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# A FINITE PRESENTATION FOR THE TWIST SUBGROUP OF THE MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE

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Abstract. Let $N_{g,s}$ denote the nonorientable surface of genus $g$ with $s$ boundary components. Recently Paris and Szepietowski [12] obtained an explicit finite presentation for the mapping class group $\mathcal{M}(N_{g,s})$ of the surface $N_{g,s}$, where $s \in \{0,1\}$ and $g+s>3$. Following this work, we obtain a finite presentation for the subgroup $\mathcal{T}(N_{g,s})$ of $\mathcal{M}(N_{g,s})$ generated by Dehn twists.

## 1. Introduction

Let $N_{g,s}$ be a smooth, nonorientable, compact surface of genus $g$ with $s$ boundary components. If $s$ is zero, then we omit it from the notation. If we do not want to emphasise the numbers $g, s$, we simply write $N$ for a surface $N_{g,s}$. Recall that $N_g$ is a connected sum of $g$ projective planes and $N_{g,s}$ is obtained from $N_g$ by removing $s$ open disks.

Let $\mathrm{Diff}(N)$ be the group of all diffeomorphisms $h\colon N \to N$ such that $h$ is the identity on each boundary component. By $\mathcal{M}(N)$ we denote the quotient group of $\mathrm{Diff}(N)$ by the subgroup consisting of maps isotopic to the identity, where we assume that isotopies are the identity on each boundary component. $\mathcal{M}(N)$ is called the *mapping class group* of $N$.

The mapping class group $\mathcal{M}(S_{g,s})$ of an orientable surface is defined analogously, but we consider only orientation preserving maps.

### 1.1. Background

One of the most important elements in mapping class groups of surfaces are Dehn twists. They were discovered by Max Dehn, who first observed that they generate the mapping class group $\mathcal{M}(S_g)$ of a closed oriented surface $S_g$. Twists were rediscovered by Lickorish [8, 10], who also proved that $\mathcal{M}(S_g)$ is

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generated by $3g-1$ Dehn twists about nonseparating circles. Later Humphries reduced this generating set to $2g+1$ twists [4].

Since Dehn twists generate the mapping class group $\mathcal{M}(S_g)$, it is natural to ask about possible relations between them. Let us mention some results in this direction. Birman [1] observed that there is a close relation between mapping class group $\mathcal{M}(S_g)$ and the mapping class group of a punctured sphere, which in fact is a quotient of the braid group $B_{2g+2}$. This correspondence leads to a number of interesting relations, for example: *braid* and *chain relations*, relations with hyperelliptic involution, relations with elements of finite order. Later Johnson [5] discovered the so-called *lantern relation*, which apparently has been used by Dehn in 1920's. It turned out that this set of relations was enough to give a full presentation of $\mathcal{M}(S_g)$, which was obtained by Wajnryb [16]. Later some other relations were discovered, for example *star relations* or relations between fundamental elements in Artin groups embedded in $\mathcal{M}(S_g)$. These relations led to some other interesting presentations of $\mathcal{M}(S_g)$ – see [3, 11].

In the nonorientable case, Lickorish [9] first observed that Dehn twists do not generate the mapping class group $\mathcal{M}(N_g)$ for $g \geq 2$. More precisely, he proved that Dehn twists generate the so-called *twist subgroup* $\mathcal{T}(N_g)$ which is of index 2 in $\mathcal{M}(N_g)$. Later Chillingworth [2] found finite generating sets for $\mathcal{T}(N_g)$ and $\mathcal{M}(N_g)$. These generating sets were extended to the case of a surface with punctures and/or boundary components in [6, 13, 14].

As for relations, recently Paris and Szepietowski [12] obtained a finite presentations for groups $\mathcal{M}(N_{g,s})$ where $s \in \{0,1\}$ and $g+s>3$.

### 1.2. Main results

The main goal of this paper is to find a complete set of relations between Dehn twists on a nonorientable surface $N$. To be more precise, we obtain a presentation for the twist subgroup $\mathcal{T}(N_{g,s})$ of the mapping class group $\mathcal{M}(N_{g,s})$ of a nonorientable surface (Theorems 3.1 and 3.2), where $s \in \{0,1\}$ and $g+s>3$. The obtained presentations may seem to be complicated, but many relations are needed only for small genera and stably the presentations are quite simple.

Our starting point is the presentation of $\mathcal{M}(N_{g,s})$ obtained by Paris and Szepietowski [12], however their presentation has $g-1$ generators which are not elements of $\mathcal{T}(N_{g,s})$, hence it leads to a very complicated presentation of the twist subgroup. Therefore, we use a recent simplification of their presentation [15], which has only one generator not belonging to $\mathcal{T}(N_{g,s})$ (Theorems 2.1, 2.2 and 2.3).

## 2. Preliminaries

### 2.1. Notation

Let us represent surfaces $N_{g,0}$ and $N_{g,1}$ as respectively a sphere or a disc with $g$ crosscaps and let $\alpha_1,\ldots,\alpha_{g-1},\beta$ be two-sided circles indicated in Figure 1.

Small arrows in this figure indicate directions of Dehn twists $a_1, \dots, a_{g-1}, b$

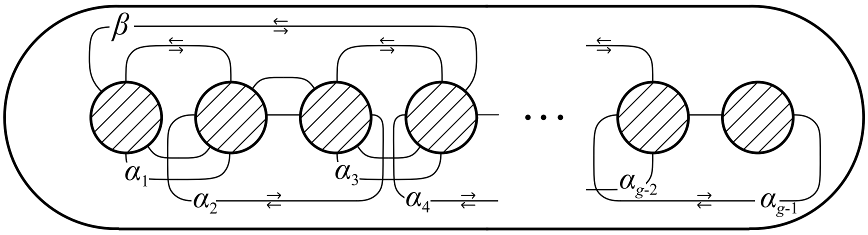


FIGURE 1. Surface $N$ as a sphere/disc with crosscaps.

associated with these circles. Observe that $\beta$ (hence also $b$) is defined only if $g \geq 4$. From now on whenever we use $b$, we silently assume that $g \geq 4$.

Moreover, for any unoriented one-sided circle $\mu$ and oriented two-sided circle $\alpha$ which intersects $\mu$ in one point (Figure 2), we define a *crosscap slide* (or Y-homeomorphism) $Y_{\mu,\alpha}$, that is the effect of pushing $\mu$ along the curve $\alpha$ – for precise definition see Section 2.2 of [12].

