

## GEOMETRIC LORENZ ATTRACTOR AND ORBITAL SHADOWING PROPERTY

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ABSTRACT. Komuro [3] proved that geometric Lorenz attractor does not satisfy the shadowing property. In this paper we study the condition under which geometric Lorenz attractor has the orbital shadowing property that is weaker than the shadowing property.

Geometric Lorenz attractor is a dynamical system constructed by Guckenheimer in order to analyse Lorenz system of equations which is related to fluid convection. Numerical observation indicates that all solutions of the Lorenz system pass transversely through a square which will be denoted by  $\Sigma$ , and so a two-dimensional invertible Poincaré map  $F$  can be defined on  $\Sigma$  as in Figure 1 (for more details, see [1]).

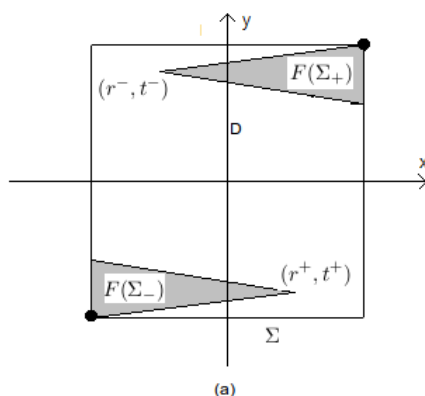


FIGURE 1. Poincaré map

Received March 30, 2015.

2010 *Mathematics Subject Classification*. Primary 37C50, 37D45, 37D50; Secondary 34C28.

*Key words and phrases*. geometric Lorenz model, strange attractor, orbital shadowing property.

In [2], Guckenheimer assume that there is a system of coordinate on  $\Sigma$  such that the Poincaré map  $F$  has the following properties:

1. There is a family  $\mathcal{F}$  of curves in  $\Sigma$  which contains  $D$  and has the property that  $\mathcal{F}$  is invariant under the map  $F$  and the curves in the family  $\mathcal{F}$  are given by  $x = \text{constant}$ , where  $D$  is the set with  $x = 0$ .
2. There are functions  $f$  and  $g$  such that  $F$  has the form  $F(x, y) = (f(x), g(x, y))$  for  $x \neq 0$  and  $F(-x, -y) = -F(x, y)$ .
3.  $f'(x) > \sqrt{2}$  for  $x \neq 0$ , and  $f'(x) \rightarrow \infty$  as  $x \rightarrow 0$ .
4.  $0 < \frac{\partial g}{\partial y} < 1$  for  $x \neq 0$ , and  $g'_y \rightarrow 0$  as  $x \rightarrow 0$ .

The above assumptions imply

$$DF(x, y) = \begin{pmatrix} f' & 0 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}, \quad f'_x > \sqrt{2}, \quad g'_y < 1.$$

Consequently,  $F$  contracts  $\Sigma$  in  $y$ -direction and expands  $\Sigma$  in  $x$ -direction. Now one can makes the geometric Lorenz attractor by using suspension flow as in Figure 2 (for more details see [1, 2]). Komuro [3] proved geometric Lorenz attractor does not satisfy the shadowing property except the case of  $f(0) = 1$  and  $f(1) = 1$ . Roughly speaking, it means that if  $f(0) \neq 0$  or  $f(1) \neq 1$ , there is a pseudo orbit (or chain) in geometric Lorenz attractor which cannot be shadowed by a real orbit such that the direction of the real orbit and the pseudo orbit is the same. As we can see in Section 2, the orbital shadowing property does not require the same direction between the real orbit and the pseudo orbit, and just require a small Hausdorff distance between their closures.

