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SOME IDENTITIES OF DEGENERATE GENOCCHI POLYNOMIALS

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Abstract. L. Carlitz introduced higher order degenerate Euler polynomials in [4, 5] and studied a degenerate Staudt-Clausen theorem in [4]. D. S. Kim and T. Kim gave some formulas and identities of degenerate Euler polynomials which are derived from the fermionic p-adic integrals on $\mathbb{Z}_p$ (see [9]). In this paper, we introduce higher order degenerate Genocchi polynomials. And we give some formulas and identities of degenerate Genocchi polynomials which are derived from the fermionic p-adic integrals on $\mathbb{Z}_p$.

1. Introduction

Let p be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $|\cdot|_p$ be the p-adic norm with $|p|_p = 1/p$. For f in the space of continuous functions on $\mathbb{Z}_p$, the fermionic p-adic integrals on $\mathbb{Z}_p$ is introduced by Kim to be

$$(1)\qquad I_{-1}(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x \quad \text{(see [15, 18, 22])}.$$

There are many works related with fermionic p-adic integrals (see [15, 18, 22]). From (1), we note the integral equation as follows:

$$(2)\qquad I_{-1}(f_1)+I_{-1}(f)=2f(0), \quad \text{where } f_1(x)=f(x+1),$$

and iterated integral equation:

$$(3)\qquad I_{-1}(f_n)+(-1)^{n-1}I_{-1}(f)=2\sum_{l=0}^{n-1}(-1)^{n-1-l}f(l) \quad \text{(see [15, 18, 22])},$$

where $n\in\mathbb{N}$ and $f_n(x)=f(x+n)$.

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By (1), we easily get

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \sum_{a=0}^{d-1}(-1)^a \int_{\mathbb{Z}_p} f(a+dx)d\mu_{-1}(x), \tag{4}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$.

For $r \in \mathbb{Z}_+(=\mathbb{N}\cup\{0\})$ and $\lambda, t \in \mathbb{Z}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$, the degenerate Genocchi polynomials $\mathcal{G}_n^{(r)}(\lambda, x)$ of order r are defined by the generating function to be

$$\left(\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(r)}(\lambda, x)\frac{t^n}{n!}. \tag{5}$$

When $x=0$, $\mathcal{G}_n^{(r)}(\lambda) = \mathcal{G}_n^{(r)}(\lambda, 0)$ are called the degenerate Genocchi numbers of order r.

From (2), we note that

$$\begin{aligned} &t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ &= \left(\frac{2t}{e^t+1}\right)^r e^{xt} \\ &= \sum_{n=0}^{\infty} G_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [7, 21, 24, 27, 31])}, \end{aligned} \tag{6}$$

where $G_n^{(r)}(x)$ are called the Genocchi polynomials of order r. When $x=0$, $G_n^{(r)} = G_n^{(r)}(0)$ are called the Genocchi numbers of order r.

By (6), we have $G_0^{(r)}(x) = G_1^{(r)}(x) = \cdots = G_{r-1}^{(r)}(x) = 0$, thus we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r+x)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = \frac{G_{n+r}^{(r)}(x)}{(n+r)_r}, \tag{7}$$

where $n \geq 0$ and $(n)_r = n(n-1)\cdots(n-r+1) = \sum_{l=0}^{r} S_1(r,l)n^l$.

There have been many works related with various degenerate polynomials. For example, many authors apply degenerate polynomials to Boole polynomials in [8] and to Barnes-type Bernoulli polynomials in [11]. Degenerate polynomials related with higher order Euler polynomials is investigated by D. S. Kim and T. Kim in [10]. Also, Genocchi polynomials are studied by many authors (see [1-3, 6, 15-32]).The first paper, which introduces the q-extension of Genocchi numbers and polynomials, is [18] by Kim.

With the viewpoint of (7), we consider the degenerate Genocchi polynomials which can be represented by the multivariate fermionic p-adic integrals on $\mathbb{Z}_p$. The purpose of this paper is to give some formulas and identities of higher order degenerate Genocchi polynomials which are derived from the multivariate fermionic p-adic integrals on $\mathbb{Z}_p$.