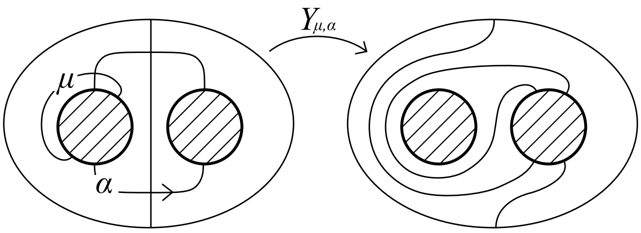


FIGURE 2. Crosscap slide.

In particular, let $y = Y_{\mu_1,\alpha_1}$, where $\mu_1, \alpha_1$ are curves indicated in Figure 3.

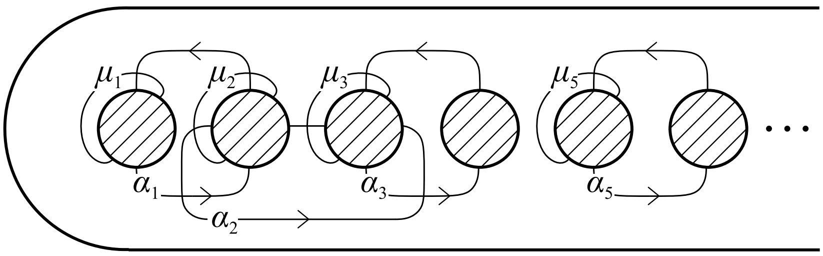


FIGURE 3. Circles $\mu_i$ and $\alpha_i$.

The following three theorems are the main results of [15].

**Theorem 2.1.** *If $g \geq 3$ is odd or $g = 4$, then $\mathcal{M}(N_{g,1})$ admits a presentation with generators $a_1, \ldots, a_{g-1}, y$ and $b$ for $g \geq 4$. The defining relations are*

(A1) $a_i a_j = a_j a_i$ *for* $g \geq 4$, $|i - j| > 1$,
(A2) $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ *for* $i = 1, \ldots, g-2$,
(A3) $a_i b = b a_i$ *for* $g \geq 4, i \neq 4$,
(A4) $b a_4 b = a_4 b a_4$ *for* $g \geq 5$,
(A5) $(a_2 a_3 a_4 b)^{10} = (a_1 a_2 a_3 a_4 b)^6$ *for* $g \geq 5$,
(A6) $(a_2 a_3 a_4 a_5 a_6 b)^{12} = (a_1 a_2 a_3 a_4 a_5 a_6 b)^9$ *for* $g \geq 7$,
(B1) $y(a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1}) = (a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1}) y$ *for* $g \geq 4$,
(B2) $y(a_2 a_1 y^{-1} a_2^{-1} y a_1 a_2) y = a_1 (a_2 a_1 y^{-1} a_2^{-1} y a_1 a_2) a_1$,
(B3) $a_i y = y a_i$ *for* $g \geq 4$, $i = 3, 4, \ldots, g-1$,
(B4) $a_2 (y a_2 y^{-1}) = (y a_2 y^{-1}) a_2$,
(B5) $y a_1 = a_1^{-1} y$,
(B6) $byby^{-1} = [a_1 a_2 a_3 (y^{-1} a_2 y) a_3^{-1} a_2^{-1} a_1^{-1}][a_2^{-1} a_3^{-1} (y a_2 y^{-1}) a_3 a_2]$ *for* $g \geq 4$,
(B7) $(a_4 a_5 a_3 a_4 a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1} a_4^{-1} a_3^{-1} a_5^{-1} a_4^{-1}) b$
$= b (a_4 a_5 a_3 a_4 a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1} a_4^{-1} a_3^{-1} a_5^{-1} a_4^{-1})$ *for* $g \geq 6$,
(B8) $[(y a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1}) b (a_4 a_3 a_2 a_1 y^{-1})][(a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1}) b^{-1} (a_4 a_3 a_2 a_1)]$
$= [(a_4^{-1} a_3^{-1} a_2^{-1}) y (a_2 a_3 a_4)][a_3^{-1} a_2^{-1} y^{-1} a_2 a_3][a_2^{-1} y a_2] y^{-1}$ *for* $g \geq 5$.

*If $g \geq 6$ is even, then $\mathcal{M}(N_{g,1})$ admits a presentation with generators $a_1, \ldots, a_{g-1}$, $y$, $b$ and additionally $b_0, b_1, \ldots, b_{\frac{g-2}{2}}$. The defining relations are relations* (A1)–(A6), (B1)–(B8) *above and additionally*

(A7) $b_0 = a_1$, $b_1 = b$,
(A8) $b_{i+1} = (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3} b_i)^5 (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3})^{-6}$
*for* $1 \leq i \leq \frac{g-4}{2}$,
(A9a) $b_2 b = b b_2$ *for* $g = 6$,
(A9b) $b_{\frac{g-2}{2}} a_{g-5} = a_{g-5} b_{\frac{g-2}{2}}$ *for* $g \geq 8$.

**Theorem 2.2.** *If $g \geq 4$, then the group $\mathcal{M}(N_{g,0})$ is isomorphic to the quotient of the group $\mathcal{M}(N_{g,1})$ with presentation given in Theorem 2.1 obtained by adding a generator $\varrho$ and relations*

(C1a) $(a_1 a_2 \cdots a_{g-1})^g = \varrho$ *for* $g$ *odd*,
(C1b) $(a_1 a_2 \cdots a_{g-1})^g = 1$ *for* $g$ *even*,
(C2$_1$) $\varrho a_1 = a_1 \varrho$,
(C3) $\varrho^2 = 1$,
(C4a) $(y^{-1} a_2 a_3 \cdots a_{g-1} y a_2 a_3 \cdots a_{g-1})^{\frac{g-1}{2}} = 1$ *for* $g$ *odd*,
(C4b) $(y^{-1} a_2 a_3 \cdots a_{g-1} y a_2 a_3 \cdots a_{g-1})^{\frac{g-2}{2}} y^{-1} a_2 a_3 \cdots a_{g-1} = \varrho$ *for* $g$ *even*.

**Theorem 2.3.** *Relations* (C4a), (C4b) *and* (C2$_1$) *in the presentation given by Theorem 2.2 may be replaced by*

(C2) $\varrho a_i = a_i \varrho$ *for* $i = 1, \ldots, g-1$,
(C5) $y \varrho = \varrho y^{-1}$,
(C4) $(y \varrho a_2 a_3 \cdots a_{g-1})^{g-1} = 1$.

