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Comparison of Offset Approximation Methods of Conics with Explicit Error Bounds

Sung Chul Bae[1] and Young Joon Ahn[2†]

Abstract

In this paper the approximation methods of offset curve of conic with explicit error bound are considered. The quadratic approximation of conic(QAC) method, the method based on quadratic circle approximation(BQC) and the Pythagorean hodograph cubic(PHC) approximation have the explicit error bound for approximation of offset curve of conic. We present the explicit upper bound of the Hausdorff distance between the offset curve of conic and its PHC approximation. Also we show that the PHC approximation of any symmetric conic is closer to the line passing through both endpoints of the conic than the QAC.

Key words: Conic, Offset Approximation, Explicit Error Bound, Convolution Curve, Pythagorean Hodograph.

1. Introduction

Conic is a widely used curve in CAD/CAM system or in the field of CAGD (Computer Aided Geometric Design). The conic can be represented in the rational quadratic Bézier form[1,2]. The necessary and sufficient condition of the conic with monotone curvature has been found[3,4]. Also, the necessary and sufficient condition for the rational cubic or quartic Bézier curve to be a conic is presented[5,6]. The geometric characteristics of the rational quadratic Bézier curve including center, asymptotes, foci, axes and eccentricity are obtained[2,7-10].

Conic can not be represented in Bézier form, and so it is needed to be approximated when it is used in system which admits only polynomial curves. Floater[11,12] found the G^2 quadratic approximation of conic with an explicit error bound and the G^{n-1} spline approximation of odd degree $n \geq 3$ of conic with an explicit error bound and with the approximation order $2n$. Conic is approximated by G^3 quintic spline[13] and G^2 quartic spline[14] with an explicit error bound.

Offset curves of generic conics except for circle or parabola do not admit rational parameterizations[15] and are irreducible (non-rational) algebraic curve of degree eight[15,16]. The offset curve of circle is a circle, and that of parabola is a rational Bézier curve of degree six which is shown by Lü[17]. Thus it is an important task to approximate the offset curve of generic conic by polynomial or rational curves in CAGD. Early, Farin[18] presented a G^1 endpoint interpolation of offset curve of conic by conic together with matching an intermediate tangent. Farouki[15] obtained the conic approximation interpolating the endpoints, tangents at the endpoints and parametric midpoint of the offset of conic, and the exact error analysis.

In this paper we collect three approximation methods of offset of conic which have explicit error bounds. They are the quadratic approximation of conic(QAC) method, the method based on quadratic circle approximation(BQC) and the Pythagorean hodograph cubic (PHC) approximation method. We present the error bound of PHC approximation in a closed form. We compare them with their error bounds and illustrate that the best approximation method from the three methods can be obtained easily by the error bounds. Also, we show that the PHC approximation is closer to the line passing through both endpoints of the symmetric conic than the QAC method.

Our manuscript is constructed as follows. In Section 2, three approximation methods of offset curve of conic with the explicit error bounds are explained and the

[1]Department of Mathematics Education, Korea University, Seoul, Korea
[2]Department of Mathematics Education, Chosun University, Gwangju, Korea

†Corresponding author : ahn@chosun.ac.kr
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error bound of PHC approximation is presented in a closed form. In Section 3, we illustrate that the best approximation from the three method for each given conic and given offset distance can be obtained, and our work is summarized in Section 4.

2. Offset Approximation Method of Conics

In this section we introduce three methods of offset approximation of conics by rational Bézier curve which have the explicit bound of the Hausdorff distance between the approximation curve and the offset of conic.

2.1. Quadratic Approximation of Conics

The G^2 quadratic spline approximation of conics is founded by Floater[11]. For given conic $\boldsymbol{b}(t) = \frac{\sum_{i=0}^{2} B_i^2(t) w_i \boldsymbol{b}_i}{\sum_{i=0}^{2} B_i^2(t) w_i}$, the quadratic approximation is $\sum_{i=0}^{2} B_i^2(t)\boldsymbol{b}_i$, where $\boldsymbol{b}_0$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ are control points of the conic, $w_0=1, w_1>0, w_2=1$ are weights of the conic in standard form, and $B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$, $i = 0,...,n$, is the Bernstein polynomial of degree n. In general this approximation is only G^1 endpoint interpolation of conic[19]. But subdivision at the shoulder point of conic makes the quadratic approximate spline curvature-continuous(G^2). This is one of the merits of Floater's approximation method. The other merit is the sharp error bound as follows[11]: the Hausdorff distance between the conic $\boldsymbol{b}$ and its quadratic approximation $\boldsymbol{q}$ is

$$d_H(\boldsymbol{b},\boldsymbol{q}) \le \frac{|w_1 - 1|}{4(w_1+1)} |\boldsymbol{b}_0 - 2\boldsymbol{b}_1 + \boldsymbol{b}_2|. \tag{2.1}$$

