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ON SOLUTION SET FOR CONVEX OPTIMIZATION PROBLEM WITH CONVEX INTEGRABLE OBJECTIVE FUNCTION AND GEOMETRIC CONSTRAINT SET

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Abstract. In this paper, we consider a convex optimization problem with a convex integrable objective function and a geometric constraint set. We characterize the solution set of the problem when we know its one solution.

1. Introduction and preliminaries

Convex optimization problems often have multiple solutions. Recently, Mangasarian [10] established simple and complete characterizations for the solution set of the problem when we knew one solution of the problem. Since then, many authors have studied such characterizations for solution sets of several classes of optimization problems [2, 3, 5, 6, 8, 9, 11, 12, 13, 15]. In particular, Jeyakumar, Lee and Dinh [5] showed that the Lagrangian function of a cone-constrained convex optimization problem, which has a cone-inequality constraint, is constant on its solution set, and then derived the Lagrange multiplier based characterizations of the solution set when we know one solution of the solution set. Moreover, Jeyakumar, Lee and Li [7] developed the characterizations of the solution sets to convex optimization problems in the face of data uncertainty.

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In this paper, we characterize all the solutions of a convex optimization problem with a convex integrable objective function and a geometric constraint set when we know its one solution.

We begin this section by fixing notation and definitions. Throughout this paper, we denote the Euclidean space with dimension n by $\mathbb{R}^n$. The inner product $\langle \cdot, \cdot \rangle$ is defined on $\mathbb{R}^n$. The norm of $x \in \mathbb{R}^n$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if for all $\mu \in [0,1]$, $\varphi((1-\mu)x + \mu y) \leq (1-\mu)\varphi(x) + \mu\varphi(y)$ for all $x, y \in \mathbb{R}^n$. Let A be a closed and convex set in a Hilbert space H which has the inner product $\langle \cdot, \cdot \rangle$ defined on H. The indicator function δ_A respect to a subset A of H, is defined by

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

The (convex) normal cone of A at a point $x \in H$ is defined as

$$N_A(x) := \begin{cases} \{y \in H : \langle y, a - x \rangle \leq 0 \text{ for any } a \in A\}, & \text{if } x \in A, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The (convex) subdifferential of $f : H \to \mathbb{R}$ at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) := \{z \in H : \langle z, y - x \rangle \leq f(y) - f(x) \text{ for any } y \in H\}.$$

2. Solution sets

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a globally Lipchitz function, i.e., there exists $K > 0$ such that for all $x, y \in \mathbb{R}^n$, $|\varphi(x) - \varphi(y)| \leqq K\|x - y\|$. We suppose that φ is convex. Let C be a nonempty closed convex subset of $L^2_n[a,b]$, where $L^2_n[a,b]$ refers to the set of 2-integrable functions from $[a,b]$ to $\mathbb{R}^n$ with the inner product $\langle \cdot, \cdot \rangle$ defined by for any $x, y \in L^2_n[a,b]$, $\langle x, y \rangle = \int_a^b x(t)^T y(t) dt$.

Consider the following convex optimization problem:

$$\text{(P)} \quad \min_{x \in C} f(x) := \int_a^b \varphi(x(t)) dt.$$

Let $S := \{x \in C \mid f(y) \geqq f(x),\ \forall y \in C\}$. Then S is the set of solutions of (P). We assume that $S \neq \emptyset$. We note that f is globally Lipchitz on $L^2_n[a,b]$ and f is convex.

The following Proposition is well-known in [14]. But we give its proof for completeness.

PROPOSITION 2.1. *$\bar{x} \in S$ if and only if there exists $\xi \in \partial f(\bar{x})$ such that $\langle \xi, x - \bar{x} \rangle \geqq 0$ for any $x \in C$, that is, $\int_a^b \xi(t)^T (x(t) - \bar{x}(t)) dt \geqq 0$ for any $x \in C$.*

Proof. $\bar{x} \in S$ if and only if $0 \in \partial(f + \delta_C)(\bar{x})$, i.e., $0 \in \partial f(\bar{x}) + N_C(\bar{x})$. So, the result of the proposition holds. □

THEOREM 2.2. *Let $\bar{x} \in S$. Then the solution set S of* (P) *is characterized as follows:*

$$S = \{x \in C \mid \exists \xi \in L_n^2[a,b] \ s.t. \ \xi(t) \in \partial\varphi(x(t)) \cap \partial\varphi(\bar{x}(t)) \ a.e. \ on \ [a,b] \\ \text{and} \int_a^b \xi(t)^T (x(t) - \bar{x}(t)) dt = 0\}.$$