## 3. Presentation for the twist subgroup

Recall that for $s \leq 1$ and $g \geq 3$ the twist subgroup $\mathcal{T}(N_{g,s})$ has index 2 in $\mathcal{M}(N_{g,s})$ (for details see [9, 13]), hence we can obtain its presentation using Reidemeister–Schreier rewriting process. To be more precise, we define a Schreier transversal $U = \{1, y\}$ for $\mathcal{T}(N_{g,s})$ in $\mathcal{M}(N_{g,s})$ and for any $h \in \mathcal{M}(N_{g,s})$ we define

$$\overline{h} = \begin{cases} 1 & \text{if } h \in \mathcal{T}(N_{g,s}), \\ y & \text{if } h \notin \mathcal{T}(N_{g,s}). \end{cases}$$

The Reidemeister–Schreier theorem states that $\mathcal{T}(N_{g,s})$ admits a presentation with generators $ux\overline{ux}^{-1}$, where $x$ is a generator of $\mathcal{M}(N_{g,s})$, $u \in U$ and $ux \notin U$. The set of defining relations consists of relations of the form $uru^{-1}$, where $u \in U$ and $r$ is a defining relation for $\mathcal{M}(N_{g,s})$.

**Theorem 3.1.** *If $g \geq 3$ is odd or $g = 4$, then $\mathcal{T}(N_{g,1})$ admits a presentation with generators $a_1, \ldots, a_{g-1}, e, f, y^2$ and $b, c$ for $g \geq 4$. The defining relations are* (A1)–(A6) *and*

($\overline{\text{A1}}_1$) $ea_j = a_je$ *for* $g \geq 5$, $j \geq 4$,
($\overline{\text{A1}}_2$) $fa_j = a_jf$ *for* $g \geq 5$, $j \geq 4$,
($\overline{\text{A2}}_1$) $a_1ea_1 = ea_1e$,
($\overline{\text{A2}}_2$) $a_3^{-1}ea_3^{-1} = ea_3^{-1}e$ *for* $g \geq 4$,
($\overline{\text{A2}}_3$) $a_1fa_1 = fa_1f$,
($\overline{\text{A3}}_1$) $a_1c = ca_1$ *for* $g = 4, 5$,
($\overline{\text{A3}}_2$) $ec = ce$ *for* $g = 4, 5$,
($\overline{\text{A4}}$) $ca_4c = a_4ca_4$ *for* $g = 5, 6$,
($\overline{\text{A5}}$) $(e^{-1}a_3a_4c)^{10} = (a_1^{-1}e^{-1}a_3a_4c)^6$ *for* $g = 5, 6$,
($\overline{\text{A6}}$) $(e^{-1}a_3a_4a_5a_6c)^{12} = (a_1^{-1}e^{-1}a_3a_4a_5a_6c)^9$ *for* $g = 7, 8$,
($\overline{\text{B1}}$) $(a_2a_3a_1a_2ea_1a_3^{-1}e)(a_2a_3a_1a_2fa_1a_3^{-1}f) = 1$ *for* $g \geq 4$,
($\overline{\text{B2}}_1$) $y^2 = a_2a_1ea_1a_2a_1a_2a_1a_2fa_1a_2$,
($\overline{\text{B2}}_2$) $(a_2a_1ea_1a_2a_1a_2a_1a_2fa_1a_2)(a_2a_1fa_1a_2a_1a_2a_1a_2ea_1a_2) = 1$,
($\overline{\text{B3}}$) $y^2a_3 = a_3y^2$ *for* $g \geq 4$,
($\overline{\text{B4}}_1$) $ea_2 = a_2e$,
($\overline{\text{B4}}_2$) $fa_2 = a_2f$,
($\overline{\text{B6}}_1$) $bc = [a_1a_2a_3f^{-1}a_3^{-1}a_2^{-1}a_1^{-1}][a_2^{-1}a_3^{-1}e^{-1}a_3a_2]$ *for* $g \geq 4$,
($\overline{\text{B6}}_2$) $c(y^2by^{-2}) = [a_1^{-1}e^{-1}a_3a_2a_3^{-1}ea_1][ea_3^{-1}(y^2a_2y^{-2})a_3e^{-1}]$ *for* $g = 4, 5$,
($\overline{\text{B7}}_1$) $(a_4a_5a_3a_4a_2a_3a_1a_2ea_1a_3^{-1}ea_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1})c$
$= b(a_4a_5a_3a_4a_2a_3a_1a_2ea_1a_3^{-1}ea_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1})$ *for* $g \geq 6$,
($\overline{\text{B7}}_2$) $(a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1})b(a_4a_5a_3a_4a_2a_3a_1a_2)y^2$
$= y^2(a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1})b(a_4a_5a_3a_4a_2a_3a_1a_2)$ *for* $g \geq 6$,
($\overline{\text{B8}}_1$) $\left[(a_1ea_3^{-1}a_4^{-1})c(a_4a_3e^{-1}a_1^{-1})\right]\left[(a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})b^{-1}(a_4a_3a_2a_1)\right]$
$= a_4^{-1}\left[(a_3^{-1}a_2^{-1}e^{-1}a_3)a_4(a_3^{-1}ea_2a_3)\right]a_2^{-1}e^{-1}$ *for* $g \geq 5$,
($\overline{\text{B8}}_2$) $\left[(a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})b(a_4a_3a_2a_1)\right]\left[(a_1fa_3^{-1}a_4^{-1})y^{-2}c^{-1}y^2(a_4a_3f^{-1}a_1^{-1})\right]$
$= a_4^{-1}\left[(a_3^{-1}fa_2a_3)a_4(a_3^{-1}a_2^{-1}f^{-1}a_3)\right]fa_2$ *for* $g = 5, 6$.

*If $g \geq 6$ is even, then $\mathcal{T}(N_{g,1})$ admits a presentation with generators $a_1, \ldots, a_{g-1}$, $e, f, y^2, b, c$ and additionally $b_0, b_1, \ldots, b_{\frac{g-2}{2}}$, $\overline{b}_{\frac{g-6}{2}}, \overline{b}_{\frac{g-4}{2}}, \overline{b}_{\frac{g-2}{2}}$. The defining relations are relations* (A1)–(A9), $(\overline{\text{A1}}_1)$–$(\overline{\text{A6}})$, $(\overline{\text{B1}})$–$(\overline{\text{B8}}_2)$ *and additionally*