2. Some identities of higher order degenerate Genocchi polynomials

In this section, we assume that $\lambda, t \in \mathbb{Z}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. Let us take

$$f(x_1, x_2, \ldots, x_r, x) = (1+\lambda t)^{\frac{x_1+\cdots+x_r+x}{\lambda}}.$$

Then, by (2), we get

$$\begin{aligned} t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x_1+\cdots+x_r+x}{\lambda}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ = \left(\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \right)^r (1+\lambda t)^{\frac{x}{\lambda}}. \end{aligned} \tag{8}$$

From (5) and (8), we have

$$\begin{aligned} \sum_{n=0}^{\infty} t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{x_1+\cdots+x_r+x}{\lambda} \right)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \frac{\lambda^n t^n}{n!} \\ = \sum_{n=0}^{\infty} \mathcal{G}_n^{(r)}(\lambda, x) \frac{t^n}{n!}. \end{aligned} \tag{9}$$

Now, we define $(x|\lambda)_n$ as

$$\begin{aligned} (x|\lambda)_n &= x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda) \\ &= \lambda^n \left(\frac{x}{\lambda}\right)\left(\frac{x}{\lambda}-1\right)\left(\frac{x}{\lambda}-2\right)\cdots\left(\frac{x}{\lambda}-n+1\right) \\ &= \lambda^n \left(\frac{x}{\lambda}\right)_n, \quad (n \geq 0). \end{aligned} \tag{10}$$

From (9), (10) and the fact that $\mathcal{G}_0^{(r)}(\lambda, x) = \mathcal{G}_1^{(r)}(\lambda, x) = \cdots = \mathcal{G}_{r-1}^{(r)}(\lambda, x) = 0$, we can derive the Witt-type formula for $\mathcal{G}_n^{(r)}(\lambda, x)$ as follows:

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r+x|\lambda)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = \frac{\mathcal{G}_{n+r}^{(r)}(\lambda, x)}{(n+r)_r}, \quad (n \geq 0). \tag{11}$$

Therefore, by (11), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$\begin{aligned} \frac{\mathcal{G}_{n+r}^{(r)}(\lambda, x)}{(n+r)_r} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r+x|\lambda)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ &= \lambda^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{x_1+\cdots+x_r+x}{\lambda} \right)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \end{aligned}$$

Now, we observe that

$$\left(\frac{x_1+\cdots+x_r+x}{\lambda} \right)_n = \sum_{l=0}^{n} S_1(n,l) \left(\frac{x_1+\cdots+x_r+x}{\lambda} \right)^l. \tag{12}$$

By Theorem 2.1 and (12), we get

$$\begin{aligned}\frac{\mathcal{G}_{n+r}^{(r)}(\lambda,x)}{(n+r)_r} &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^l d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\frac{G_{l+r}^{(r)}(x)}{(l+r)_r}.\end{aligned}\tag{13}$$

Therefore, by (13), we obtain the following corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$\frac{\mathcal{G}_{n+r}^{(r)}(\lambda,x)}{(n+r)_r} = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\frac{G_{l+r}^{(r)}(x)}{(l+r)_r}.$$

By replacing t with $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (5), we get

$$\begin{aligned}\left(\frac{2t}{e^t+1}\right)^r e^{xt} &= \sum_{n=0}^{\infty}\mathcal{G}_n^{(r)}(\lambda,x)\frac{1}{n!\lambda^n}(e^{\lambda t}-1)^n\\ &= \sum_{n=0}^{\infty}\mathcal{G}_n^{(r)}(\lambda,x)\frac{1}{\lambda^n}\sum_{m=n}^{\infty}S_2(m,n)\frac{\lambda^m t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{G}_n^{(r)}(\lambda,x)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!},\end{aligned}\tag{14}$$

where $S_2(m,n)$ is the Stirling number of the second kind.