In this paper we call this quadratic approximation method of conic by QAC. The QAC can be applied to offset approximation, since quadratic Bézier curve has rational offset[17]. Lüshowed that the offset curve of any quadratic Bézier curve can be expressed by rational Bézier curve of degree six[17]. Thus the offset of the quadratic Bézier approximation is an approximation of offset of the conic. Moreover, the quadratic spline approximation is G^2 continuous, so is its offset approximation. If the offset curve of conic, the quadratic approximation and its offset curve have no cusp, then the Hausdorff distance is invariant under convolution[20]. Thus, for the offset distance $r \in \mathbb{R}$,

$$d_H(\boldsymbol{b} * r\boldsymbol{c}, \boldsymbol{q} * r\boldsymbol{c}) = d_H(\boldsymbol{b},\boldsymbol{q}) \tag{2.2}$$

where $\boldsymbol{c}$ is a circular arc which is the Gauss map of $\boldsymbol{b}$ and $\boldsymbol{q}$, and * means the convolution curve of two compatible curves. Thus if the offset curve $\boldsymbol{b}*r\boldsymbol{c}$ has a cusp, then $\boldsymbol{b}$ should be subdivided at the point where $\boldsymbol{b}*r\boldsymbol{c}$ has the cusp, and then Equation (2.2) can be applied.

If the error bound in Equation (2.1) is larger than the given tolerance *TOL*, then the conic should be subdivided repeatedly until the error bound is less than the tolerance. Let $\boldsymbol{Q}$ be the G^2 quadratic spline curve which is the composition curve of the quadratic Bézier approximations obtained by subdivision at shoulder points of segments of the conic. If $\boldsymbol{Q}$ has n segments, then the error bound[11] is

$$d_H(\boldsymbol{b},\boldsymbol{Q}) \le \frac{|w_{n,1} - 1|}{4(w_{n,1}+1)} \max_{k=1,\cdots,n} |\boldsymbol{b}_{k,0} - 2\boldsymbol{b}_{k,1} + \boldsymbol{b}_{k,2}| \tag{2.3}$$

where $\boldsymbol{b}_{k,0}$, $\boldsymbol{b}_{k,1}$, $\boldsymbol{b}_{k,2}$, $k = 1,...,n$, are control points of the k-th subdivided segment of conic and 1, $w_{n,1}$, 1 are the weights of the segment. By elevating the number of segment n one by one, the minimum required number of segments within the error tolerance can be obtained.

2.2. Based on Quadratic Circle Approximation

In this section we explain the offset approximation based on quadratic circle approximation which was presented firstly by Lee *et al.*[21]. This method can be extended to G^2 approximation[20,22]. For given conic $\boldsymbol{b}$ and offset distance r, its offset curve is $\boldsymbol{b}*r\boldsymbol{c}$. If the circular arc $\boldsymbol{c}$ is approximated by quadratic Bézier curve $\boldsymbol{c}^a$, then the offset curve $\boldsymbol{b}*r\boldsymbol{c}$ is approximated by $\boldsymbol{b}*r\boldsymbol{c}^a$ which is a rational Bézier curve of degree six[21]. If these curves are cusp-free, then the error analysis is obtained by

$$d_H(\boldsymbol{b} * r\boldsymbol{c}, \boldsymbol{b} * r\boldsymbol{c}^a) = r d_H(\boldsymbol{c},\boldsymbol{c}^a) \tag{2.4}$$

[20] and the Hausdorff distance $d_H(\boldsymbol{c},\boldsymbol{c}^a)$ between circular arc $\boldsymbol{c}$ of angle θ and its G^1 quadratic Bézier approximation $\boldsymbol{c}^a$ can be easily obtained by

$$d_H(\boldsymbol{c},\boldsymbol{c}^a) = 2\sin^4\frac{\theta}{4}sec\frac{\theta}{2}$$