$(\overline{\text{A7a}})$ $\overline{b}_0 = a_1^{-1}$, $\overline{b}_1 = c$ *for* $g = 6$,
$(\overline{\text{A7b}})$ $\overline{b}_1 = c$ *for* $g = 8$,
$(\overline{\text{A7c}})$ $\overline{b}_i = z_{g-1} b_i z_{g-1}^{-1}$ *where* $i = \frac{g-6}{2}, \frac{g-4}{2}$, $i \geq 2$ *and*

$$z_{g-1} = (a_{g-1}a_g a_{g-2}a_{g-1} \cdots a_3 a_4 e^{-1} a_3 a_1^{-1} e^{-1})(a_2^{-1} a_1^{-1} \cdots a_{g-1}^{-1} a_{g-2}^{-1} a_g^{-1} a_{g-1}^{-1}),$$

$(\overline{\text{A8a}})$ $\overline{b}_2 = (\overline{b}_0 e^{-1} a_3 a_4 a_5 \overline{b}_1)^5 (\overline{b}_0 e^{-1} a_3 a_4 a_5)^{-6}$ *for* $g = 6$,
$(\overline{\text{A8b}})$ $\overline{b}_{\frac{g-2}{2}} = (\overline{b}_{\frac{g-6}{2}} a_{g-4} a_{g-3} a_{g-2} a_{g-1} \overline{b}_{\frac{g-4}{2}})^5 (\overline{b}_{\frac{g-6}{2}} a_{g-4} a_{g-3} a_{g-2} a_{g-1})^{-6}$ *for* $g \geq 8$,
$(\overline{\text{A9a}})$ $\overline{b}_2 c = c \overline{b}_2$ *for* $g = 6$,
$(\overline{\text{A9b}})$ $\overline{b}_{\frac{g-2}{2}} a_{g-5} = a_{g-5} \overline{b}_{\frac{g-2}{2}}$ *for* $g \geq 8$.

*Proof.* As noted before, we apply Reidemeister–Schreier theorem to the presentation given by Theorem 2.1. Hence as generators of the twist subgroup $\mathcal{T}(N_{g,1})$ we obtain $a_1, \ldots, a_{g-1}, ya_1y^{-1}, \ldots, ya_{g-1}y^{-1}, y^2$ and $b, yby^{-1}$ for $g \geq 4$. Moreover, if $g \geq 6$ is even, we have additional generators: $b_0, b_1, \ldots, b_{\frac{g-2}{2}}$, $yb_0y^{-1}, yb_1y^{-1}, \ldots, yb_{\frac{g-2}{2}}y^{-1}$. Let us name some of these generators:

$$e = ya_2^{-1}y^{-1},\ c = yby^{-1},\ \overline{b}_i = yb_iy^{-1} \text{ for } i = 0, \ldots, \frac{g-2}{2}.$$

We also add one generator $f = y^{-1}a_2^{-1}y$ with defining relation

(D1) $f = y^{-2}ey^2$

(see Figure 4).

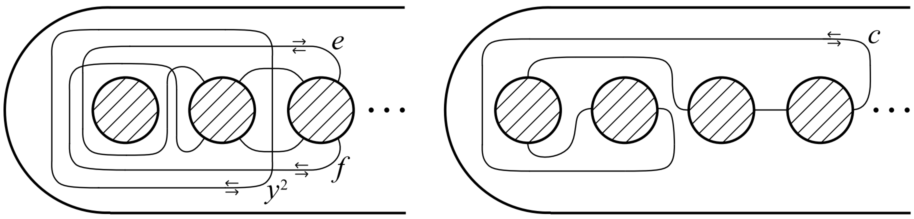


FIGURE 4. Twists $e, f, y^2, c$.

**(B3)** Observe first that relation (B3) rewrites as

$$ya_iy^{-1} = a_i \text{ for } i = 3, 4, \ldots, g-1.$$

This means that we can remove generators $ya_3y^{-1}, \ldots, ya_{g-1}y^{-1}$ from the presentation, hence from now on we will silently identify $ya_iy^{-1}$ with $a_i$ for $i = 3, 4, \ldots, g-1$.

**(B5)** Similarly, (B5) allows us to identify $ya_1y^{-1}$ with $a_1^{-1}$.

Observe also that conjugations of (B3) and (B5) by $y$ give

($\overline{\text{B3}}$) $y^2a_i = a_iy^2$ for $i = 1, 3, 4, \dots, g-1$.

We will show later that this relation can be replaced by ($\overline{\text{A1}}_2$) if $i \neq 3$.

**(A1)–(A9)** Relations which do not contain $y$, that is (A1)–(A9) does not need rewriting, however we need to add their versions conjugated by $y$. This gives relations ($\overline{\text{A1}}_1$), ($\overline{\text{A2}}_1$), ($\overline{\text{A2}}_2$), ($\overline{\text{A3}}_2$), ($\overline{\text{A4}}$)–($\overline{\text{A6}}$) and

($\overline{\text{A3}}_1$) $a_ic = ca_i$ for $g \geq 4$, $i \neq 2, 4$.

If $g \geq 6$ is even, then we have additionally

($\overline{\text{A7}}$) $\bar{b}_0 = a_1^{-1}$, $\bar{b}_1 = c$,
($\overline{\text{A8a}}$) $\bar{b}_2 = (\bar{b}_0e^{-1}a_3a_4a_5\bar{b}_1)^5(\bar{b}_0e^{-1}a_3a_4a_5)^{-6}$,
($\overline{\text{A8b}}$) $\bar{b}_{i+1} = (\bar{b}_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3}\bar{b}_i)^5(\bar{b}_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3})^{-6}$ for $2 \leq i \leq \frac{g-4}{2}$,
($\overline{\text{A9a}}$) $\bar{b}_2c = c\bar{b}_2$ for $g = 6$,
($\overline{\text{A9b}}$) $\bar{b}_{\frac{g-2}{2}}a_{g-5} = a_{g-5}\bar{b}_{\frac{g-2}{2}}$ for $g \geq 8$.

**(B4)** Relation (B4) and its conjugation by $y^{-1}$ rewrite respectively as ($\overline{\text{B4}}_1$) and ($\overline{\text{B4}}_2$). It is also useful to note that relations (D1), ($\overline{\text{B3}}$), ($\overline{\text{A2}}_1$) and ($\overline{\text{A2}}_2$) imply that

($\overline{\text{A2}}_3$) $a_1fa_1 = fa_1f$,
($\overline{\text{A2}}_4$) $a_3^{-1}fa_3^{-1} = fa_3^{-1}f$ for $g \geq 4$.

