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Multivariate CUSUM control charts for monitoring the covariance matrix

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Abstract

This paper is a study on the multivariate CUSUM control charts using three different control statistics for monitoring covariance matrix. We get control limits and ARLs of the proposed multivariate CUSUM control charts using three different control statistics by using computer simulations. The performances of these proposed multivariate CUSUM control charts have been investigated by comparing ARLs. The purpose of control charts is to detect assignable causes of variation so that these causes can be found and eliminated from process, variability will be reduced and the process will be improved. We show that the charts based on three different control statistics are very effective in detecting shifts, especially shifts in covariances when the variables are highly correlated. When variables are highly correlated, our overall recommendation is to use the multivariate CUSUM control charts using trace for detecting changes in covariance matrix.

Keywords: Average run length, covariance matrix, multivariate CUSUM control chart.

1. Introduction

The quality of production product must maintain a constant level in a continuous industrial production process. But many problems in quality control involve a vector of observations of several characteristics rather than a single characteristic. Although one of variables could monitor the process using separate control charts to the extent that these measurements are mutually correlated, it will obtain better sensitivity using multivariate methods that exploit the correlations. Chang and Heo (2011), Jeong and Cho (2012a, 2012b), Reynolds and Cho (2006, 2011) studied the multivariate control charts for monitoring covariance matrix.

The cumulative sum (CUSUM) control chart was proposed by Page (1954). This chart is good alternative to the Shewhart control chart when small or moderate shifts are interest. The CUSUM chart is maintained by taking sample and plotting a cumulative sum of differences between the sample mean and the target value in time order in the chart. Up to the present, multivariate control charts have been widely used for monitoring process mean

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vector. But relatively little attention has been given to the use of multivariate charts for monitoring covariance matrix.

The purpose of this study is to construct and evaluate the multivariate CUSUM control chart for monitoring the covariance matrix Σ. For monitoring process covariance matrix, we construct multivariate CUSUM control chart based on three different control statistics proposed by Hotelling V_t, Hui L_t and likelihood ratio test (LRT) statistic W_t and evaluate the proposed multivariate CUSUM control chart in terms of average run length (ARL).

2. Description of control procedures

Suppose that the process of interest has p quality variables presented by the random vector $\underline{X}' = (X_1, X_2, \cdots, X_p)$ and we take a sequence of samples of size n at each sampling occasion t $(t = 1, 2, \cdots)$. It will be assumed that the successive observation vectors are independent and have multivariate normal distribution with $N_p(\mu, \Sigma)$ where the mean vector $\mu = \mu_0$ is known.

2.1. Notations, assumptions and properties

Suppose that measurement is $X^{'}$, a p -component vector, which is assumed to follow a multivariate normal distribution. Let σ represent the vector of standard deviations of the p variables. Let Σ_0 and σ_0 be the in-control values of Σ and σ, respectively.

Suppose that the objective is to monitor Σ where the target values Σ_0 and μ_0 are known. It is assumed that the in-control process covariance matrix is as follows;

$$\Sigma_0 = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{bmatrix}$$

In reality, some of the in-control parameter values would need to be estimated during a Phase I period when process data are collected for purposes of parameter estimation. But, we consider control charts in Phase II under the simplifying assumption that the in-control parameter values are known. We will usually refer to Σ_0 as the target, even though, in practice, some of the components of Σ_0 may correspond to estimated values.

Assume that the process will be monitored by taking a sample of $n \geq p$ independent observation vectors at sampling point t, where the sampling points are d time units apart. Let X_{tij} represent observation j $(j = 1, 2, \cdots, n)$ for variable i $(i = 1, 2, \cdots, p)$ at sampling point t $(t = 1, 2, \cdots)$, and let the corresponding standardized observation b, and let the corresponding standardized observation be

$$Z_{tij} = \frac{(X_{tij} - \mu_{0i})}{\sigma_{0i}}$$

where μ_{0i} is the i th component of μ_0 and σ_{0i} is the i th component of σ_0. Also let

$$Z_{tj} = (Z_{t1j}, Z_{t2j}, \cdots, Z_{tpj})^{'}, j = 1, 2, \cdots, n$$

be the vector of standardized observations for observation vector j at sampling point t. Let Σ be the covariance matrix of Z_{tj}, and let Σ_0 be the in-control value of Σ_Z. The in-control distribution of Z_{tij} is standard normal, so Σ_0 is also the in-control correlation matrix of the unstandardized observations.

Some control statistics used for monitoring Σ are functions of the sample estimates of Σ_Z. At sampling point t, let $\widehat{\Sigma}_t$ be the maximum likelihood estimator of Σ_Z, where the $(i, i^{'})$ element of element of $\widehat{\Sigma}_t$ is $\sum_{j=1}^{n} Z_{tij} Z_{ti^{'}j}/n$

2.2. Control statistics

We consider the case in which the primary purpose is to detect changes in the covariances. Three different Multivariate CUSUM control charts will be presented.