Therefore by (6) and (14), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$G_m^{(r)}(x) = \sum_{n=0}^{m}\mathcal{G}_n^{(r)}(\lambda,x)\lambda^{m-n}S_2(m,n).$$

When $r=1$, $G_n(\lambda,x) = G_n^{(1)}(\lambda,x)$ are called the degenerate Genocchi polynomials. In particular, $x=0$, $G_n(\lambda) = G_n(\lambda,0)$ are called the degenerate Genocchi numbers.

Thus by Theorem 2.1 and (7), we get

$$\begin{aligned}\frac{\mathcal{G}_{n+1}(\lambda,x)}{n+1} &= \int_{\mathbb{Z}_p}(y+x|\lambda)_n d\mu_{-1}(y)\\ &= \lambda^n\int_{\mathbb{Z}_p}\left(\frac{y+x}{\lambda}\right)_n d\mu_{-1}(y)\\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\int_{\mathbb{Z}_p}(y+x)^l d\mu_{-1}(y)\end{aligned}\tag{15}$$

$$= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\frac{G_{l+1}(x)}{l+1},$$

and

$$\frac{\mathcal{G}_{n+1}(\lambda)}{n+1} = \int_{\mathbb{Z}_p} (y|\lambda)_n d\mu_{-1}(y) \tag{16}$$
$$= \lambda^n \int_{\mathbb{Z}_p} \left(\frac{y}{\lambda}\right)_n d\mu_{-1}(y)$$
$$= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\frac{G_{l+1}}{l+1}.$$

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, from (3), (15) and (16), we have

$$\int_{\mathbb{Z}_p} (y+d|\lambda)_n d\mu_{-1}(y) + \int_{\mathbb{Z}_p} (y|\lambda)_n d\mu_{-1}(y) = 2\sum_{l=0}^{d-1}(-1)^l (l|\lambda)_n. \tag{17}$$

Thus, by (17), we get

$$\frac{\mathcal{G}_{n+1}(\lambda,d)}{n+1} + \frac{\mathcal{G}_{n+1}(\lambda)}{n+1} = 2\sum_{l=0}^{d-1}(-1)^l (l|\lambda)_n \tag{18}$$
$$= 2\sum_{m=0}^{n}\sum_{l=0}^{d-1}(-1)^l \lambda^{n-m} S_1(n,m) l^m.$$

Therefore, by (18), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have*

$$\frac{\mathcal{G}_{n+1}(\lambda,d) + \mathcal{G}_{n+1}(\lambda)}{2(n+1)} = \sum_{l=0}^{d-1}(-1)^l (l|\lambda)_n$$
$$= \sum_{m=0}^{n}\sum_{l=0}^{d-1}(-1)^l \lambda^{n-m} S_1(n,m) l^m.$$

From (4), we have

$$\int_{\mathbb{Z}_p} (y|\lambda)_n d\mu_{-1}(y) = \sum_{a=0}^{d-1}(-1)^a \int_{\mathbb{Z}_p} (a+dy|\lambda)_n d\mu_{-1}(y) \tag{19}$$
$$= d^n \sum_{a=0}^{d-1}(-1)^a \int_{\mathbb{Z}_p} \left(\frac{a}{d} + y\middle|\frac{\lambda}{d}\right)_n d\mu_{-1}(y)$$
$$= d^n \sum_{a=0}^{d-1}(-1)^a \frac{\mathcal{G}_{n+1}(\frac{\lambda}{d},\frac{a}{d})}{n+1},$$

where $n \geq 0$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$.

