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ON ABSOLUTE VALUES OF $\mathcal{Q}_K$ FUNCTIONS

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Abstract. In this paper, the effect of absolute values on the behavior of functions f in the spaces $\mathcal{Q}_K$ is investigated. It is clear that $g \in \mathcal{Q}_K(\partial\mathbb{D}) \Rightarrow |g| \in \mathcal{Q}_K(\partial\mathbb{D})$, but the converse is not always true. For f in the Hardy space H^2, we give a condition involving the modulus of the function only, such that the condition together with $|f| \in \mathcal{Q}_K(\partial\mathbb{D})$ is equivalent to $f \in \mathcal{Q}_K$. As an application, a new criterion for inner-outer factorisation of $\mathcal{Q}_K$ spaces is given. These results are also new for $\mathcal{Q}_p$ spaces.

1. Introduction

Denote by $\partial\mathbb{D}$ the boundary of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of functions analytic in $\mathbb{D}$. Throughout this paper, we assume that $K : [0, \infty) \to [0, \infty)$ is a right-continuous and increasing function. A function $f \in H(\mathbb{D})$ belongs to the space $\mathcal{Q}_K$ if

$$\|f\|^2_{\mathcal{Q}_K} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K\left(g(a, z)\right) dA(z) < \infty,$$

where dA is the area measure on $\mathbb{D}$ and $g(a, z)$ is the Green function in $\mathbb{D}$ with singularity at $a \in \mathbb{D}$. By [5, Theorem 2.1], we know that $\|f\|^2_{\mathcal{Q}_K}$ is equivalent to

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K\left(1 - |\sigma_a(z)|^2\right) dA(z),$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is a Möbius transformation of $\mathbb{D}$. If $K(t) = t^p$, $0 \le p < \infty$, then the space $\mathcal{Q}_K$ gives the space $\mathcal{Q}_p$ (cf. [11, 13]). In particular, $\mathcal{Q}_0$ is the Dirichlet space; $\mathcal{Q}_1 = BMOA$, the space of functions with bounded mean oscillation on $\mathbb{D}$; $\mathcal{Q}_p$ is the Bloch space for all $p > 1$. See [5] and [6] for more

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results on $\mathcal{Q}_K$ spaces. Let $\mathcal{Q}_K(\partial\mathbb{D})$ be the space of $f \in L^2(\partial\mathbb{D})$ with

$$\|f\|^2_{\mathcal{Q}_K(\partial\mathbb{D})} = \sup_{I\subset\partial\mathbb{D}} \int_I \int_I \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^2} K\left(\frac{|\zeta - \eta|}{|I|}\right) |d\zeta||d\eta| < \infty.$$

Clearly, if $K(t) = t^2$, then $\mathcal{Q}_K(\partial\mathbb{D})$ is equal to $BMO(\partial\mathbb{D})$, the space of functions having bounded mean oscillation on $\partial\mathbb{D}$ (see [7]).

To study $\mathcal{Q}_K$ and $\mathcal{Q}_K(\partial\mathbb{D})$, we usually need two constraints on K as follows.

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1.1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{1.2}$$

where

$$\varphi_K(s) = \sup_{0<t\le 1} K(st)/K(t), \quad 0 < s < \infty.$$

If K satisfies (1.2), then $\mathcal{Q}_K \subsetneqq BMOA \subsetneqq H^2$, where H^2 denotes the Hardy space in $\mathbb{D}$ (see [3, 7]). Thus, if K satisfies (1.2), then the function $f \in \mathcal{Q}_K$ has its non-tangential limit $\widetilde{f}$ almost everywhere on $\partial\mathbb{D}$. We also know that for $f \in H^2$ if K satisfies (1.1) and (1.2), then $f \in \mathcal{Q}_K$ if and only if $\widetilde{f} \in \mathcal{Q}_K(\partial\mathbb{D})$. Using the triangle inequality, one gets that if $g \in \mathcal{Q}_K(\partial\mathbb{D})$, then $|g|$ also belongs to $\mathcal{Q}_K(\partial\mathbb{D})$. In general, the converse is not true. Consider

$$g(e^{it}) = \begin{cases} \log t, & 0 < t < \pi, \\ -\log|t|, & -\pi < t < 0. \end{cases}$$

By [8, p. 66], $|g| \in BMO(\partial\mathbb{D})$, but $g \notin BMO(\partial\mathbb{D})$. For $g \in H^2$, it is natural to seek a condition which together with $|\widetilde{g}| \in \mathcal{Q}_K(\partial\mathbb{D})$ is equivalent to $\widetilde{g} \in \mathcal{Q}_K(\partial\mathbb{D})$. Our main result, Theorem 1.1, is even new for $\mathcal{Q}_p$ spaces.