**(B2)** Using (A2), ($\overline{\text{A2}}_1$), ($\overline{\text{A2}}_2$) and ($\overline{\text{B4}}_1$) we rewrite (B2).

$$\underset{\rightarrow}{[\underline{y}]}(a_2a_1\underline{y^{-1}}a_2^{-1}\underset{\rightarrow}{[y]}a_1a_2)y = a_1(a_2a_1y^{-1}a_2^{-1}ya_1a_2)a_1,$$
$$[e^{-1}a_1^{-1}\underset{\leftarrow}{a_2^{-1}}a_1^{-1}e^{-1}]y^2 = a_1a_2a_1f[a_1a_2a_1],$$
$$y^2 = e[a_1a_2a_1]ea_1a_2a_1fa_2a_1a_2,$$
$$y^2 = \underset{\rightarrow}{[e]}a_2a_1a_2\underset{\leftarrow}{[e]}a_1a_2a_1fa_2a_1a_2,$$
$$y^2 = a_2[ea_1e]a_2a_1a_2a_1a_2fa_1a_2,$$
$$y^2 = a_2a_1ea_1a_2a_1a_2a_1a_2fa_1a_2.$$

In the above computations we introduced the notation which should help the reader to follow our transformations. The underlined parts indicate expressions which will be reduced, and parts with small arrows indicate expressions which will be moved to the left/right.

As a conjugation of (B2) we can take

$$(a_2a_1y^{-1}a_2^{-1}ya_1a_2) = y^{-1}a_1(a_2a_1y^{-1}a_2^{-1}ya_1a_2)a_1y^{-1}.$$

By a straightforward computation this gives

$$y^{-2} = a_2a_1fa_1a_2a_1a_2a_1a_2ea_1a_2,$$

which together with ($\overline{\text{B2}}_1$) gives ($\overline{\text{B2}}_2$).

Observe that ($\overline{\mathrm{B}2}_1$) together with (A1) and ($\overline{\mathrm{A}1}_1$) imply that we can replace ($\overline{\mathrm{B}3}$) for $i \geq 4$ with ($\overline{\mathrm{A}1}_2$).

Observe also that ($\overline{\mathrm{B}2}_1$), (A2), ($\overline{\mathrm{B}4}_1$) and ($\overline{\mathrm{B}4}_2$) imply that ($\overline{\mathrm{B}3}$) for $i = 1$ is superfluous.

We will now show that (D1) is superfluous – we will need here ($\overline{\mathrm{A}2}_3$), hence we add this relation to the statement. Using ($\overline{\mathrm{B}2}_1$) we substitute for $y^2$.

$$\begin{aligned} f &= (\underset{\leftarrow}{[\underline{a_2^{-1}}]}a_1^{-1}f^{-1}a_2^{-1}a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-1}\underline{a_1^{-1}}e^{-1}a_1^{-1}\underline{a_2^{-1}})\underset{\leftarrow}{[\underline{e}]} \\ &\quad (\underline{a_2}a_1e\underset{\rightarrow}{[a_1]}a_2a_1a_2a_1a_2fa_1\underset{\rightarrow}{[\underline{a_2}]}), \\ \underline{f} &= \underline{(a_1^{-1}f^{-1}a_2^{-1}a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-1}e^{-1}a_1^{-1})(a_1ea_2a_1a_2a_1a_2fa_1)}\underline{f}. \end{aligned}$$

**(B1)** If we use (B1) in the form

$$(a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1})y^{-1} = y^{-1}(a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}),$$

after rewriting we get ($\overline{\mathrm{B}1}$). Conjugating this relation by $y$ gives

$$(\overline{\mathrm{B}1}_2) \quad \begin{aligned} &y(a_2a_3a_1a_2\underset{\leftarrow}{[\underline{y}]}a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1})\underset{\rightarrow}{[\underline{y^{-2}}]} = (a_2a_3a_1a_2\underline{y}a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1})\underset{\leftarrow}{[\underline{y^{-1}}]}, \\ &y^2(f^{-1}a_3a_1^{-1}f^{-1}a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}) = (a_2a_3a_1a_2ea_1a_3^{-1}e)y^2. \end{aligned}$$

Now we will show that this relation is superfluous – it is a consequence of relations (A1), (A2), ($\overline{\mathrm{A}2}_1$)–($\overline{\mathrm{A}2}_4$), ($\overline{\mathrm{B}1}$), ($\overline{\mathrm{B}2}_1$), ($\overline{\mathrm{B}2}_2$), ($\overline{\mathrm{B}4}_1$), ($\overline{\mathrm{B}4}_2$). We substitute for $y^2$ using ($\overline{\mathrm{B}2}_1$) and ($\overline{\mathrm{B}2}_2$).

$$\begin{aligned} &(\underline{a_2a_1}ea_1a_2a_1\underline{a_2}a_1a_2fa_1a_2)(\underset{\leftarrow}{[\underline{f^{-1}}]}a_3a_1^{-1}f^{-1}a_2^{-1}\underline{a_1^{-1}}a_3^{-1}\underline{a_2^{-1}}) \\ &= (\underline{a_2}a_3\underline{a_1}a_2ea_1a_3^{-1}\underset{\rightarrow}{[\underline{e}]})(a_2^{-1}a_1^{-1}e^{-1}\underline{a_1^{-1}}a_2^{-1}a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-1}f^{-1}\underline{a_1^{-1}a_2^{-1}}), \\ &(ea_1a_2\underset{\leftarrow}{[a_1]}\underset{\rightarrow}{[a_1]}\underline{a_2}fa_1a_2)(a_3\underset{\leftarrow}{[\underline{a_1^{-1}}]}f^{-1}a_2^{-1}a_3^{-1}) \\ &= (a_3a_2e\underset{\rightarrow}{[\underline{a_1}]}a_3^{-1})(a_2^{-1}a_1^{-1}e^{-1}\underline{a_2^{-1}}\underset{\leftarrow}{[a_1^{-1}]}a_2^{-1}a_1^{-1}\underset{\rightarrow}{[a_2^{-1}]}f^{-1}), \\ &(\underset{\leftarrow}{[e]}a_1a_2fa_1a_2)(\underline{f}a_3f^{-1}a_2^{-1}a_3^{-1})\underset{\leftarrow}{[\underline{a_2}]} \\ &= \underset{\rightarrow}{[\underline{a_2^{-1}}]}(a_3a_2ea_3^{-1})(\underline{e^{-1}}a_2^{-1}a_1^{-1}e^{-1}a_2^{-1}a_1^{-1}\underset{\rightarrow}{[f^{-1}]}), \\ &(a_1a_2fa_1a_2)(\underline{a_3}\underset{\rightarrow}{[f^{-1}]}\underset{\leftarrow}{[a_2^{-1}]}\underline{a_3^{-1}}f) = (e^{-1}\underline{a_3}\underset{\rightarrow}{[a_2]}\underset{\leftarrow}{[e]}\underline{a_3^{-1}})(a_2^{-1}a_1^{-1}e^{-1}a_2^{-1}a_1^{-1}), \\ &a_1a_2fa_1a_3^{-1}\underset{\rightarrow}{[a_2]}f\underset{\rightarrow}{[a_3]} = \underset{\leftarrow}{[a_3^{-1}]}e^{-1}\underset{\leftarrow}{[a_2^{-1}]}a_3a_1^{-1}e^{-1}a_2^{-1}a_1^{-1}, \\ &a_2a_3a_1a_2fa_1a_3^{-1}f = e^{-1}a_3a_1^{-1}e^{-1}a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}. \end{aligned}$$

What we get is ($\overline{\mathrm{B}1}$).