Hotelling (1947) proposed the use of the Lawley-Hotelling V_t statistic for monitoring covariance matrix. The distribution of V_t was studied by Lawley (1938) and Hotelling (1951). Hotelling proposed the following control statistic using trace for monitoring Σ

$$V_t = \sum_{j=1}^{n}(X_{tj} - \mu_0)'\Sigma_0^{-1}(X_{tj} - \mu_0) = ntr(\widehat{\Sigma}_t\Sigma_0^{-1}) \tag{2.1}$$

where V_t has a chi-square distribution with np degrees of freedom.

Hui (1980) studied the use of the sample generalized variance for monitoring the process covariance matrix using the following statistic L_t

$$L_t = \frac{|\widehat{\Sigma}_t|}{|\Sigma_0|} \tag{2.2}$$

The last control chart can be constructed by using the likelihood ratio statistic for testing $\mathrm{H}_0 : \Sigma = \Sigma_0$ vs $\mathrm{H}_1 : \Sigma \neq \Sigma_0$. For the t th sample $(t = 1, 2, \cdots)$, the likelihood ratio statistic is

$$\lambda = \left(\frac{e}{n}\right)^{\frac{1}{2}np} |\widehat{\Sigma}_t\Sigma_0^{-1}|^{\frac{1}{2}n} \exp\left\{\frac{1}{2}tr(\Sigma_0^{-1}\widehat{\Sigma}_t)\right\} \tag{2.3}$$

Nagarsenker and Pillai (1973) developed a method for obtaining the exact null distribution of $L = \lambda^{2/n}$ in a series form and computed percentage point of L to any degree of accuracy even for small sample size. Thus the statistic W_t can be a control statistic for monitoring Σ.

$$W_t = tr(\Sigma_0^{-1}\widehat{\Sigma_t}) - n\ln|\widehat{\Sigma_t}| + n\ln|\Sigma_0| + np\ln n - np \tag{2.4}$$

where $W_t = -2\ln\lambda$.

In general, if the process shifts from Σ_0 to Σ_1 then it is difficult to obtain the distributions of V_t, L_t and W_t. Thus, in order to evaluate the performance of the CUSUM control charts for monitoring Σ it is necessary to use Markov chain approach, integral equation approach and computer simulations.

2.3. Multivariate CUSUM Control Charts

CUSUM control chart was suggested by Page (1954). This chart is a good alternative to the Shewhart chart when detection of small or moderate shifts in a production process is important. A CUSUM chart directly incorporates all of the information in the sequence of sample values by plotting the cumulative sum of the deviation of the sample values from the target value.

A multivariate CUSUM statistic based on the control statistic S_t is given by

$$Y_t = \max(Y_{t-1}, 0) + (S_t - k) \tag{2.5}$$

where the statistics in (2.1), (2.2) and (2.4) can be replaced by S_t respectively, $Y_0 = 0$ and reference value $k \geq 0$. This chart for dispersion matrix signals whenever $Y_t \geq h$.

The control limit h can be obtained by Markov chain or integral equation approach to satisfy a specified in-control ARL when the process parameters are on-target. And when the process parameters in Σ have been changed, the performance of this chart can be evaluated by simulation.

We will compare multivariate CUSUM control charts in terms of the average run length (ARL) required to detect shifts in process parameters when three CUSUM control charts have the same false alarm rate. If there is a shift in a process parameter the ARL is the appropriate measure of detection time for this shift.

The Markov chain and integral equation methods can be used to evaluate properties of the multivariate CUSUM chart. Thus simulation with 10,000 runs was used. All of the schemes being compared have an in-control ARL of 800 hours where the sampling interval d is assumed to be 1 hour is assumed to be 1 hour.

3. Performances of the multivariate CUSUM control charts

We can use a Markov chain approach, integral equations or simulations to get control limit h value. If the process shifts from Σ_0, then it is necessary to use simulations to get ARLs. Also control limits h and ARLs for three different CUSUM charts are obtained by 10,000 runs.

The performance of the multivariate CUSUM control charts of three control statistics for monitoring the covariance matrix have been investigated in the types of shifts in Σ. When the production process changes, the following types of shifts were considered;

(1) covariances are change and variances are not change,
(2) variances and covariances are simultaneously change.

The ability of a control chart to detect any shifts in the production process is determined by the length of time required to signal. Thus, a good control chart detects shifts quickly in the process when the process is out-of-control state, and produces few false alarms when the process is in-control state.

We consider three multivariate CUSUM control charts based on the trace, determinant and likelihood ratio statistics respectively.