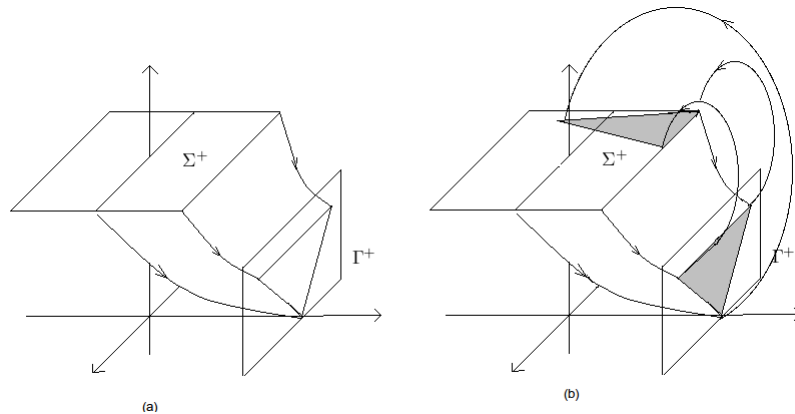


FIGURE 2. Suspension flow

On the other hand, Williams [4] described geometric Lorenz attractor as the inverse limit of a semi-flow on a 2-dimensional branched manifold, and showed

that the system is conjugate to the system constructed by Guckenheimer in [2]. In this paper we use the system by Williams to prove that geometric Lorenz attractor does not satisfy the orbital shadowing property. For this we introduce the notion of geometric Lorenz attractor by Williams.

Let  $X$  be a compact metric space with a distance function  $d$ . A semi-flow  $\phi = \{\phi^t\}_{t \geq 0}$  on  $X$  is a continuous map

$$\phi : X \times [0, \infty) \rightarrow X, (x, t) \mapsto \phi(x, t) = \phi^t(x) = x \cdot t$$

such that  $\phi^0$  is the identity map,  $\phi^t : X \rightarrow X$  is surjective and  $\phi^{t+s} = \phi^t \circ \phi^s$  holds for every  $t, s \geq 0$ . We define

$$\tilde{X} = \{\tilde{x} = (x^s)_{s \leq 0} \in \prod_{s \leq 0} X : x^t = \phi^{t-s}(x^s), s \leq t \leq 0\},$$

and

$$\tilde{\phi}^t(\tilde{x}) = \begin{cases} (\phi^t(x^s))_{s \leq 0} & t \geq 0, \\ (x^{s+t})_{s \leq 0} & t < 0. \end{cases}$$

A distance function on  $\tilde{X}$  is defined by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \int_0^\infty e^{-t} d(x^{-t}, y^{-t}) dt.$$

Then  $(\tilde{X}, \tilde{d})$  is a compact metric space, and  $\{\tilde{\phi}^t\}_{t \in \mathbb{R}}$  is a flow on  $\tilde{X}$ . The flow  $(\tilde{X}, \tilde{\phi})$  is called the *inverse limit* of  $(X, \phi)$ . We denote this by

$$(\tilde{X}, \tilde{\phi}) = \lim_{\leftarrow} (X, \phi).$$

For any  $0 \leq \alpha < \beta \leq \infty$ , the set  $\{\phi(x, t) \mid \alpha \leq t \leq \beta\}$  will be denoted by  $x \cdot [\alpha, \beta]$  if  $\beta < \infty$ , and by  $x \cdot [\alpha, \infty)$  if  $\beta = \infty$ . Let  $K$  be a 2-dimensional compact branched manifold illustrated as in Figure 3. We suppose that a  $C^1$ -semi-flow  $\phi$  on  $K$  is given as illustrated by arrows in Figure 3. Then  $(\tilde{K}, \tilde{\phi}) = \lim_{\leftarrow} (K, \phi)$  is called the *geometric Lorenz attractor* induced by  $(K, \phi)$ .

*Remark 1.* In Figure 3, we denote  $I = [b, c]$ ,  $I_a^+ = (a, c]$ ,  $I_a^- = [b, a)$  and  $I_a = I_a^+ \cup I_a^-$ . Let  $f : I_a \rightarrow I$  be the return map of  $\phi$ . More precisely, for each  $x \in I_a$ ,  $f(x)$  is defined by  $f(x) = \phi^T(x)$ , where  $T = \inf\{s > 0 : \phi^s(x) \in I\}$ . The point  $e$  is a singularity for  $\phi$  and so the point “ $a$ ” does not return to  $I$ , hence  $f$  is not defined on the point “ $a$ ”. Moreover,  $\cup_{i \in \mathbb{N}} \{f^{-i}(a)\}$  is a dense subset of  $[b, c]$  (for more details, see [3]).

