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