3.1. Control statistic using trace

There are control limits h for multivariate CUSUM control charts for monitoring by using the statistic V_t given by (2.1). We consider multivariate CUSUM control charts for two variables and two observations, four variables and four observations. Table 3.1 gives the values of h for p =2, 4, n =2, 4, $k = E(V_t) + (1/2)i$ $(i = 1, 2, 3, 4)$ when the in-control ARL is approximately 800.

Table 3.1 Values of control limit h for CUSUM charts using trace when the in-control ARL is 800

	$k = E(V_t) + 0.5$	$k = E(V_t) + 1$	$k = E(V_t) + 1.5$	$k = E(V_t) + 2$
$n = 2, p = 2$	32.2800	22.7800	18.4201	15.9000
$n = 4, p = 4$	84.9000	60.9990	48.6830	40.8624

Tables 3.2∼3.4 give the ARLs of multivariate CUSUM control charts using trace for $p = 2, 4$, $n = 2, 4$, $k = E(V_t) + (1/2)i$ $(i = 1, 2, 3, 4)$ and three different in-control correlation coefficients for ρ_0 =0.9, 0.5, 0.3 when covariances are change and variances are not change. Also ρ considered is decreased from 10 percent to 90 percent. As shown in Tables 3.2-3.4 the multivariate CUSUM control charts using trace for monitoring the covariance matrix are effective in terms of ARLs in detecting changes in covariances.

Table 3.2 ARLs for CUSUM charts using trace when covariances are change (ρ_0 = 0.9)

	$n = 2, p = 2$				$n = 4, p = 4$			
k	$E(V_t)$+0.5	$E(V_t)$+1	$E(V_t)$+1.5	$E(V_t)$+2	$E(V_t)$+0.5	$E(V_t)$+1	$E(V_t)$+1.5	$E(V_t)$+2
ρ_0 =0.90	800.43	800.65	800.52	800.62	800.99	800.46	800.12	800.10
ρ =0.81	27.35	26.94	28.62	32.11	10.34	8.29	7.3	6.69
ρ =0.72	13.53	11.66	11.35	11.47	5.87	4.78	4.24	3.90
ρ =0.63	9.33	7.90	7.53	7.36	4.40	3.66	3.31	3.06
ρ =0.54	7.45	6.33	5.91	5.74	3.68	3.14	2.84	2.66
ρ =0.45	6.32	5.36	5.00	4.85	3.25	2.81	2.58	2.43
ρ =0.36	5.60	4.77	4.41	4.35	2.99	2.59	2.42	2.30
ρ =0.27	5.04	4.33	4.08	3.93	2.78	2.43	2.29	2.20
ρ =0.18	4.68	4.03	3.83	3.66	2.62	2.33	2.21	2.14
ρ =0.09	4.37	3.82	3.61	3.47	2.49	2.25	2.16	2.10

Table 3.3 ARLs for CUSUM charts using trace when covariances are change (ρ_0 = 0.5)

	$n = 2, p = 2$				$n = 4, p = 4$			
k	$E(V_t)$+0.5	$E(V_t)$+1	$E(V_t)$+1.5	$E(V_t)$+2	$E(V_t)$+0.5	$E(V_t)$+1	$E(V_t)$+1.5	$E(V_t)$+2
ρ_0 =0.50	806.77	806.53	796.78	806.48	802.82	797.82	793.87	797.39
ρ =0.45	436.87	483.45	515.63	543.93	126.31	136.58	154.76	173.34
ρ =0.40	256.87	304.33	333.44	355.79	56.95	53.80	56.42	62.13
ρ =0.35	169.23	200.00	225.36	247.69	36.57	32.15	31.41	32.49
ρ =0.30	116.58	138.25	157.25	174.71	27.08	22.61	21.07	21.15
ρ =0.25	87.70	99.31	112.39	125.3	21.61	17.82	16.27	15.62
ρ =0.20	68.84	77.19	86.68	93.91	18.04	14.74	13.26	12.63
ρ =0.15	56.41	60.98	67.47	73.62	15.59	12.69	11.3	10.49
ρ =0.10	47.34	49.23	55.65	60.25	13.76	11.09	9.87	9.912
ρ =0.05	41.08	42.11	45.35	50.03	12.25	9.91	8.85	8.08

Table 3.4 ARLs for CUSUM charts using trace when covariances are change (ρ_0 = 0.3)