[21,22]. If the error is larger than the given tolerance *TOL*, then the conic should be subdivided. If the circular arc $\boldsymbol{c}$ is subdivided with the same length and each segment is approximated by the G^1 quadratic interpolation $\boldsymbol{c}^a$, then the composition curve $\boldsymbol{C}^a$ of these G^1 quadratic Bézier curves is G^2 approximation of the circular arc $\boldsymbol{c}$[11]. Moreover, the offset approximation $\boldsymbol{b}*r\boldsymbol{C}^a$ is curvature continuous and

$$d_H(\boldsymbol{b}*r\boldsymbol{c},\boldsymbol{b}*r\boldsymbol{C}^a) = 2r\sin^4\frac{\theta}{4n}sec\frac{\theta}{2n} \tag{2.5}$$

[21,22] where n is the number of quadratic segments of $\boldsymbol{C}^a$.

2.3. PH Cubic Approximation of Conic

Pythagorean hodograph curves are firstly presented by Farouki and Sakkalis[23]. In this section we explain Pythagorean hodograph cubic approximation of conic and present its error bound analysis.

A cubic curve $\boldsymbol{p}(t) = \sum_{i=0}^{3}\boldsymbol{p}_i B_i^3(t)$ has a Pythagorean hodograph(PH) if and only if

$$|\Delta \boldsymbol{p}_0|^2 = |\Delta \boldsymbol{p}_1||\Delta \boldsymbol{p}_2| \text{ and } \theta_1 = \theta_2 \tag{2.6}$$

[23-25], where $\Delta\boldsymbol{p}_i = \boldsymbol{p}_i - \boldsymbol{p}_{i-1}$ for $i = 0,\ 1,\ 2$ and $\theta_i = \angle \boldsymbol{p}_{i-1}\boldsymbol{p}_i\boldsymbol{p}_{i+1}$ for $i = 0,\ 1$. The PH cubic curve $\boldsymbol{p}(t)$ is a G^1 endpoint interpolation of the conic $\boldsymbol{b}$ contained in the (closed) triangle $\Delta\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{b}_2$ if and only if

$$\begin{aligned}
\boldsymbol{p}_0 &= \boldsymbol{b}_0 \\
\boldsymbol{p}_1 &= (1-\delta_0)\boldsymbol{b}_1 + \delta_0\boldsymbol{b}_0 \\
\boldsymbol{p}_2 &= (1-\delta_1)\boldsymbol{b}_1 + \delta_1\boldsymbol{b}_2 \\
\boldsymbol{p}_3 &= \boldsymbol{b}_2
\end{aligned}$$

for some real numbers $\delta_0,\ \delta_1 \in (0,\ 1)$ satisfying Equation (2.6). Equation (2.6) holds if and only if

$$(4\sin^2(\frac{\theta_2-\theta_1}{2})-1)\lambda\delta_1^2 + (\lambda+1)\delta_1 - 1 = 0 \text{ and } \delta_0 = \lambda\delta_1$$

where $\lambda = \frac{|\Delta \boldsymbol{b}_1|}{|\Delta \boldsymbol{b}_0|}$. Putting $\psi(\delta_1) = \left(4\sin^2\left(\frac{\theta_2-\theta_1}{2}\right)-1\right)\lambda\delta_1^2 + (\lambda+1)\delta_1 - 1$, we have $\psi(0) = -1 < 0$ and $\psi(1) = 4\lambda\sin^2\frac{\theta_2-\theta_1}{2} > 0$, so that the equation $\psi(\delta_1) = 0$ has the unique solution in the open interval (0,1). If $|\theta_2-\theta_1| \neq \frac{\pi}{3}$, then the quadratic equation $\psi(\delta_1) = 0$ of δ_1 has the unique solution in the open interval (0,1) by

$$\delta_1 = \frac{-(\lambda+1)+\sqrt{(\lambda+1)^2+4(4\sin^2(\frac{\theta_2-\theta_1}{2})-1)}}{2(4\sin^2(\frac{\theta_2-\theta_1}{2})-1)}, \tag{2.7}$$

and if $|\theta_2-\theta_1| = \frac{\pi}{3}$, then the linear equation $\psi(\delta_1) = 0$ of δ_1 has the solution