Theorem 1.1. *Suppose that K satisfies* (1.1) *and* (1.2). *Let $f \in H^2$. Set*

$$d\mu_z(\zeta) = \frac{1 - |z|^2}{2\pi|\zeta - z|^2}|d\zeta|, \quad z \in \mathbb{D}, \quad \zeta \in \partial\mathbb{D}.$$

Then the following conditions are equivalent.

(i) $f \in \mathcal{Q}_K$.
(ii) $\widetilde{f} \in \mathcal{Q}_K(\partial\mathbb{D})$.
(iii) $|\widetilde{f}| \in \mathcal{Q}_K(\partial\mathbb{D})$ *and*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)| d\mu_z(\zeta) - |f(z)|\right)^2 \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2} dA(z) < \infty. \tag{1.3}$$

Applying Theorem 1.1, in Section 4, we will show a new criterion for inner-outer factorisation of $\mathcal{Q}_K$ spaces.

In this article, the symbol $A \approx B$ means that $A \lesssim B \lesssim A$. We say that $A \lesssim B$ if there exists a constant C such that $A \le CB$.

2. Preliminaries

Given $f \in L^2(\partial\mathbb{D})$, let $\widehat{f}$ be the Poisson extension of f. Namely,

$$\widehat{f}(z) = \int_{\partial\mathbb{D}} f(\zeta) d\mu_z(\zeta), \;\; z \in \mathbb{D}.$$

We first give the following characterization of $\mathcal{Q}_K(\partial\mathbb{D})$ spaces. In particular, if $K(t) = t^p$, $0 < p < 1$, the corresponding result was proved in [12].

Theorem 2.1. *Suppose that K satisfies* (1.1) *and* (1.2). *Let $f \in L^2(\partial\mathbb{D})$. Then $f \in \mathcal{Q}_K(\partial\mathbb{D})$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\int_{\partial\mathbb{D}} |f(\zeta)|^2 d\mu_z(\zeta) - |\widehat{f}(z)|^2 \right) \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2} dA(z) < \infty. \tag{2.1}$$

To prove Theorem 2.1, we need the following estimate.

Lemma 2.2. *Let* (1.1) *and* (1.2) *hold for K. If $s < 1 + c$ and $2s + r - 4 \geq 0$, then*

$$\int_{\mathbb{D}} \frac{K\left(1-|\sigma_a(w)|^2\right)}{(1-|w|^2)^s |1-\overline{w}z|^r} dA(w) \approx \frac{K\left(1-|\sigma_a(z)|^2\right)}{(1-|z|^2)^{s+r-2}}$$

for all $a, z \in \mathbb{D}$. Here c is a small enough positive constant which depends only on (1.1) *and* (1.2).

Proof. We point out that

$$\int_{\mathbb{D}} \frac{K\left(1-|\sigma_a(w)|^2\right)}{(1-|w|^2)^s |1-\overline{w}z|^r} dA(w) \lesssim \frac{K\left(1-|\sigma_a(z)|^2\right)}{(1-|z|^2)^{s+r-2}}$$

was proved in [1]. So we need only to prove the reverse. For any $z \in \mathbb{D}$, let

$$E(z, 1/2) = \{w \in \mathbb{D} : |\sigma_z(w)| < 1/2\}$$

be the pseudo-hyperbolic disk. It is well known that

$$1 - |z| \approx 1 - |w| \approx |1 - \overline{w}z|$$

for all $w \in E(z, 1/2)$. Furthermore, by [14, Lemma 4.30], we have that $|1 - a\overline{w}| \approx |1 - a\overline{z}|$ for all $a \in \mathbb{D}$ and $w \in E(z, 1/2)$. Since K satisfies (1.2), $K(2t) \approx K(t)$ for all $t \in (0, 1)$. We obtain

$$\begin{aligned}\int_{\mathbb{D}} \frac{K\left(1-|\sigma_a(w)|^2\right)}{(1-|w|^2)^s |1-\overline{w}z|^r} dA(w) &\geq \int_{E(z,1/2)} \frac{K\left(1-|\sigma_a(w)|^2\right)}{(1-|w|^2)^s |1-\overline{w}z|^r} dA(w) \\ &\approx \frac{K\left(1-|\sigma_a(z)|^2\right)}{(1-|z|^2)^{s+r-2}},\end{aligned}$$

which gives the desired result. □

Proof of Theorem 2.1. For any $f \in L^2(\partial\mathbb{D})$, the Littlewood-Paley identity ([7, p. 228]) shows that

$$\int_{\mathbb{D}} |\nabla \widehat{f}(w)|^2 \log \frac{1}{|w|} dA(w) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} |f(\zeta) - \widehat{f}(0)|^2 |d\zeta|. \tag{2.2}$$