	$n = 2, p = 2$				$n = 4, p = 4$			
k	$E(V_t)$+0.5	$E(V_t)$+1	$E(V_t)$+1.5	$E(V_t)$+2	$E(V_t)$+0.5	$E(V_t)$+1	$E(V_t)$+1.5	$E(V_t)$+2
ρ_0 =0.30	806.28	806.95	809.17	800.83	801.01	803.26	808.2	800.44
ρ =0.27	676.44	700.02	702.83	716.42	354.16	388.89	425.67	450.07
ρ =0.24	543.45	599.21	593.74	619.83	188.32	215.53	239.79	263.65
ρ =0.21	455.36	495.51	522.63	553.25	123.02	132.75	150.21	167.93
ρ =0.18	384.84	419.28	451.2	476.77	88.049	89.41	99.72	113.28
ρ =0.15	326.96	362.01	390.76	411.33	69.114	66.26	72.20	79.78
ρ =0.12	278.44	316.55	337.99	358.24	55.74	52.68	54.29	59.52
ρ =0.09	233.77	274.5	291.24	312.54	47.68	42.69	43.25	46.56
ρ =0.06	204.46	234.35	255.43	269.61	40.69	36.08	36.46	37.16
ρ =0.03	178.89	203.83	217.35	234.27	36.08	31.33	30.26	31.55

For each c, $p = 2, 4$, $n = 2, 4$, $k = E(V_t)+(1/2)i$ $(i = 1, 2, 3, 4)$, Table 3.5 gives ARLs in each cell when one, two, p variances and p covariances are simultaneously change, respectively.

Here standard deviations are changed from σ_0 to $\sigma = \sqrt{c\sigma_0}$ for $c = 1.21, 1.44, 1.69, 4.00$ and covariances are changed from $\rho_0 = 0.9$ to $\rho = 0.72, 0.54$. As shown in Table 3.5 multivariate CUSUM control charts using the statistic V_t given by (2.1) for monitoring the covariance matrix are also effective in detecting simultaneous changes in variances and covariances.

Table 3.5 ARLs for CUSUM charts using trace when variances and covariances are change ($\rho_0 = 0.9$)

		$n=2, p=2$				$n=4, p=4$			
	k	$E(V_t)$+0.5	$E(V_t)$+1	$E(V_t)$+1.5	$E(V_t)$+2	$E(V_t)$+0.5	$E(V_t)$+1	$E(V_t)$+1.5	$E(V_t)$+2
c=1.00	ρ_0 =0.90	800.43	800.65	800.52	800.62	800.99	800.46	800.12	800.10
c=1.21	ρ =0.72	10.92	9.37	8.84	8.87	5.45 5.13	4.46 4.21	3.96 3.77	3.66 3.48
		9.42	8.01	7.66	7.70	4.90 4.69	4.02 3.87	3.61 3.47	3.31 3.22
	ρ =0.54	6.63	5.61	5.19	5.06	3.52 3.42	3.02 2.93	2.76 2.67	2.57 2.50
		5.96	5.10	4.78	4.61	3.30 3.21	2.84 2.76	2.61 2.55	2.45 2.40
c=1.44	ρ =0.72	9.10	7.72	7.29	7.05	5.02 4.52	4.15 3.76	3.7 3.38	3.42 3.13
		7.39	6.26	5.87	5.76	4.18 3.96	3.50 3.32	3.16 3.02	2.93 2.80
	ρ =0.54	5.86	4.98	4.63	4.51	3.38 3.17	2.89 2.74	2.66 2.53	2.49 2.39
		5.08	4.33	4.07	3.93	3.00 2.87	2.62 2.51	2.43 2.34	2.30 2.24
c=1.69	ρ =0.72	7.56	6.48	6.00	5.91	4.63 4.03	3.85 3.39	3.46 3.07	3.22 2.85
		6.20	5.23	4.89	4.71	3.69 3.46	3.14 2.97	2.85 2.7	2.67 2.55
	ρ =0.54	5.23	4.51	4.25	4.07	3.24 2.96	2.79 2.57	2.58 2.39	2.43 2.29
		4.42	3.85	3.64	3.53	2.77 2.64	2.45 2.34	2.29 2.21	2.19 2.14
c=4.00	ρ =0.72	3.92	3.44	3.27	3.19	3.10 2.54	2.71 2.30	2.52 2.19	2.42 2.13
		3.20	2.87	2.74	2.64	2.31 2.21	2.14 2.07	2.08 2.04	2.05 2.02
	ρ =0.54	3.26	2.93	2.83	2.73	2.50 2.19	2.28 2.08	2.18 2.05	2.13 2.02
		2.79	2.57	2.48	2.41	2.08 2.04	2.02 2.01	2.01 2.00	2.01 2.00

3.2. Control statistic using determinant

There are control limits h for multivariate CUSUM control charts for monitoring by using the statistic L_t given by (2.2). Table 3.6 gives the values of h for $p = 2, 4$, $n = 2, 4$, $k = E(L_t) + (1/2)i$ $(i = 1, 2, 3, 4)$ when the in-control ARL is approximately 800.