Therefore, by (19), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have*

$$\mathcal{G}_{n+1}(\lambda) = d^n \sum_{a=0}^{d-1} (-1)^a \mathcal{G}_{n+1}\left(\frac{\lambda}{d}, \frac{a}{d}\right).$$

Moreover,

$$\mathcal{G}_{n+1}(\lambda, x) = d^n \sum_{a=0}^{d-1} (-1)^a \mathcal{G}_{n+1}\left(\frac{\lambda}{d}, \frac{a+x}{d}\right).$$

Now, we consider the degenerate Genocchi polynomials of the second kind as follows:

$$\frac{\hat{\mathcal{G}}_{n+1}(\lambda, x)}{n+1} = \int_{\mathbb{Z}_p} (-(y+x)|\lambda)_n d\mu_{-1}(y), \quad (n \geq 0, \quad \hat{\mathcal{G}}_0(\lambda, x) = 0). \tag{20}$$

Then, by (20), we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \hat{\mathcal{G}}_n(\lambda, x) \frac{t^n}{n!} &= t \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \lambda^n \binom{-\frac{y+x}{\lambda}}{n} t^n d\mu_{-1}(y) \\ &= t(1+\lambda t)^{-\frac{x}{\lambda}} \int_{\mathbb{Z}_p} (1+\lambda t)^{-\frac{y}{\lambda}} d\mu_{-1}(y) \\ &= \frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} (1+\lambda t)^{\frac{1-x}{\lambda}}. \end{aligned} \tag{21}$$

Thus, we see that the generating function for the degenerate Genocchi polynomials of the second is as follows:

$$\sum_{n=0}^{\infty} \hat{\mathcal{G}}_n(\lambda, x) \frac{t^n}{n!} = \frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} (1+\lambda t)^{\frac{1-x}{\lambda}}, \tag{22}$$

where $\hat{\mathcal{G}}_0(\lambda, x) = 0$. When $x = 0$, $\hat{\mathcal{G}}_n(\lambda) = \hat{\mathcal{G}}_n(\lambda, 0)$ are called the degenerate Genocchi numbers of the second kind.

From (20) and the fact $\hat{\mathcal{G}}_0(\lambda, x) = 0$, we note that

$$\begin{aligned} \frac{\hat{\mathcal{G}}_{n+1}(\lambda, x)}{n+1} &= \int_{\mathbb{Z}_p} (-(y+x)|\lambda)_n d\mu_{-1}(y) \\ &= \lambda^n \int_{\mathbb{Z}_p} \left(-\frac{y+x}{\lambda}\right)_n d\mu_{-1}(y) \\ &= \lambda^n \sum_{l=0}^{n} S_1(n, l) \frac{(-1)^l}{\lambda^l} \int_{\mathbb{Z}_p} (y+x)^l d\mu_{-1}(y) \\ &= \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} (-1)^l \frac{G_{l+1}(x)}{l+1} \\ &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} \lambda^{n-l} (-1)^n \frac{G_{l+1}(x)}{l+1}, \end{aligned} \tag{23}$$

where ${n \brack l} = (-1)^{n-l}S_1(n,l) = |S_1(n,l)|$.

Therefore, by (23), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$(-1)^n \frac{\hat{\mathcal{G}}_{n+1}(\lambda, x)}{n+1} = \sum_{l=0}^{n} {n \brack l} \lambda^{n-l} \frac{G_{l+1}(x)}{l+1}.$$

By replacing t with $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (22), we get

$$\begin{aligned}(24)\qquad &\sum_{n=0}^{\infty} \hat{\mathcal{G}}_n(\lambda, x) \frac{1}{n!} \frac{1}{\lambda^n} (e^{\lambda t} - 1)^n \\ &= \frac{2t}{e^t + 1} e^{(1-x)t} \frac{e^{\lambda t - 1}}{\lambda t} \\ &= \left(\sum_{n=0}^{\infty} G_n(1-x) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{S_2(m+1,1)\lambda^m}{m+1} \frac{t^m}{m!} \right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \binom{m}{n} G_{m-n}(1-x) \frac{\lambda^n}{n+1} S_2(n+1,1) \right) \frac{t^m}{m!}\end{aligned}$$

and

$$\begin{aligned}(25)\qquad \sum_{n=0}^{\infty} \hat{\mathcal{G}}_n(\lambda, x) \frac{1}{n!} \frac{1}{\lambda^n} (e^{\lambda t} - 1)^n &= \sum_{n=0}^{\infty} \hat{\mathcal{G}}_n(\lambda, x) \frac{1}{\lambda^n} \sum_{m=n}^{\infty} S_2(m,n) \frac{\lambda^m t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \hat{\mathcal{G}}_n(\lambda, x) \lambda^{m-n} S_2(m,n) \right) \frac{t^m}{m!}.\end{aligned}$$

Therefore, by (24) and (25), we obtain the following theorem.