$$\delta_1 = \frac{1}{\lambda+1}. \tag{2.8}$$

Since $\psi\left(\frac{1}{\lambda}\right) = \frac{4}{\lambda}\sin^2\frac{\theta_2-\theta_1}{2} > 0$, the solution δ_1 in Equations (2.7)-(2.8) is contained in the open interval $\left(0,\frac{1}{\lambda}\right)$, so that $\delta_0 = \lambda\delta_1$ is also in the open interval (0,1). Hence the PH-cubic $\boldsymbol{p}(t)$ is contained in the (closed) triangle $\Delta\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{b}_2$. Using Floater's error analysis between conic and its approximation curve contained the triangle as

$$d_H(\boldsymbol{b},\boldsymbol{p}) \le \frac{1}{4}max\{1,\frac{1}{w_1^2}\} |\boldsymbol{b}_0 - 2\boldsymbol{b}_1 + \boldsymbol{b}_2| \max_{t\in[0,1]}|f(\boldsymbol{p}(t))| \tag{2.9}$$

where $f: \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \tau_1^2 - 4w^2\tau_0\tau_2$ and $\tau_0,\ \tau_1,\ \tau_2$ are the barycentric coordinates of the points (x, y) with respect to the triangle $\Delta\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{b}_2 : (x, y) = \tau_0\boldsymbol{b}_0 + \tau_1\boldsymbol{b}_1 + \tau_2\boldsymbol{b}_2$ with $\tau_0 + \tau_1 + \tau_2 = 1$, we present the error bound of the PH-cubic approximation of conic as follows.

Proposition 2.1

The Hausdorff distance between a conic and its G^1 PH-cubic approximation has the upper bound explicitly as

$$d_H(\boldsymbol{b},\boldsymbol{p}) \le \frac{1}{4}max\{1,\frac{1}{w_1^2}\} |\boldsymbol{b}_0 - 2\boldsymbol{b}_1 + \boldsymbol{b}_2| \max\{|g(t_i)| : t_1,t_2,t_3 \in [0,1]\}$$

where $g(t) = t^2(1-t)^2 g_1(t)$ and $g_1(t)$ is a quadratic polynomial

$$g_1(t) = 9\{(1-\delta_0)(1-t)+(1-\delta_1)t\}^2$$
$$-4w_1^2\{(1-t)+3\delta_0 t\}\{t+3(1-\delta_1)(1-t)\}$$

and t_1, t_2, t_3 are the roots of cubic polynomial $2(1-2t)g_1(t) + t(1-t)g_1'(t)$.

Proof. Since

$$b(t) = \{B_0^3(t)+\delta_0 B_1^3(t)\}b_0 + \{(1-\delta_0)B_1^3(t)$$
$$+(1-\delta_1)B_2^3(t)\}b_1 + \{B_3^3(t)+\delta_1 B_2^3(t)\}b_2$$

we have

$$f(p(t)) = 9t^2(1-t)^2\{(1-\delta_0)(1-t)+(1-\delta_1)t\}^2$$
$$-4w_1^2 t^2(1-t)^2\{(1-t)+3\delta_0 t\}\{t+3\delta_1(1-t)\}$$
$$= t^2(1-t)^2 g_1(t) = g(t)$$

and its derivative is a polynomial of degree five as

$$g'(t) = 2t(1-t)^2 g_1(t) - 2t^2(1-t)g_1(t)$$
$$+t^2(1-t)^2 g_1'(t)$$
$$= t(1-t)\{2(1-2t)g(t) + t(1-t)g_1'(t)\}$$

Since $g(t)$ has the local extrema at $t = 0, 1, t_1, t_2, t_3$, the assertion follows. □

3. Comparison of Offset Approximation Methods of Conics

In this section we compare three approximation methods of offset curve of conic, QAC(Qaudratic approximation of conic) q^*rc, BQC(Based on quadratic circle approximation) b^*rc^a, and PHC(Pythagorean cubic interpolation) p^*rc. They yield the rational Bézier approximation of degree 6, 6, and 5, and with the geometric continuity of order two, two, and one, in order.

In the first example, b is a conic with control points b_0=(0,0), b_1=(2,1), b_2=(3,0), and weight $w_1 = 2$, as shown in Fig. 1. For the offset distance $r = -1$, $b^*(-c)$ (black color) is the offset curve of b, and QAC $q^*(-c)$ (green), BQC $b^*(-c^a)$(magenta) and PHC $p^*(-c)$(blue) have the error bounds 0.19, 0.02 and 0.52, respectively. In this case, the method of BQC is the best approximation, and the upper bound of PHC is overestimated.