Table 3.6 Values of control limit h for CUSUM charts using determinant when the in-control ARL is 800

	$k = E(L_t) + 0.5$	$k = E(L_t) + 1$	$k = E(L_t) + 1.5$	$k = E(L_t) + 2$
$n = 2, p = 2$	10.5201	9.6298	8.9506	8.3700
$n = 4, p = 4$	2.4190	1.9151	1.4250	0.9250

Tables 3.7~3.9 give the ARLs of multivariate CUSUM control charts using determinant for $p = 2, 4$, $n = 2, 4$, $k = E(L_t) + (1/2)i (i = 1, 2, 3, 4)$ and three different in-control correlation coefficients $\rho_0 = 0.9, 0.5, 0.3$ when covariances are change and variances are not change. Also ρ considered is decreased from 10 percent to 90 percent. As shown in Tables 3.7~3.9 the multivariate CUSUM control charts using determinant for monitoring the variance-covariance matrix are effective in detecting changes in covariances.

Table 3.7 ARLs for CUSUM charts using determinant when covariances are change (ρ_0 =0.9)

	$n=2, p=2$				$n=4, p=4$			
k	$E(L_t)$+0.5	$E(L_t)$+1	$E(L_t)$+1.5	$E(L_t)$+2	$E(L_t)$+0.5	$E(L_t)$+1	$E(L_t)$+1.5	$E(L_t)$+2
ρ_0 =0.90	800.41	800.70	800.29	800.57	800.89	800.75	800.03	800.09
ρ =0.81	88.67	114.54	132.08	132.20	28.64	31.35	32.60	32.23
ρ =0.72	38.90	48.74	59.91	58.22	9.64	10.44	10.94	11.02
ρ =0.63	24.61	30.68	37.84	37.76	5.91	6.20	6.50	6.65
ρ =0.54	19.58	23.36	28.19	27.89	4.58	4.79	4.91	4.94
ρ =0.45	16.34	19.07	22.86	23.11	3.91	4.03	4.20	4.19
ρ =0.36	14.73	16.84	20.20	19.94	3.55	3.63	3.74	3.70
ρ =0.27	13.88	15.62	18.40	18.31	3.35	3.40	3.46	3.43
ρ =0.18	13.01	14.48	17.30	17.15	3.18	3.23	3.32	3.29
ρ =0.09	12.58	14.16	16.49	16.78	3.09	3.15	3.20	3.22

Table 3.8 ARLs for CUSUM charts using determinant when covariances are change ($\rho_0 = 0.5$)

	$n=2, p=2$				$n=4, p=4$			
k	$E(L_t)+0.5$	$E(L_t)+1$	$E(L_t)+1.5$	$E(L_t)+2$	$E(L_t)+0.5$	$E(L_t)+1$	$E(L_t)+1.5$	$E(L_t)+2$
ρ_0 =0.50	800.53	807.44	801.62	805.18	804.05	805.02	806.13	803.66
ρ =0.45	643.9	652.43	642.35	662.43	481.28	490.95	493.26	482.12
ρ =0.40	530.37	539.55	546.83	546.84	324.23	326.51	331.79	326.63
ρ =0.35	456.04	467.15	465.74	484.95	236.45	235.21	235.25	236.37
ρ =0.30	404.56	408.52	425.03	430.96	175.71	174.97	180.06	177.44
ρ =0.25	358.99	374.30	379.82	381.20	141.53	141.02	142.77	145.45
ρ =0.20	336.87	349.14	358.44	366.09	119.40	119.15	120.49	121.68
ρ =0.15	313.87	325.18	337.02	347.10	102.62	103.18	104.99	103.52
ρ =0.10	299.39	309.40	325.63	332.84	92.39	93.02	94.28	95.54
ρ =0.05	287.36	304.61	311.72	316.68	86.79	89.43	89.22	88.97

Table 3.9 ARLs for CUSUM charts using determinant when covariances arechange ($\rho_0 = 0.3$)

	$n=2, p=2$				$n=4, p=4$			
k	$E(L_t)+0.5$	$E(L_t)+1$	$E(L_t)+1.5$	$E(L_t)+2$	$E(L_t)+0.5$	$E(L_t)+1$	$E(L_t)+1.5$	$E(L_t)+2$
ρ_0 =0.30	803.22	807.79	799.03	802.54	800.53	800.90	800.47	800.86
ρ =0.27	758.16	744.45	735.54	761.33	663.33	673.30	679.16	673.91
ρ =0.24	706.31	713.40	710.71	714.11	585.02	576.24	581.90	577.80
ρ =0.21	671.01	685.31	676.63	680.93	493.71	502.28	508.28	510.03
ρ =0.18	640.47	648.92	651.08	654.18	447.82	448.63	450.87	448.22
ρ =0.15	620.40	619.40	627.58	641.25	404.11	409.39	421.57	411.77
ρ =0.12	594.50	609.97	613.47	622.03	376.33	378.14	379.46	379.36
ρ =0.09	590.53	584.92	595.53	602.12	348.40	347.93	354.19	344.91
ρ =0.06	573.87	590.41	588.64	593.62	333.01	338.57	334.00	330.41
ρ =0.03	576.82	582.42	576.28	589.78	320.00	324.52	328.23	320.69