Proof. (1) Since f is continuous and convex, $\partial f(\bar{x}) \neq \emptyset$. Since $\bar{x} \in S$, there exists $\xi \in \partial f(\bar{x})$ such that $\langle \xi, x - \bar{x} \rangle \geqq 0$ for all $x \in C$. Let $\tilde{x} \in S$. Then $\langle \xi, \tilde{x} - \bar{x} \rangle \geqq 0$. Since $\xi \in \partial f(\bar{x})$, $0 = f(\tilde{x}) - f(\bar{x}) \geqq \langle \xi, \tilde{x} - \bar{x} \rangle$. Thus, $\langle \xi, \tilde{x} - \bar{x} \rangle = 0$. So, for any $x \in L_n^2[a,b]$, we have

$$\begin{aligned} f(x) &\geqq f(\bar{x}) + \langle \xi, x - \bar{x} \rangle \\ &= f(\bar{x}) + \langle \xi, x - \tilde{x} \rangle + \langle \xi, \tilde{x} - \bar{x} \rangle \\ &= f(\tilde{x}) + \langle \xi, x - \tilde{x} \rangle, \end{aligned}$$

and hence $\xi \in \partial f(\tilde{x})$. Consequently, if $\tilde{x} \in S$, then there exists $\xi \in \partial f(\bar{x}) \cap \partial f(\tilde{x})$ such that $\langle \xi, \tilde{x} - \bar{x} \rangle = 0$.

(2) Let $x, v \in L_n^2[a,b]$. We will prove that $f'(x; v) \leqq \int_a^b \varphi'(x(t); v(t)) dt$, where $f'(x; v) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha v) - f(x)}{\alpha}$ and $\varphi'(x(t); v(t)) = \lim_{\alpha \downarrow 0} \frac{\varphi(x(t) + \alpha v(t)) - \varphi(x(t))}{\alpha}$. We can fine a sequence $\{\alpha_n\} \downarrow 0$ such that

$$f'(x; v) = \lim_{n \to \infty} \frac{f(x + \alpha_n v) - f(x)}{\alpha_n}.$$

Thus we have

$$f'(x; v) = \lim_{n \to \infty} \int_a^b \frac{\varphi(x(t) + \alpha_n v(t)) - \varphi(x(t))}{\alpha_n} dt.$$

Since φ is globally Lipschitz, there exists $K > 0$ such that

$$\frac{\varphi(x(t) + \alpha_n v(t)) - \varphi(x(t))}{\alpha_n} \leqq K \|v(t)\| \text{ for any } t \in [a,b].$$

So, we have

$$
\begin{aligned}
&f'(x;v)\\
&= -\lim_{n\to\infty}\int_a^b \left(-\frac{\varphi(x(t)+\alpha_n v(t))-\varphi(x(t))}{\alpha_n}\right)dt\\
&= -\lim_{n\to\infty}\int_a^b \left(-K\|v(t)\|+K\|v(t)\|-\frac{\varphi(x(t)+\alpha_n v(t))-\varphi(x(t))}{\alpha_n}\right)dt\\
&= \int_a^b K\|v(t)\|dt-\lim_{n\to\infty}\int_a^b \left(K\|v(t)\|-\frac{\varphi(x(t)+\alpha_n v(t))-\varphi(x(t))}{\alpha_n}\right)dt.
\end{aligned}
$$

By Fatou's Lemma,

$$
\begin{aligned}
&f'(x;v)\\
&\leqq \int_a^b K\|v(t)\|dt-\int_a^b \liminf_{n\to\infty}\left(K\|v(t)\|-\frac{\varphi(x(t)+\alpha_n v(t))-\varphi(x(t))}{\alpha_n}\right)dt\\
&= -\int_a^b \liminf_{n\to\infty}\left(-\frac{\varphi(x(t)+\alpha_n v(t))-\varphi(x(t))}{\alpha_n}\right)dt\\
&= \int_a^b \limsup_{n\to\infty}\frac{\varphi(x(t)+\alpha_n v(t))-\varphi(x(t))}{\alpha_n}dt\\
&= \int_a^b \lim_{n\to\infty}\frac{\varphi(x(t)+\alpha_n v(t))-\varphi(x(t))}{\alpha_n}dt\\
&= \int_a^b \varphi'(x(t);v(t))dt.
\end{aligned}
$$

Hence we have,

$$
f'(x;v)\leqq \int_a^b \varphi'(x(t);v(t))dt. \tag{2.1}
$$

Let $x, v \in L_n^2[a,b]$. For each $t\in[a,b]$, $\varphi'(x(t);v(t))=\max\{\langle y,v(t)\rangle : y\in\partial\varphi(x(t))\}$. Since $\partial\varphi$ is an upper semicontinuous multifunction (see Proposition 1.5 (e) in page 73 of [4] and Proposition 4.3 in page 80 of [4]), $\partial\varphi$ is measurable. Since $x(\cdot)$ is a measurable function, a multifunction $t\mapsto\partial\varphi(x(t))$ is measurable. Following the proofs of Lemma 8.2.3, Theorem 8.2.9, Lemma 8.2.12 and Theorem 8.2.11 in [1], we can prove that a multifunction

$$
t\mapsto\{r'\in\partial\varphi(x(t)) : \langle r',v(t)\rangle=\max_{y\in\partial\varphi(x(t))}\langle y,v(t)\rangle\}
$$

is measurable and closed-valued. So, by measurable selection theorem (see Theorem 5.3 in page 151 of [4]), there exists a measurable function