Table 1. Degree of rational Bézier approximation of offset curve of conic and order of geometric continuity

	QAC	BQC	PHC
Degree of rational Bézier approximation	6	6	5
Order of geometric continuity	2	2	1

In the second example, an ellipse with control points b_0=(2,0), b_1=(2,1), b_2=(0,1) and weight $w_1=1/\frac{1}{\sqrt{2}}$ is given. For the offset distance $r = 1$, the approximation methods of QAC, BQC and PHC have the error bounds 9.59×10^{-2}, 6.07×10^{-2}, 4.34×10^{-2}, respectively, as shown in Fig. 2. The PHC method is the best approximation and is overestimated.

Proposition 3.1

If the conic $b(t)$ is not a line segment with $\|b_1-b_0\|=\|b_2-b_1\|$, then the PHC p is closer to the line seg-

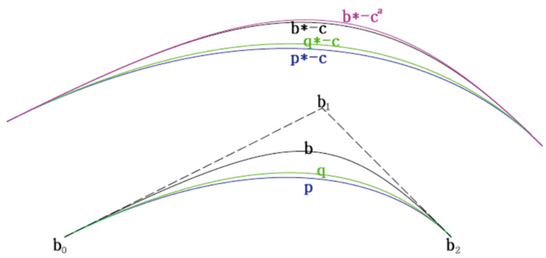

Fig. 1. The hyperbola(black) with $w_1 = 2$ and offset distance $r = -1$, and its approximations, QAC(q^*-c, green), BQC (b^*-c^a), magenta), and PHC(p^*-c, blue).

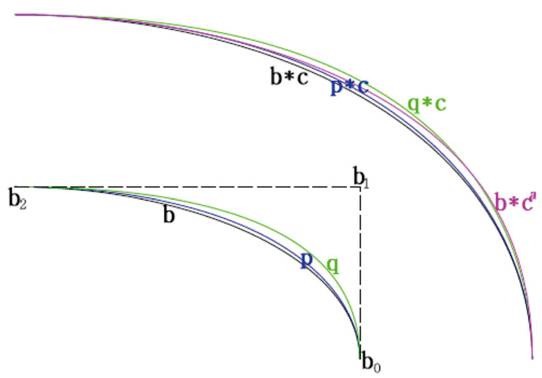

Fig. 2. Ellipse(black) with control points b_0, b_1, b_2, its offset curve(b^*c, black) with offset distance $r = 1$, and the approximation methods by QAC(q^*c, green), BQC(b^*c^a, magenta) and PHC(p^*c, blue).

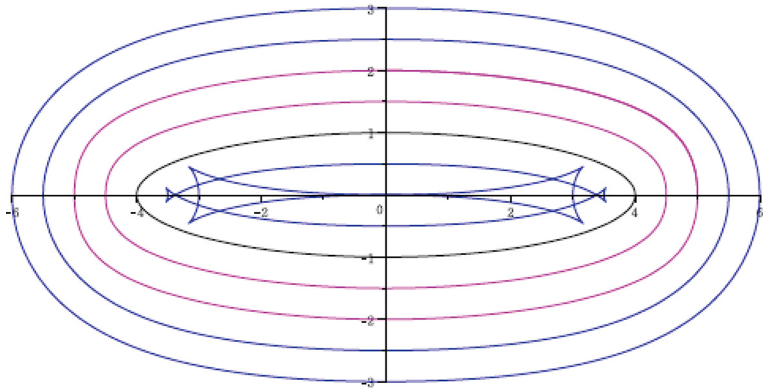

Fig. 3. For given ellipse $\frac{x^2}{4^2}+y^2=1$ (black) and offset distance $r=-1,-\frac{1}{2},\frac{1}{2},1,\frac{3}{2},2$, the best approximation is $\boldsymbol{p}^*(-\boldsymbol{c})$(blue), $\boldsymbol{p}^*(-\frac{1}{2}\boldsymbol{c})$(blue), $\boldsymbol{b}^*\frac{1}{2}\boldsymbol{c}^a$ (magenta), $\boldsymbol{b}^*\boldsymbol{c}^a$(magenta), $\boldsymbol{p}^*\frac{3}{2}\boldsymbol{c}$ (blue), $\boldsymbol{p}^*2\boldsymbol{c}$(blue), respectively.

ment $\boldsymbol{b}_0\boldsymbol{b}_2$ than the QAC $\boldsymbol{q}$.