For $p = 2, 4$, $n = 2, 4$, $k = E(L_t) + (1/2)i$ $(i = 1, 2, 3, 4)$, Table 3.10 gives p ARLs in each cell when one, two, p variances and p covariances are simultaneously change, respectively. Here standard deviations are changed from σ_0 to $\sigma = \sqrt{c\sigma_0}$ for $c = 1.21, 1.44, 1.69, 4.00$ and covariances are changed from $\rho_0 = 0.9$ to $\rho = 0.72, 0.54$. As shown in Table 3.10 multivariate CUSUM control charts using the statistic L_t given by (2.2) for monitoring the covariance matrix are also effective in detecting simultaneously changes in variances and covariances.

Table 3.10 ARLs for CUSUM charts using determinant when variances and covariances are change (ρ_0 =0.9)

		$n=2, p=2$				$n=4, p=4$			
	k	$E(L_t)+0.5$	$E(L_t)+1$	$E(L_t)+1.5$	$E(L_t)+2$	$E(L_t)+0.5$	$E(L_t)+1$	$E(L_t)+1.5$	$E(L_t)+2$
c=1.00	ρ_0 =0.90	800.41	800.70	800.29	800.57	800.89	800.75	800.03	800.09
c=1.21	ρ =0.72	32.82	37.11	39.31	42.47	9.18 7.98	9.30 8.27	9.38 8.33	9.52 8.31
		23.13	26.16	28.45	30.22	7.01 6.24	7.28 6.49	7.36 6.53	7.29 6.44
	ρ =0.54	17.43	19.11	20.72	21.63	4.45 4.17	4.55 4.20	4.50 4.21	4.58 4.29
		13.24	14.63	15.6	16.27	3.90 3.65	3.94 3.7	3.96 3.72	3.94 3.73
c=1.44	ρ =0.72	24.05	27.22	29.01	30.55	8.12 6.51	8.28 6.52	8.47 6.6	8.42 6.72
		14.01	15.45	16.56	17.25	5.33 4.53	5.41 4.63	5.41 4.62	5.41 4.62
	ρ =0.54	13.70	15.10	16.07	16.72	4.16 3.68	4.22 3.74	4.24 3.73	4.24 3.73
		9.27	9.92	10.39	10.97	3.33 3.07	3.37 3.13	3.39 3.10	3.40 3.11
c=1.69	ρ =0.72	18.57	20.97	22.25	23.61	7.40 5.49	7.31 5.53	7.53 5.58	7.61 5.64
		9.89	10.54	11.09	11.52	4.37 3.63	4.42 3.67	4.40 3.63	4.47 3.69
	ρ =0.54	11.30	12.35	13.07	13.71	3.94 3.35	4.00 3.38	4.01 3.43	4.05 3.41
		7.12	7.32	7.75	7.86	3.03 2.73	3.03 2.74	3.02 2.73	3.03 2.73
c=4.00	ρ =0.72	9.03	9.82	10.21	10.6	5.91 3.66	5.87 3.63	5.90 3.65	5.90 3.70
		3.67	3.66	3.67	3.71	2.64 2.23	2.65 2.23	2.65 2.23	2.63 2.23
	ρ =0.54	5.72	5.84	5.99	6.21	3.11 2.50	3.15 2.51	3.16 2.53	3.16 2.51
		3.25	3.26	3.24	3.30	2.25 2.12	2.25 2.11	2.25 2.13	2.25 2.12

3.3. Control statistic using likelihood ratio test

There are control limits h for multivariate CUSUM control charts for monitoring by using the statistic W_t given by (2.4). Table 3.11 gives the values of h for $p = 2, 4$, $n = 2, 4$, $k = E(W_t) + (1/2)i$ $(i = 1, 2, 3, 4)$ when the in-control ARL is approximately 800.