**(B6)** If we rewrite (B6) we get ($\overline{\mathrm{B}6}_1$), and (B6) conjugated by $y$ gives ($\overline{\mathrm{B}6}_2$).

**(B7)** If we use (B7) in the form

$$\begin{aligned}&a_4a_5a_3a_4a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1}by^{-1}\\ &= ba_4a_5a_3a_4a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1}y^{-1},\end{aligned}$$

after rewriting we get ($\overline{\mathrm{B7}}_1$). By conjugating this relation by $y^{-1}$, taking inverses of both sides and using (D1), we get

$$\begin{aligned}&(\underset{\leftarrow}{[a_4a_5a_3a_4a_2a_3a_1a_2]}fa_1a_3^{-1}fa_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1})\underset{\leftarrow}{[y^{-2}]}cy^2\\ &= b(a_4a_5a_3a_4a_2a_3a_1a_2\underset{\rightarrow}{[fa_1a_3^{-1}fa_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1}]}),\\ &y^{-2}(ea_1a_3^{-1}ea_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1})c(a_4a_5a_3a_4e^{-1}a_3a_1^{-1}e^{-1})y^2\\ &= (a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1})b(a_4a_5a_3a_4a_2a_3a_1a_2).\end{aligned}$$

This together with ($\overline{\mathrm{B7}}_1$) gives ($\overline{\mathrm{B7}}_2$). For further reference observe that using ($\overline{\mathrm{B1}}$) the above relation can be also rewritten as

($\overline{\mathrm{B7}}_3$)
$$\begin{aligned}&(a_4a_5a_3a_4e^{-1}a_3a_1^{-1}e^{-1})(a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1})y^{-2}cy^2\\ &= b(a_4a_5a_3a_4e^{-1}a_3a_1^{-1}e^{-1})(a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1}).\end{aligned}$$

Observe that we can use ($\overline{\mathrm{B7}}_1$) and ($\overline{\mathrm{B6}}_1$) as definitions of $c$. It is straightforward to check that the first of these relations imply ($\overline{\mathrm{A3}}_2$) and ($\overline{\mathrm{A3}}_1$) for $i = 1$. The second one imply ($\overline{\mathrm{A3}}_1$) for $i = 3$ and $i \geq 5$.

**(B8)** If we rewrite (B8) we get ($\overline{\mathrm{B8}}_1$) and (B8) conjugated by $y^{-1}$ gives ($\overline{\mathrm{B8}}_2$).

**Further reductions.** For any $3 \leq k \leq g-1$ define

$$z_k = (a_{k-1}a_ka_{k-2}a_{k-1}\cdots a_3a_4e^{-1}a_3a_1^{-1}e^{-1})(a_2^{-1}a_1^{-1}\cdots a_{k-1}^{-1}a_{k-2}^{-1}a_k^{-1}a_{k-1}^{-1}).$$

Geometrically $z_k$ is the product of crosscap slides $yY^{\pm 1}_{\mu_k,\alpha_k}$, where $\mu_k$ and $\alpha_k$ are circles indicated in Figure 3 (see Section 4 of [15]), hence on the left of $\mu_k$, conjugation by $z_k$ has the same effect as conjugation by $y$. More precisely,

(D2) $z_ka_1z_k^{-1} = a_1^{-1}$,
(D3) $z_ka_2z_k^{-1} = e^{-1}$ for $k \geq 4$,
(D4) $z_ka_iz_k^{-1} = a_i$ for $3 \leq i \leq k-2$,
(D5) $z_kbz_k^{-1} = c$ for $k \geq 5$,
(D6) $z_ky^2z_k^{-1} = y^2$,
(D7) $z_kfz_k^{-1} = a_2^{-1}$ for $k \geq 4$,
(D8) $z_kez_k^{-1} = y^2a_2^{-1}y^{-2}$ for $k \geq 4$,
(D9) $z_kcz_k^{-1} = y^2by^{-2}$ for $k \geq 5$.

Relations (D2)–(D4) are straightforward consequences of (A1), (A2), ($\overline{\mathrm{A1}}_1$), ($\overline{\mathrm{A2}}_1$), ($\overline{\mathrm{A2}}_2$). For (D5) we need additionally (A3), ($\overline{\mathrm{A3}}_1$) and ($\overline{\mathrm{B7}}_1$).

Let us prove (D6) – we will use (A1), ($\overline{\mathrm{A1}}_1$), ($\overline{\mathrm{B1}}$), ($\overline{\mathrm{B3}}$) and ($\overline{\mathrm{B1}}_2$) (hence we need all relations that we used to reduce ($\overline{\mathrm{B1}}_2$)).

$$z_ky^2 = (a_{k-1}a_ka_{k-2}a_{k-1}\cdots a_3a_4e^{-1}a_3a_1^{-1}e^{-1})(a_2^{-1}a_1^{-1}\cdots a_{k-1}^{-1}a_{k-2}^{-1}a_k^{-1}a_{k-1}^{-1})y^2$$

$$\begin{aligned}
&= (a_{k-1}a_k \cdots a_3a_4)[e^{-1}a_3a_1^{-1}e^{-1}a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}]y^2(a_4^{-1}a_3^{-1}\cdots a_k^{-1}a_{k-1}^{-1})\\
&= (a_{k-1}a_k \cdots a_3a_4)y^2[e^{-1}a_3a_1^{-1}e^{-1}a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}](a_4^{-1}a_3^{-1}\cdots a_k^{-1}a_{k-1}^{-1})\\
&= y^2z_k.
\end{aligned}$$

Now we will prove (D7) – we will use (A1), (A2), ($\overline{\text{A1}}_2$), ($\overline{\text{A2}}_4$), ($\overline{\text{B1}}$).