**Definition.** A  $(\delta, T)$ -pseudo-orbit ( $T > 0$ ) of a flow  $\phi$  on a compact metric space  $X$  is defined as a sequence  $\{(x_i, t_i) : t_i \geq T, i \in \mathbb{Z}\} \subset X \times \mathbb{R}$  such that

$$d(\phi(x_i, t_i), x_{i+1}) < \delta$$

for all  $i \in \mathbb{Z}$ .

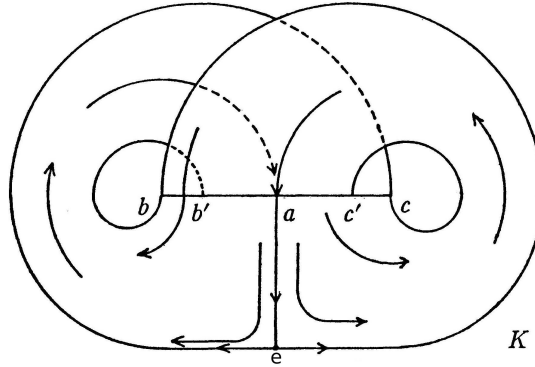


FIGURE 3. Branched manifold

For all  $i \in \mathbb{Z}$ , put

$$S_i = \begin{cases} \sum_{j=0}^{i-1} t_j & i > 0 \\ 0 & i = 0 \\ -\sum_{j=i}^{-1} t_j & i < 0. \end{cases}$$

For any  $(\delta, T)$ -pseudo-orbit  $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$  of  $\phi$ , we let  $x_0 * t = \phi(x_i, t - S_i)$  for any  $t \in [S_i, S_{i+1}]$ .

**Definition.** We say that a flow  $\phi$  on a compact metric space  $X$  has the *shadowing property* if for any  $\varepsilon > 0$  there exist  $\delta > 0$  and  $T \geq 1$  such that for any  $(\delta, T)$ -pseudo-orbit  $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$  of  $\phi$ , there exist a point  $x \in X$  and an increasing homeomorphism  $h : \mathbb{R} \rightarrow \mathbb{R}$  with  $h(0) = 0$ , such that

$$d(\phi(x, h(t)), x_0 * t) < \varepsilon$$

for all  $t \in \mathbb{R}$ .

**Definition.** We say that a flow  $\phi$  on a compact metric space  $X$  has the *orbital shadowing property* if for any  $\varepsilon > 0$ , there exist  $\delta > 0$  and  $T \geq 1$  such that for any  $(\delta, T)$ -pseudo-orbit  $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$  of  $\phi$ , there exists a point  $x \in X$  such that

$$d_H(\overline{\{x_0 * t : t \in \mathbb{R}\}}, \overline{\{\phi(x, t) : t \in \mathbb{R}\}}) < \varepsilon,$$

where  $d_H$  is the Hausdorff metric on the collection of closed subsets of  $X$ . Here we say that the  $(\delta, T)$ -pseudo-orbit  $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$  is  $\varepsilon$ -*orbitally shadowed* by the real orbit of  $O(x, \phi)$ .

Note that if a flow  $\phi$  has the shadowing property, then it has the orbital shadowing property, but the converse does not hold in general. Now we are going to investigate the condition under which geometric Lorenz attractor has the orbital shadowing property. The philosophy of the technique is similar to that in [3], even though we face with some difficulties in technical details. For simplicity, we assume  $a = \frac{1}{2}$ ,  $b = 0$  and  $c = 1$ . Suppose that  $f : [0, 1] - \{\frac{1}{2}\} \rightarrow$

$[0, 1]$  is the return map of the semi-flow  $\phi$  illustrated in Figure 3. Hereafter we use  $(K_f, \phi_f)$  instead of  $(K, \phi)$ . So  $(\tilde{K}_f, \tilde{\phi}_f)$  is the geometric Lorenz attractor induced by  $(K_f, \phi_f)$ . To prove our main result, we need the following lemma.