Table 3.11 Values of control limit h for CUSUM charts using likelihood ratio test statistic when the in-control ARL is approximately 800

	$k = E(W_t)+0.5$	$k = E(W_t)+1$	$k = E(W_t)+1.5$	$k = E(W_t)+2$
$n = 2, p = 2$	1444.90	1046.00	650.00	268.00
$n = 4, p = 4$	81.31	75.40	70.70	66.69

Tables 3.12∼3.14 give the ARLs of multivariate CUSUM control charts using likelihood ratio test statistic for $p = 2, 4$, $n = 2, 4$, $k = E(W_t) + (1/2)i(i = 1, 2, 3, 4)$ and three different in-control correlation coefficients $\rho_0 = 0.9, 0.5, 0.3$ when covariances are change and variances are not change. Also ρ considered is decreased from 10 percent to 90 percent. As shown in Tables 3.12-3.14 the multivariate CUSUM control charts using likelihood ratio test statistic for monitoring the variance-covariance matrix are effective in detecting changes in covariances.

Table 3.12 ARLs for CUSUM charts using LRT statistic when covariances are change (ρ_0 =0.9)

	$n = 2, p = 2$				$n = 4, p = 4$			
k	$E(W_t)+0.5$	$E(W_t)+1$	$E(W_t)+1.5$	$E(W_t)+2$	$E(W_t)+0.5$	$E(W_t)+1$	$E(W_t)+1.5$	$E(W_t)+2$
ρ_0 =0.90	800.63	800.49	800.75	800.02	800.07	800.18	800.07	800.47
ρ =0.81	622.62	574.18	490.53	320.53	83.46	92.21	102.84	113.85
ρ =0.72	431.94	367.95	277.54	145.19	14.27	14.26	14.44	14.76
ρ =0.63	314.95	255.99	181.39	87.47	6.94	6.74	6.61	6.53
ρ =0.54	242.39	191.73	131.56	60.99	4.59	4.40	4.26	4.15
ρ =0.45	194.87	151.55	102.01	46.44	3.44	3.30	3.19	3.11
ρ =0.36	161.93	124.47	82.70	37.12	2.78	2.67	2.58	2.50
ρ =0.27	137.95	105.10	69.24	30.85	2.36	2.26	2.18	2.12
ρ =0.18	119.76	90.69	59.34	26.36	2.06	1.97	1.91	1.85
ρ =0.09	105.52	79.63	51.93	23.00	1.84	1.77	1.71	1.66

Table 3.13 ARLs for CUSUM charts using LRT statistic when covariances are change (ρ_0 = 0.5)

	$n = 2, p = 2$				$n = 4, p = 4$			
k	$E(W_t)+0.5$	$E(W_t)+1$	$E(W_t)+1.5$	$E(W_t)+2$	$E(W_t)+0.5$	$E(W_t)+1$	$E(W_t)+1.5$	$E(W_t)+2$
ρ_0 =0.50	796.12	794.21	790.49	776.75	762.01	761.96	771.21	771.01
ρ =0.45	783.37	776.74	763.45	717.93	658.50	671.95	684.29	691.70
ρ =0.40	764.38	751.20	724.54	642.59	528.50	547.01	567.32	586.02
ρ =0.35	740.78	720.23	679.30	563.58	389.52	415.43	438.10	459.07
ρ =0.30	713.99	685.71	631.13	489.46	269.43	293.55	320.24	341.89
ρ =0.25	685.08	649.38	583.33	425.18	183.67	202.72	223.08	241.12
ρ =0.20	655.13	612.89	536.83	370.33	121.75	135.52	150.66	166.99
ρ =0.15	624.78	576.69	493.61	324.18	79.90	88.61	98.94	110.92
ρ =0.10	594.57	541.66	453.31	284.21	53.93	59.46	66.42	73.51
ρ =0.05	564.97	508.10	416.85	250.74	37.81	40.70	44.40	43.18

Table 3.14 ARLs for CUSUM charts using LRT statistic when covariances are change (ρ_0 = 0.3)

	$n = 2, p = 2$				$n = 4, p = 4$			
k	$E(W_t)+0.5$	$E(W_t)+1$	$E(W_t)+1.5$	$E(W_t)+2$	$E(W_t)+0.5$	$E(W_t)+1$	$E(W_t)+1.5$	$E(W_t)+2$
ρ_0 =0.30	799.70	799.02	798.28	794.84	790.33	788.53	794.52	791.34
ρ =0.27	796.70	795.04	791.77	779.43	765.23	763.16	771.66	771.01
ρ =0.24	791.89	788.37	781.27	756.26	721.01	724.11	735.41	740.91
ρ =0.21	785.34	779.51	767.32	726.45	661.18	675.59	688.15	692.67
ρ =0.18	777.24	768.55	750.49	691.53	597.67	613.94	633.21	642.12
ρ =0.15	767.81	755.77	731.23	654.22	524.15	543.44	566.32	583.67
ρ =0.12	757.05	741.48	710.01	615.50	448.65	471.11	495.84	519.03
ρ =0.09	745.18	725.97	687.40	576.91	373.49	401.79	424.95	449.47
ρ =0.06	732.37	709.26	663.91	538.20	305.26	331.36	359.15	381.69
ρ =0.03	718.82	691.84	639.52	501.23	243.71	267.26	293.16	317.03