Theorem 2.7. *For $m \geq 0$, we have*

$$\sum_{n=0}^{m} \binom{m}{n} G_{m-n}(1-x) \frac{\lambda^n}{n+1} S_2(n+1,1) = \sum_{n=0}^{m} \hat{\mathcal{G}}_n(\lambda, x) \lambda^{m-n} S_2(m,n).$$

We observe that

$$(26)\qquad \binom{x+y}{n} = \sum_{l=0}^{n} \binom{x}{l} \binom{y}{n-l}, \quad (n \geq 0).$$

Now, we consider

$$\begin{aligned}(27)\qquad \frac{(-1)^n \mathcal{G}_{n+1}(\lambda)}{(n+1)!} &= \frac{(-1)^n}{n!} \int_{\mathbb{Z}_p} (y|\lambda)_n d\mu_{-1}(y) \\ &= \lambda^n \int_{\mathbb{Z}_p} \binom{-\frac{y}{\lambda} + n - 1}{n} d\mu_{-1}(y) \\ &= \lambda^n \sum_{l=0}^{n} \binom{n-1}{n-l} \int_{\mathbb{Z}_p} \binom{-\frac{y}{\lambda}}{l} d\mu_{-1}(y)\end{aligned}$$

$$= \lambda^n \sum_{l=0}^{n} \binom{n-1}{l-1} \frac{1}{\lambda^l l!} \int_{\mathbb{Z}_p} (-y|\lambda)_l d\mu_{-1}(y)$$
$$= \sum_{l=0}^{n} \binom{n-1}{l-1} \lambda^{n-l} \frac{\hat{\mathcal{G}}_{l+1}(\lambda)}{(l+1)!}.$$

By the same method as (27), we also get

$$\frac{(-1)^n}{(n+1)!}\hat{\mathcal{G}}_{n+1}(\lambda) = \sum_{l=0}^{n} \binom{n-1}{l-1} \lambda^{n-l} \frac{\mathcal{G}_{l+1}(\lambda)}{(l+1)!}. \tag{28}$$

Therefore, by (27) and (28), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$\frac{(-1)^n}{(n+1)!}\mathcal{G}_{n+1}(\lambda) = \sum_{l=0}^{n} \binom{n-1}{l-1} \lambda^{n-l} \frac{\hat{\mathcal{G}}_{l+1}(\lambda)}{(l+1)!}$$

and

$$\frac{(-1)^n}{(n+1)!}\hat{\mathcal{G}}_{n+1}(\lambda) = \sum_{l=0}^{n} \binom{n-1}{l-1} \lambda^{n-l} \frac{\mathcal{G}_{l+1}(\lambda)}{(l+1)!}.$$

Remark 2.9. Note that

$$\lim_{\lambda\to 0} \frac{1}{2}\left(\frac{\mathcal{G}_{n+1}(\lambda, d) + \mathcal{G}_{n+1}(\lambda)}{n+1}\right) = \lim_{\lambda\to 0}\left(\sum_{m=0}^{n}\sum_{l=0}^{d-1}(-1)^l \lambda^{n-m} S_1(n,m) l^m\right)$$
$$= \sum_{l=0}^{d-1}(-1)^l l^n$$
$$= \frac{1}{2}\left(\frac{G_{n+1}(d) + G_{n+1}}{n+1}\right).$$