**Lemma 2.** *If  $(\tilde{K}_f, \tilde{\phi}_f)$  satisfies the orbital shadowing property, then  $(K_f, \phi_f)$  satisfies the orbital shadowing property.*

*Proof.* Put  $\varepsilon > 0$ . Let  $\delta > 0$  and  $T > 1$  be such that any  $(\delta, T)$ -pseudo-orbit of  $\tilde{\phi}_f$  can be  $\varepsilon$ -orbitally shadowed by a real orbit of  $\tilde{\phi}_f$ . Let  $N$  be such that  $\int_N^\infty e^{-r} dr < \frac{\delta}{2D}$ , where  $D$  is the diameter of  $K_f$ . Let  $\delta' > 0$  be such that if  $x, y \in K_f$  and  $d(x, y) < \delta'$ , then

$$d(\phi_f(x, t), \phi_f(y, t)) < \frac{\delta}{2}$$

for all  $0 \leq t \leq N$ . Assume that  $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$  is a  $(\delta', T)$ -pseudo orbit of  $\phi_f$ . Since  $\phi(\cdot, t)$  is surjective for all  $t \in \mathbb{R}$ , for any  $i \in \mathbb{Z}$ , there is  $\tilde{y}_i \in \tilde{K}_f$  such that  $\tilde{y}_i^{-t} = \phi(x_i, N - t)$  for  $0 \leq t \leq N$ . So we get

$$\begin{aligned} & \tilde{d}(\tilde{\phi}_f(\tilde{y}_i, t_i), \tilde{y}_{i+1}) \\ &= \int_0^\infty e^{-r} d((\tilde{\phi}_f(\tilde{y}_i, t_i))^{-r}, \tilde{y}_{i+1}^{-r}) dr \\ &\leq \int_0^N e^{-r} d(\phi_f((\tilde{y}_i^{-r}), t_i), \tilde{y}_{i+1}^{-r}) dr + \int_N^\infty e^{-r} D \\ &= \int_0^N e^{-r} d(\phi_f(x_i, N - r + t_i), \phi_f(x_{i+1}, N - r)) dr + \int_N^\infty e^{-r} D \\ &\leq \frac{\delta}{2} \int_0^N e^{-r} + \frac{\delta}{2} \leq \delta. \end{aligned}$$

So  $\tilde{\xi} = \{(\tilde{y}_i, t_i)\}_{i \in \mathbb{Z}}$  is a  $\delta$ -pseudo orbit of  $\phi_f$ . Since  $K_f$  is compact, there is  $\varepsilon' > 0$  such that  $d(x, y) < \varepsilon'$  implies  $d(\phi(x, t), \phi(y, t)) \leq \frac{\varepsilon}{2e}$  for all  $t \in [0, N]$ . Because  $(\tilde{K}_f, \tilde{\phi}_f)$  satisfies the orbital shadowing property, there is  $\tilde{y} \in \tilde{K}_f$  such that  $d_H(\overline{orbit(\tilde{y})}, \tilde{\xi}) < \varepsilon'$ . So if  $s, t \in \mathbb{R}$ , we get

$$\begin{aligned} & \tilde{d}(\tilde{\phi}_f(\tilde{y}, s), \tilde{y}_0 * t) = \tilde{d}(\tilde{\phi}_f(\tilde{y}, s), \tilde{\phi}_f(\tilde{y}_i, t - s_i)) \\ &= \int_0^\infty e^{-r} d((\tilde{\phi}_f(\tilde{y}, s))^{-r}, (\tilde{\phi}_f(\tilde{y}_i, t - s_i))^{-r}) dr \\ &\geq \int_0^1 e^{-r} d(\phi_f(\tilde{y}^{-r}, s), \phi_f(\tilde{y}_i^{-r}, t - s_i)) dr \\ &\geq \int_0^1 e^{-r} d(\phi_f(\tilde{y}^{-N}, N - r + s), \phi_f(x_i, N - r + t + s_i)) dr \\ &\geq e^{-1} d(\phi_f(\tilde{y}^{-N}, s + N - r_0), \phi(x_i, t - s_i + N - r_0)) \end{aligned}$$

for some  $r_0 \in [0, 1]$ . Hence we obtain

$$d(\phi_f(\tilde{y}^{-N}, s + N - r_0), \phi_f(x_i, t - s_i + N - r_0)) < e\varepsilon',$$

and so  $d(\tilde{y}^{-N} \cdot (s + N), x_i \cdot (t - s_i + N)) < \varepsilon$ . Consequently, for any  $s \in \mathbb{R}$ , ( $t \in \mathbb{R}$ ) there is  $t \in \mathbb{R}$  ( $s \in \mathbb{R}$ ) such that

$$d(\tilde{y} \cdot (s + N), y_0 * (t + N)) < \varepsilon.$$

This implies  $d_H(\overline{o(\tilde{y}^N)}, \bar{\xi}) < \varepsilon$ , and so completes the proof. □