For $p = 2, 4$, $n = 2, 4$, $k = E(W_t) + (1/2)i$ $(i = 1, 2, 3, 4)$, Table 3.15 gives p ARLs in each cell when one, two, p variances and p covariances are simultaneously change, respectively. Here standard deviations are changed from σ_0 to $\sigma = \sqrt{c\sigma_0}$ for $c = 1.21, 1.44, 1.69, 4.00$ and covariances are changed from $\rho_0 = 0.9$ to $\rho = 0.72, 0.54$. As shown in Table 3.15 multivariate CUSUM control charts using the statistic W_t given by (2.4) for monitoring the variance-covariance matrix are also effective in detecting simultaneously changes in variances and covariances.

Table 3.15 ARLs for CUSUM charts using LRT statistic when variances and covariances are change ($\rho_0 = 0.9$)

		$n=2, p=2$				$n=4, p=4$			
	k	$E(W_t)$+0.5	$E(W_t)$+1	$E(W_t)$+1.5	$E(W_t)$+2	$E(W_t)$+0.5	$E(W_t)$+1	$E(W_t)$+1.5	$E(W_t)$+2
c=1.00	ρ_0 =0.90	800.63	800.49	800.75	800.02	800.07	800.18	800.07	800.47
c=1.21	ρ =0.72	379.78	316.77	231.98	116.66	11.85 10.34	11.73 10.17	11.77 10.13	11.86 10.16
		349.86	288.31	207.89	102.44	9.26 8.54	9.08 8.32	9.00 8.22	8.98 8.18
	ρ =0.54	213.88	167.40	113.53	52.06	4.17 3.87	4.01 3.71	3.88 3.59	3.78 3.49
		193.80	150.64	101.34	46.12	3.61 3.41	3.46 3.26	3.35 3.17	3.27 3.08
c=1.44	ρ =0.72	320.63	261.25	185.53	89.83	9.58 7.55	9.37 7.33	9.30 7.20	9.26 7.12
		281.64	226.00	157.73	74.57	6.46 5.84	6.27 5.63	6.14 5.48	6.04 5.38
	ρ =0.54	185.31	143.68	96.30	43.65	3.78 3.27	3.62 3.14	3.51 3.04	3.41 2.95
		156.62	120.24	79.67	35.73	2.93 2.69	2.81 2.58	2.71 2.50	2.64 2.42
c=1.69	ρ =0.72	265.34	211.78	146.69	68.85	7.65 5.66	7.44 5.47	7.32 5.33	7.24 5.22
		228.15	179.41	122.45	56.38	4.78 4.35	4.58 4.17	4.46 4.04	4.35 3.94
	ρ =0.54	159.39	122.40	81.27	36.45	3.40 2.81	3.26 2.69	3.16 2.60	3.08 2.52
		128.40	97.58	63.99	28.50	2.45 2.22	2.35 2.13	2.27 2.05	2.20 1.99
c=4.00	ρ =0.72	89.80	67.43	43.69	19.36	2.77 1.88	2.66 1.81	2.57 1.75	2.51 1.70
		73.07	54.55	35.19	15.58	1.59 1.5	1.53 1.44	1.48 1.40	1.45 1.37
	ρ =0.54	59.97	44.62	28.70	12.77	1.84 1.36	1.78 1.32	1.72 1.29	1.68 1.27
		44.93	33.32	21.44	9.64	1.19 1.12	1.16 1.10	1.14 1.09	1.13 1.08

4. Summary and concluding remarks

This paper is a study on the multivariate CUSUM control charts using three different control statistics for monitoring covariance matrix.

We consider multivariate CUSUM control charts using three different control statistics for $p = 2, 4$, $n = 2, 4$, $k = E(S_t) + (1/2)i$ $(i = 1, 2, 3, 4)$. We got control limits and ARLs of the proposed multivariate CUSUM control charts using three different control statistics by using computer simulations. The performance of these proposed multivariate CUSUM control charts is compared in terms of their ARLs.

The objective of monitoring is assumed to be the detection of small as well as large shifts in Σ and as quickly as highly correlated variables shifts. The conclusions from this investigation can be summarized as follows. As shown in Tables 3.1-3.15, we conclude that the multivariate CUSUM control charts based on the trace in (2.1) are very effective in terms of ARLs for detecting changes in covariance, variance and covariance in Σ, especially changes in covariances when the variables are highly correlated.

We recommend to use the multivariate CUSUM control charts with $\rho_0 = 0.9$ using trace for detecting of changes in Σ.

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