Proof. Let θ be the angle $\angle \boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{b}_2$. The quadratic Bézier curve $\boldsymbol{q}(t)$ can be expressed in cubic Bézier form

$$\boldsymbol{q}(t)=\sum_{i=0}^{3}\boldsymbol{q}_i B_i^3(t)$$

where $\boldsymbol{q}_0=\boldsymbol{b}_0$, $\boldsymbol{q}_1=\frac{1}{3}\boldsymbol{b}_0+\frac{2}{3}\boldsymbol{b}_1$, $\boldsymbol{q}_2=\frac{2}{3}\boldsymbol{b}_2+\frac{1}{3}\boldsymbol{b}_3$, $\boldsymbol{q}_3=\boldsymbol{b}_2$. Since $\|\boldsymbol{b}_1-\boldsymbol{b}_0\|=\|\boldsymbol{b}_2-\boldsymbol{b}_1\|$, $\angle \boldsymbol{q}_0\boldsymbol{q}_1\boldsymbol{q}_2=\angle \boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3$. Since $\|\boldsymbol{b}_1-\boldsymbol{q}_1\|=\|\boldsymbol{b}_1-\boldsymbol{q}_2\|=\frac{1}{2}\|\Delta\boldsymbol{q}_0\|=\frac{1}{2}\|\Delta\boldsymbol{q}_2\|$ and $\theta<\pi$, we have $\|\Delta\boldsymbol{q}_1\|<\|\boldsymbol{b}_1-\boldsymbol{q}_1\|+\|\boldsymbol{b}_1-\boldsymbol{q}_2\|=\|\Delta\boldsymbol{q}_0\|=\|\Delta\boldsymbol{q}_2\|$ and $\|\Delta\boldsymbol{q}_1\|^2<\|\Delta\boldsymbol{q}_0\|\cdot\|\Delta\boldsymbol{q}_2\|$. The two lines $\boldsymbol{q}_1\boldsymbol{q}_2$ and $\boldsymbol{p}_1\boldsymbol{p}_2$ are parallel to the line $\boldsymbol{b}_0\boldsymbol{b}_2$. Since $\|\Delta\boldsymbol{p}_1\|^2=\|\Delta\boldsymbol{p}_0\|\cdot\|\Delta\boldsymbol{p}_2\|$, the line $\boldsymbol{p}_1\boldsymbol{p}_2$ is closer to the line $\boldsymbol{b}_0\boldsymbol{b}_2$ than the line $\boldsymbol{q}_1\boldsymbol{q}_2$, so that the PHC $\boldsymbol{p}$ is closer to the line $\boldsymbol{p}_1\boldsymbol{p}_2$ than the QAC $\boldsymbol{q}$. □

In the third example, for given ellipse $\frac{x^2}{4^2}+y^2=1$ and for offset distance $r=-1,-\frac{1}{2},\frac{1}{2},1,\frac{3}{2},2$, we find the best approximation among three methods, QAC, BQC and PHC. For $r=\frac{1}{2},1$, the BQC method($\boldsymbol{b}^*\frac{1}{2}\boldsymbol{c}^a,\boldsymbol{b}^*\boldsymbol{c}^a$) has the smallest error bound, and for $r=\frac{3}{2},2$, the PHC method($\boldsymbol{p}^*\frac{3}{2}\boldsymbol{c},\boldsymbol{p}^*2\boldsymbol{c}$) is the best approximation even if it is overestimated. Since the error of BQC method depends on the offset distance r, linearly, and the error bound of PHC method is constant which is independent of r, if r is larger than two, then PHC has smaller error bound than BQC. For $r=-\frac{1}{2},-1$, the offset curve has cusp, so that the Equations (2.2) and (2.4) cannot be applied. The error of each approximation method is obtained by complicatedly. The BQC method($\boldsymbol{b}^*(-\frac{1}{2}\boldsymbol{c}^a)$, $\boldsymbol{b}^*(-\boldsymbol{c}^a)$) has the smallest error, as shown in Fig. 3.

4. Conclusions

The contribution of this paper as follows. We presented the explicit upper bound of the Hausdorff distance between the offset curve of conic and its PHC approximation. So, the comparison of three approximation methods, QAC, BQC and PHC which all have the error bounds in closed form can be possible. Also we showed that the PHC approximation of any symmetric conic is closer to the line $\boldsymbol{b}_0\boldsymbol{b}_2$ than the QAC method. We illustrated that the best approximation for each given conic and given offset distance can be obtained simply.

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