$\xi : [a,b] \to \mathbb{R}^n$ such that for any $t \in [a,b]$

$$\xi(t) \in \partial\varphi(x(t)) \text{ and } \langle \xi(t), v(t)\rangle = \varphi'(x(t); v(t)).$$

Here, since $\xi(t) \in \partial\varphi(x(t))$ and φ is globally Lipschitz, there exists $K \geqq 0$ such that $\|\xi(t)\| \leqq K$ for all $t \in [a,b]$. Thus $\xi \in L^2_n[a,b]$. Thus from (2.1),

$$\begin{aligned} f'(x;v) &\leqq \int_a^b \varphi'(x(t); v(t))dt \\ &= \langle \xi, v\rangle \\ &\leqq \max\{\langle \xi', v\rangle : \xi' \in L^2_n[a,b],\ \xi'(t) \in \partial\varphi(x(t)) \text{ a.e. on } [a,b]\}. \end{aligned}$$

Let $W = \{\xi' \in L^2_n[a,b] : \xi'(t) \in \partial\varphi(x(t)) \text{ a.e. on } [a,b]\}$. Then W is convex and weakly closed. Since $f'(x;v) = \max\limits_{\xi' \in \partial f(x)} \langle \xi', v\rangle \leqq \max\limits_{\xi' \in W} \langle \xi', v\rangle$, by Proposition 1.3 (c) in page 72 of [4], $\partial f(x) \subset W$. Let $\xi \in W$. Then $\xi(t) \in \partial\varphi(x(t))$ a.e. on $[a,b]$. So, for any $y \in L^2_n[a,b]$,

$$\varphi(y(t)) \geqq \varphi(x(t)) + \langle \xi(t), y(t) - x(t)\rangle \text{ a.e..}$$

Hence $\int_a^b \varphi(y(t))dt \geqq \int_a^b \varphi(x(t))dt + \int_a^b \xi(t)^T(y(t) - x(t))\rangle dt$ for any $y \in L^2_n[a,b]$, i.e., $f(y) \geqq f(x) + \langle \xi, y - x\rangle$ for any $y \in L^2_n[a,b]$. Hence $\xi \in \partial f(x)$. Thus, $W \subset \partial f(x)$. Consequently,

$$\partial f(x) = \{\xi \in L^2_n[a,b] : \xi(t) \in \partial\varphi(x(t)) \text{ a.e. on } [a,b]\}. \tag{2.2}$$

(3) From (1), if $x \in S$, then there exists $\xi \in \partial f(x) \cap \partial f(\bar{x})$ such that $\langle \xi, x - \bar{x}\rangle = 0$. Let $x \in C$ be such that there exists $\xi \in \partial f(x) \cap \partial f(\bar{x})$ such that $\langle \xi, x - \bar{x}\rangle = 0$. Then $f(\bar{x}) \geqq f(x) + \langle \xi, \bar{x} - x\rangle = f(x)$. Since $x \in C$, $x \in S$. Hence, we have

$$S = \{x \in C : \text{there exists } \xi \in \partial f(x) \cap \partial f(\bar{x}) \text{ such that } \langle \xi, x - \bar{x}\rangle = 0\}.$$

Thus, from (2.2),

$$\begin{aligned} S = \{x \in C : \exists \xi \in L^2_n[a,b] &\text{ such that } \xi(t) \in \partial\varphi(x(t)) \cap \partial\varphi(\bar{x}(t)) \\ &\text{a.e. on } [a,b] \text{ and } \int_a^b \langle \xi(t), x(t) - \bar{x}(t)\rangle dt = 0\}. \end{aligned}$$

□

Example 2.3. Let $\varphi(x) := \max\{|x| - 1, 0\}$ and let $f(x) := \int_{-1}^1 \varphi(x(t))dt$, $x \in L^2_1[-1,1]$. Let $C = \{x \in L^2_1[-1,1] : x(t) \in [-1,1]\ \forall t \in [-1,1]\}$ and let $S := \{x \in C : f(y) \geqq f(x)\ \forall y \in C\}$. Then we can check that $S = C$. Let $\bar{x} = 0 \in C$. Moreover, $\{x \in C : \exists \xi \in L^2_1[-1,1] \text{ s.t. } \xi(t) \in \partial\varphi(x(t)) \cap \partial\varphi(\bar{x}(t))$ a.e. on $[-1,1]$ and $\int_{-1}^1 \xi(t)(x(t) - \bar{x}(t))dt = 0\} = C$.

Thus, Theorem 2.2 holds.

Remark 2.4. The part (2) of the proof of Theorem 2.2 can be obtained from Theorem 5.18 in page 160 of [4]. The Theorem 5.18 states the characterizations of limiting subdifferential and generalized gradient of the integral function. But we give its proof for completeness.

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