$$\begin{aligned}
z_kf &= (a_{k-1}a_k \cdots a_4a_5)a_3a_4[e^{-1}a_3a_1^{-1}e^{-1}a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}]a_4^{-1}a_3^{-1}\\
&\quad (a_5^{-1}a_4^{-1}\cdots a_k^{-1}a_{k-1}^{-1})f\\
&= (a_{k-1}a_k \cdots a_4a_5)a_3a_4[a_2a_3a_1a_2fa_1a_3^{-1}f]a_4^{-1}a_3^{-1}[\underset{\leftarrow}{f}](a_5^{-1}a_4^{-1}\cdots a_k^{-1}a_{k-1}^{-1})\\
&= (a_{k-1}a_k \cdots a_4a_5)a_2^{-1}a_3a_4[a_2a_3a_1a_2fa_1a_3^{-1}f]a_4^{-1}a_3^{-1}(a_5^{-1}a_4^{-1}\cdots a_k^{-1}a_{k-1}^{-1})\\
&= a_2^{-1}(a_{k-1}a_k \cdots a_3a_4)[e^{-1}a_3a_1^{-1}e^{-1}a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}](a_4^{-1}a_3^{-1}\cdots a_k^{-1}a_{k-1}^{-1})\\
&= a_2^{-1}z_k.
\end{aligned}$$

Relation (D8) is a consequence of (D6), (D7) and (D1). Finally, (D9) is a consequence of ($\overline{\text{B7}}_3$) and (D6) (hence we need ($\overline{\text{B7}}_2$)).

Relations (D2)–(D9) imply that

- ($\overline{\text{A4}}$) is superfluous if $g \geq 7$,
- ($\overline{\text{A5}}$) is superfluous if $g \geq 7$,
- ($\overline{\text{A6}}$) is superfluous if $g \geq 9$,
- ($\overline{\text{B6}}_2$) is superfluous if $g \geq 6$,
- ($\overline{\text{B8}}_2$) is superfluous if $g \geq 7$.

Moreover, if $g \geq 8$, relations ($\overline{\text{A8a}}$) and ($\overline{\text{A8b}}$) for $i < \frac{g-4}{2}$ are consequences of relation (A8). Hence we can remove all these relations together with generators $\overline{b}_0, \ldots, \overline{b}_{\frac{g-8}{2}}$ and instead add the relation

$$\overline{b}_i = z_{g-1}b_iz_{g-1}^{-1} \quad \text{for } i = \frac{g-6}{2},\ \frac{g-4}{2}.$$

This is exactly ($\overline{\text{A7c}}$). □

**Theorem 3.2.** *If $g \geq 5$ is odd, then the group $\mathcal{T}(N_{g,0})$ is isomorphic to the quotient of the group $\mathcal{T}(N_{g,1})$ with presentation given in Theorem 3.1 obtained by adding a generator $\varrho$ and relations*

(C1a) $(a_1a_2\cdots a_{g-1})^g = \varrho$,
($\overline{\text{C1a}}$) $(a_1^{-1}e^{-1}a_3\cdots a_{g-1})^g = y^2\varrho$,
(C2) $a_i\varrho = \varrho a_i$ *for* $i = 1, 2, \ldots, g-1$,
($\overline{\text{C2}}$) $\varrho e = f\varrho$,
($\overline{\text{C5}}$) $\varrho y^2 = y^{-2}\varrho$,
(C3) $\varrho^2 = 1$,
($\overline{\text{C4a}}$) $(a_2a_3\cdots a_{g-1}e^{-1}a_3\cdots a_{g-1})^{\frac{g-1}{2}} = 1$.

*Moreover, relations ($\overline{\text{A1}}_2$), ($\overline{\text{B2}}_2$), ($\overline{\text{B4}}_2$) are superfluous.*

*If $g \geq 4$ is even, then the group $\mathcal{T}(N_{g,0})$ is isomorphic to the quotient of the group $\mathcal{T}(N_{g,1})$ with presentation given in Theorem 3.1 obtained by adding a generator $\overline{\varrho}$ and relations*

(C1b) $(a_1a_2\cdots a_{g-1})^g = 1$,
($\overline{\text{C2}}_1$) $\overline{\varrho}a_1 = a_1^{-1}\overline{\varrho}$,
($\overline{\text{C2}}_2$) $\overline{\varrho}a_i = a_i\overline{\varrho}$ *for* $i = 3,\dots,g-1$,
($\overline{\text{C2}}_3$) $\overline{\varrho}a_2 = e^{-1}\overline{\varrho}$,
($\overline{\text{C5}}$) $\overline{\varrho}y^2 = y^{-2}\overline{\varrho}$,
($\overline{\text{C3}}$) $\overline{\varrho}^2 = 1$,
($\overline{\text{C4}}$) $(\overline{\varrho}a_2a_3\cdots a_{g-1})^{g-1} = 1$.

*Moreover, relations* ($\overline{\text{A1}}_1$), ($\overline{\text{A2}}_1$), ($\overline{\text{A2}}_2$) *are superfluous.*

*Proof.* We follow the lines of the proof of Theorem 3.1, but as a starting point we now have Theorem 2.2. Moreover, it is convenient to add relations (C2) and (C5), so in particular (C4a) and (C4b) are equivalent to (C4) (see Theorem 2.3). Generator $\varrho$ yields two additional generators for $\mathcal{T}(N_{g,0})$, namely $\varrho, y\varrho y^{-1}$ if $g$ is odd and $\overline{\varrho} = y\varrho, \varrho y^{-1}$ if $g$ is even.

Suppose first that $g$ is odd. Then (C5) and its conjugate by $y^{-1}$ rewrite as

$$y^2\varrho = y\varrho y^{-1},$$
$$y^2\varrho y^2 = \varrho.$$

The first relation implies that we can remove generator $y\varrho y^{-1}$ – we will do this silently from now on. The second one gives ($\overline{\text{C5}}$).

Relations (C1a), (C2), (C3) does not need rewriting, and if we conjugate them by $y$ we get respectively ($\overline{\text{C1a}}$), ($\overline{\text{C2}}$) (we use here ($\overline{\text{D1}}$), hence also ($\overline{\text{A2}}_3$)) and relation equivalent to ($\overline{\text{C5}}$).

Relation (C4a) and its conjugate by $y$ rewrite respectively as

$$(f^{-1}a_3\cdots a_{g-1}a_2a_3\cdots a_{g-1})^{\frac{g-1}{2}} = 1,$$
$$(a_2a_3\cdots a_{g-1}e^{-1}a_3\cdots a_{g-1})^{\frac{g-1}{2}} = 1.$$

The second relation is ($\overline{\text{C4a}}$), and if we conjugate it by $\varrho$, by (C2) and ($\overline{\text{C2}}$) we get the first one.