**Theorem 3.** *Geometric Lorenz attractor satisfies the orbital shadowing property if and only if  $f(0) = 0$  and  $f(1) = 1$ .*

*Proof.* By applying the result of Theorem 1 in [3] and Lemma 2, it is enough to show that if  $f(0) \neq 0$  or  $f(1) \neq 1$ , then  $(K_f, \phi_f)$  does not satisfy the orbital shadowing property.

**Case 1.**  $f(0) \neq 0$  and  $f(1) \neq 1$ : Suppose  $b \in \cup_{i \in \mathbb{N}} f^{-i}(\{a\})$  and  $c \in \cup_{i \in \mathbb{N}} f^{-i}(\{a\})$ . We show that  $(K_f, \phi_f)$  does not satisfy the orbital shadowing property. Take  $r > 0$  such that  $\phi_f(c, r) = a$ , and let  $Arc(e, b)$  be the unique arc connecting  $e$  and  $b$  in  $K_f$  as in Figure 4. Let

$$\varepsilon' = d_H(Arc(e, b), \overline{c \cdot [0, r]}).$$

Choose  $s > 0$  and  $\varepsilon < \frac{\varepsilon'}{12}$  such that if  $d(x, y) < \varepsilon$ , then

$$d(\phi_f(x, s), \phi_f(y, s)) < \frac{\varepsilon'}{6} \quad \text{and} \quad d(\phi_f(b, s), e) < \frac{\varepsilon'}{6}.$$

We show that for every  $\delta > 0$  and  $T \geq 1$  there is a  $(\delta, T)$ -pseudo-orbit of  $\phi_f$  which can not be  $\varepsilon$ -orbitally shadowed by a real orbit of  $\phi_f$ .

Let  $x_0 = b$  and  $t_0 > T$  be such that  $d(\phi_f(x_0, t_0), e) < \frac{\delta}{2}$ . Take  $x_1 \in Arc(e, c)$  with  $d(x_1, e) < \frac{\delta}{2}$ . Then there exists  $t_1 > T$  such that

$$d(\phi_f(x_1, t_1), c) < \frac{\delta}{2}.$$

Put  $x_2 = c$ . Since  $c \in \cup_{i \in \mathbb{N}} f^{-i}(\{a\})$ , there is  $t_2 > T$  such that

$$d(\phi_f(x_2, t_2), e) < \frac{\delta}{2}.$$

Then  $\xi = \{\phi_f(x_i, t) \mid t \in (0, t_i); i = 0, 1, 2\}$  is a  $(\delta, T)$ -pseudo-orbit of  $\phi_f$ , and so there is  $y \in K$  such that  $d_H(\overline{y \cdot \mathbb{R}}, \bar{\xi}) < \varepsilon$ . This implies that there is  $t' > 0$  such that  $d(\phi_f(y, t'), b) < \varepsilon$ , and  $\phi_f(y, t')$  is in the right side of  $b$ . Hence  $d(\phi_f(y, t' + s), \phi(b, s)) < \frac{\varepsilon'}{6}$ , and  $\phi_f(y, t' + s)$  is in the right side of the line  $(a, e)$  (see Figure 4). Since  $d(\phi_f(b, s), e) < \frac{\varepsilon'}{6}$ , we have

$$d(\phi_f(y, t' + s), e) < \frac{\varepsilon'}{3}.$$

Let  $r_1 > 0$  and  $r_2 > 0$  be such that

$$d(\phi_f(y, t' + s + r_1), e) = \frac{\varepsilon'}{3} \quad \text{and} \quad d(\phi_f(y, t' + s + r_2), e) = \frac{2\varepsilon'}{3}.$$