It is not difficult to show that

$$\lim_{\lambda\to 0} \mathcal{G}_n(\lambda, x) = G_n(x) \quad \text{and} \quad \lim_{\lambda\to 0} \mathcal{G}_n^{(r)}(\lambda, x) = G_n^{(r)}(x).$$

From (22), we have

$$\hat{\mathcal{G}}_n(\lambda, x) = \mathcal{G}_n(\lambda, 1-x)$$
$$= (-1)^{n-1}\mathcal{G}_n(-\lambda, x), \quad (n \geq 0).$$

Remark 2.10. For $r \in \mathbb{Z}_+$ and $\lambda, t \in \mathbb{Z}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$, the degenerate mixed Genocchi polynomials $\mathcal{G}_n^*(\lambda, x)$ are defined by the generating function to be

$$\frac{2\log(1+\lambda t)}{\lambda\{(1+\lambda t)^{\frac{1}{\lambda}} + 1\}}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{G}_n^*(\lambda, x)\frac{t^n}{n!}. \tag{29}$$

When $x = 0$, $\mathcal{G}_n^*(\lambda) = \mathcal{G}_n^*(\lambda, 0)$ are called the degenerate mixed Genocchi numbers.

By replacing t with $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (29), we get

$$\begin{aligned}
\frac{2t}{e^t+1}e^{xt} &= \sum_{n=0}^{\infty} \mathcal{G}_n^*(\lambda, x)\frac{1}{n!\lambda^n}(e^{\lambda t}-1)^n \\
&= \sum_{n=0}^{\infty} \mathcal{G}_n^*(\lambda, x)\frac{1}{\lambda^n}\sum_{m=n}^{\infty} S_2(m,n)\frac{\lambda^m t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=n}^{\infty} \mathcal{G}_n^*(\lambda, x)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.
\end{aligned} \tag{30}$$

Therefore by (6) and (30), we obtain the following theorem.

Theorem 2.11. *For $n \geq 0$, we have*

$$G_m(x) = \sum_{n=0}^{m} \mathcal{G}_n^*(\lambda, x)\lambda^{m-n}S_2(m,n).$$

By (2) and (29), we get

$$\begin{aligned}
\int_{\mathbb{Z}_p} \frac{\log(1+\lambda t)}{\lambda}(1+\lambda t)^{\frac{y+x}{\lambda}}d\mu_{-1}(y) &= \left(\frac{2\log(1+\lambda t)}{\lambda\{(1+\lambda t)^{\frac{1}{\lambda}}+1\}}\right)(1+\lambda t)^{\frac{x}{\lambda}} \\
&= \sum_{n=0}^{\infty}\mathcal{G}_n^*(\lambda, x)\frac{t^n}{n!}.
\end{aligned} \tag{31}$$

We adopt the definition of λ-Daehee numbers from [12]. For $\lambda, t \in \mathbb{Z}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$, we introduce the modified λ-Daehee numbers $D_{n,\lambda}$ by the generating function

$$\frac{\log(1+\lambda t)}{\lambda t} = \sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!}.$$

On the other hand

$$\begin{aligned}
&\frac{\log(1+\lambda t)}{\lambda}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{y+x}{\lambda}}d\mu_{-1}(y) \\
&= \frac{\log(1+\lambda t)}{\lambda t}t\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{y+x}{\lambda}}d\mu_{-1}(y) \\
&= \left(\sum_{m=0}^{\infty} D_m\frac{(\lambda t)^m}{m!}\right)\left(\sum_{k=0}^{\infty}\mathcal{G}_k(\lambda, x)\frac{t^k}{k!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\lambda^{n-k}D_{n-k}\mathcal{G}_k(\lambda, x)\right)\frac{t^k}{k!}.
\end{aligned} \tag{32}$$

Thus, by (31) and (32), we have mixed degenerate Genocchi polynomials are represented by sums of products of the modified λ-Daehee numbers and the degenerate Genocchi polynomials.

$$\mathcal{G}_n^{(*)}(\lambda, x) = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} D_{n-k} \mathcal{G}_k(\lambda, x).$$

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