Finally, observe that if we conjugate relations ($\overline{\text{A1}}_1$), ($\overline{\text{B2}}_1$), ($\overline{\text{B4}}_1$) by $\varrho$ we get respectively ($\overline{\text{A1}}_2$), ($\overline{\text{B2}}_2$), ($\overline{\text{B4}}_2$).

Now assume that $g$ is even, hence $\overline{\varrho} = y\varrho \in \mathcal{T}(N_{g,0})$. Relation (C5) and its conjugate by $y$ rewrite as

$$y\varrho = \varrho y^{-1},$$
$$y^2(y\varrho) = (y\varrho)y^{-2}.$$

The first relation implies that we can remove generator $\varrho y^{-1}$ – we will do this silently from now on. The second one gives ($\overline{\text{C5}}$).

If we rewrite relation (C2) we get relations ($\overline{\text{C2}}_1$)–($\overline{\text{C2}}_3$).

Relations (C1b), (C3) and (C4) rewrite respectively as (C1b), ($\overline{\text{C3}}$) and ($\overline{\text{C4}}$). Their conjugates by $y^{\pm 1}$ are superfluous since, by ($\overline{\text{C2}}_1$)–($\overline{\text{C2}}_3$), they are the same as conjugates by $\overline{\varrho}$.

Finally, observe that if we conjugate relations (A1), (A2) by $\overline{\varrho}$ we get respectively ($\overline{\text{A1}}_1$), ($\overline{\text{A2}}_1$)–($\overline{\text{A2}}_2$). □

*Remark* 3.3. Observe that relations ($\overline{\text{B2}}_1$) and (C1a), ($\overline{\text{C4}}$) allows to remove $y^2$ and $\varrho, \overline{\varrho}$ from the generating sets, hence the generating sets of the presentations given by Theorems 3.1 and 3.2 are really Dehn twists about nonseparating circles.

## 4. Geometric interpretation

We devote this last section to the geometric interpretation of relations obtained in Theorem 3.1.

Relations (A1), (A3), (A9a), (A9b), ($\overline{\text{A1}}_1$), ($\overline{\text{A1}}_2$), ($\overline{\text{A3}}_1$), ($\overline{\text{A3}}_2$), ($\overline{\text{B3}}$), ($\overline{\text{B4}}_1$), ($\overline{\text{B4}}_2$), ($\overline{\text{B7}}_2$), ($\overline{\text{A9a}}$), ($\overline{\text{A9b}}$) are standard commutativity relations between Dehn twists with disjoint supports.

Relations (A2), (A4), ($\overline{\text{A2}}_1$)–($\overline{\text{A2}}_3$), ($\overline{\text{A4}}$) are standard braid relations between Dehn twists about circles intersecting in one point.

Relations (A5), (A6), (A8), ($\overline{\text{A5}}$), ($\overline{\text{A6}}$), ($\overline{\text{A8a}}$), ($\overline{\text{A8b}}$) came from Matsumoto [11] presentation of mapping class group of an orientable surface. They have simple interpretation as relations between fundamental elements of Artin groups – for details see [11] and [7].

Relations ($\overline{\text{B7}}_1$) and ($\overline{\text{A7c}}$) are simple conjugation relations of the form $t_{f(\alpha)} = f t_\alpha f^{-1}$, where $t_\alpha$ is the twist about a circle $\alpha$.

Relations ($\overline{\text{B6}}_2$) and ($\overline{\text{B8}}_2$) are conjugates (by $y^{\pm 1}$) of ($\overline{\text{B6}}_1$) and ($\overline{\text{B8}}_1$) respectively, and ($\overline{\text{B2}}_2$) is equivalent to the conjugation of ($\overline{\text{B2}}_1$), hence we are left with four interesting relations: ($\overline{\text{B1}}$), ($\overline{\text{B2}}_1$), ($\overline{\text{B6}}_1$) and ($\overline{\text{B8}}_1$).

Relation ($\overline{\text{B1}}$) can be rewritten in a slightly more symmetric form

$$(a_2 e a_1) a_3^{-1} (a_2 e a_1) a_3 (a_2 f a_1) a_3^{-1} (a_2 f a_1) a_3 = 1.$$

This is a relation between five Dehn twists $a_1, a_2, a_3, e, f$ illustrated in Figures 1 and 4.

Relation ($\overline{\text{B2}}_1$) can be rewritten as

$$y^2 = (a_2 e a_1)^2 (a_2 f a_1)^2.$$

This is a relation between five twists $a_1, a_2, e, f, y^2$ illustrated in Figures 1 and 4.

Relation ($\overline{\text{B6}}_1$) is a relation between four Dehn twists

$$b,\ c,\ f' = (a_1 a_2 a_3) f^{-1} (a_1 a_2 a_3)^{-1},\ e' = (a_3 a_2)^{-1} e^{-1} (a_3 a_2),$$

illustrated in Figures 1, 4 and 5.

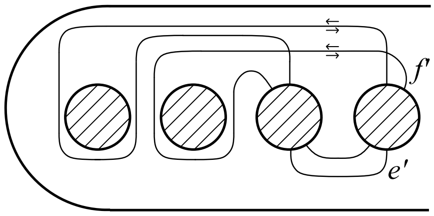


Figure 5. Dehn twists $e'$ and $f'$.

Finally, relation ($\overline{\mathrm{B}8}_1$) is a relation between six Dehn twists

$$c' = (a_1 e a_3^{-1} a_4^{-1}) c (a_1 e a_3^{-1} a_4^{-1})^{-1},\ b' = (a_4 a_3 a_2 a_1)^{-1} b^{-1} (a_4 a_3 a_2 a_1),$$
$$a_4,\ a' = (a_3^{-1} e a_2 a_3)^{-1} a_4 (a_3^{-1} e a_2 a_3),\ a_2,\ e.$$

illustrated in Figures 1, 4 and 6.

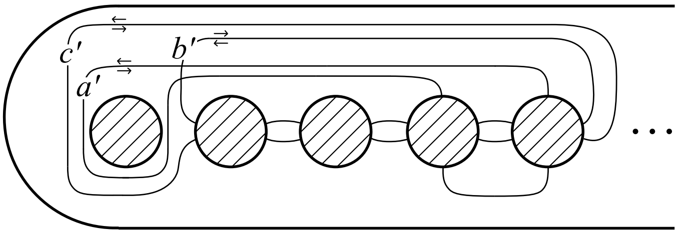


Figure 6. Dehn twists $a'$, $b'$ and $c'$.

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