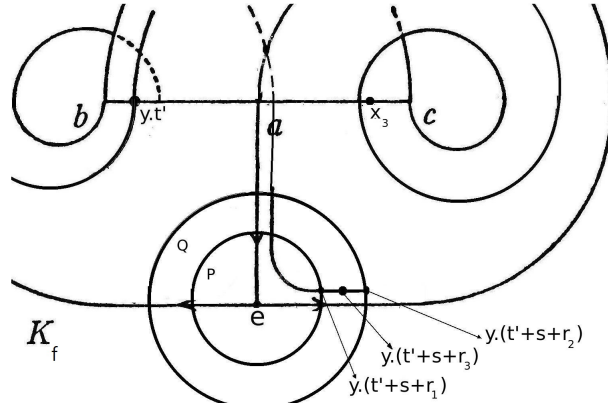


FIGURE 4. Two circles  $P$  and  $Q$  of radii  $\frac{\varepsilon'}{3}$  and  $\frac{2\varepsilon'}{3}$

Take  $r_3 > 0$  such that

$$d(\phi_f(y, t' + s + r_3), \phi_f(y, t' + s + r_1)) > \frac{\varepsilon'}{6} \quad \text{and}$$

$$d(\phi_f(y, t' + s + r_3), \phi_f(y, t' + s + r_2)) < \frac{\varepsilon'}{6}.$$

Notice that  $r_i$  ( $i = 1, 2, 3$ ) exists because of the direction of flow (see Figure 4). Since  $d(\phi_f(y, t' + s + r_1), e) = \frac{\varepsilon'}{3}$  and  $\varepsilon' = d_H(\text{Arc}(e, b), \overline{c \cdot [0, r]})$ , we have

$$d(\phi_f(y, t' + s + r_3), \overline{c \cdot [0, r]}) > d(\phi_f(y, t' + s + r_2), \overline{c \cdot [0, r]}) - d(\phi_f(y, t' + s + r_3), \phi_f(y, t' + s + r_2)) > \varepsilon.$$

Since  $(\phi_f(y, t' + s + r_3), \phi_f(y, t' + s + r_1)) > \frac{\varepsilon'}{6}$ , we have

$$d(\phi_f(y, t' + s + r_3), \overline{a \cdot [0, \infty)}) > (\phi_f(y, t' + s + r_3), \phi_f(y, t' + s + r_1)) > \varepsilon.$$

Consequently we have  $d_H(\{\phi_f(y, t' + s + r_3)\}, \bar{\xi}) > \varepsilon$ . This is a contradiction because  $\{\phi_f(y, (t' + s + r_3))\} \subset \overline{y \cdot \mathbb{R}}$ . If  $b \notin \cup_{i \in \mathbb{N}} f^{-i}(\{a\})$ , take  $t_0$  such that  $\phi_f(x_0, t_0) \in [b, a]$ , and choose  $x_1 \in [b, a] \cap \cup_{i \in \mathbb{N}} f^{-i}(\{a\})$  which is close enough to  $\phi_f(x_0, t_0)$  and  $x_1$  is in the left side of  $\phi_f(x_0, t_0)$ . Then we can drive a contradiction as we did in the above.

Similarly if  $c \notin \cup_{i \in \mathbb{N}} f^{-i}(\{a\})$ , take  $t_2 > T$  such that  $\phi_f(x_2, t_2) \in [a, c]$ , and choose  $x_3 \in \cup_{i \in \mathbb{N}} f^{-i}(\{a\})$  in the right side and close enough to  $\phi_f(x_2, t_2) \in [a, c]$ . Then we can drive a contradiction as we did in the above (see Figure 4).

**Case 2.**  $f(1) = 1$ : Put  $x_2 = c$ . Then we do not need to find  $t_2$  and  $x_3$  because  $\{x_0, x_1, x_2\}$  gives us a  $(\delta, T)$ -pseudo-orbit of  $\phi_f$  for every  $\delta > 0$  and  $T > 1$ . This completes the proof of Case 2.

**Case 3.**  $f(0) = 0$ : By assumption we have  $f(1) \neq 1$ . Put  $x_0 = c$ . Then the proof of Case 3 is similar to that of Case 2.  $\square$

**Acknowledgement.** The second author was supported by the National Research Foundation (NRF) of Korea funded by the Korean Government (No. 2011-0015193).

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