Replacing $\widehat{f}$ by $\widehat{f\circ\sigma_z}$ in (2.2) for $z\in\mathbb{D}$, one obtains

$$\int_{\partial\mathbb{D}}|f(\zeta)|^2d\mu_z(\zeta)-|\widehat{f}(z)|^2\approx\int_{\mathbb{D}}|\nabla\widehat{f}(w)|^2(1-|\sigma_z(w)|^2)dA(w).$$

Using Fubini's theorem and Lemma 2.2, we obtain, for all $a\in\mathbb{D}$, that

$$\begin{aligned}&\int_{\mathbb{D}}\left(\int_{\partial\mathbb{D}}|f(\zeta)|^2d\mu_z(\zeta)-|\widehat{f}(z)|^2\right)\frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z)\\ \approx&\int_{\mathbb{D}}\left(\int_{\mathbb{D}}|\nabla\widehat{f}(w)|^2(1-|\sigma_z(w)|^2)dA(w)\right)\frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z)\\ \approx&\int_{\mathbb{D}}|\nabla\widehat{f}(w)|^2dA(w)\int_{\mathbb{D}}\frac{(1-|\sigma_z(w)|^2)K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z)\\ \approx&\int_{\mathbb{D}}|\nabla\widehat{f}(w)|^2K(1-|\sigma_a(w)|^2)dA(w).\end{aligned}$$

By [9], we know that $f\in\mathcal{Q}_K(\partial\mathbb{D})$ if and only if

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|\nabla\widehat{f}(z)|^2K(1-|\sigma_a(z)|^2)dA(z)<\infty.$$

Therefore, $f\in\mathcal{Q}_K(\partial\mathbb{D})$ if and only if

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(\int_{\partial\mathbb{D}}|f(\zeta)|^2d\mu_z(\zeta)-|\widehat{f}(z)|^2\right)\frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z)<\infty.$$ □

By [6], for $f\in H^2$, if (1.1) and (1.2) hold for K, then $f\in\mathcal{Q}_K$ if and only if $\widetilde{f}\in\mathcal{Q}_K(\partial\mathbb{D})$. This, together with Theorem 2.1, gives the following result immediately which was also obtained in [10] by a different method.

Corollary 2.3. *Suppose that K satisfies* (1.1) *and* (1.2). *Let $f\in H^2$. Then $f\in\mathcal{Q}_K$ if and only if*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(\int_{\partial\mathbb{D}}|\widetilde{f}(\zeta)|^2d\mu_z(\zeta)-|f(z)|^2\right)\frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z)<\infty.$$

3. Proof of Theorem 1.1

Recall that $B\in H(\mathbb{D})$ is called an inner function if B is bounded in $\mathbb{D}$ and $|\widetilde{B}(\zeta)|=1$ for almost every $\zeta\in\partial\mathbb{D}$. An outer function for the Hardy space H^2 is the function of the form

$$O(z)=\eta\exp\left(\int_{\partial\mathbb{D}}\frac{\zeta+z}{\zeta-z}\log\psi(\zeta)\frac{|d\zeta|}{2\pi}\right),\quad \eta\in\partial\mathbb{D},$$

where $\psi>0$ a.e. on $\partial\mathbb{D}$, $\log\psi\in L^1(\partial\mathbb{D})$ and $\psi\in L^2(\partial\mathbb{D})$. See [3] for more results on inner and outer functions. Using a technique in [2], we give the proof of Theorem 1.1 as follows.

Proof of Theorem 1.1. Note that (i)⇔(ii) was proved in [6].

(i)⇒(iii). For $f \in \mathcal{Q}_K$, we have that $\widetilde{f} \in \mathcal{Q}_K(\partial\mathbb{D})$. The triangle inequality gives that $|\widetilde{f}| \in \mathcal{Q}_K(\partial\mathbb{D})$. For any $z \in \mathbb{D}$, it follows by Hölder's inequality that

$$\begin{aligned}\left(\int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)|d\mu_z(\zeta) - |f(z)|\right)^2 &\le \left(\int_{\partial\mathbb{D}} |\widetilde{f}(\zeta) - f(z)|d\mu_z(\zeta)\right)^2 \\ &\le \int_{\partial\mathbb{D}} |\widetilde{f}(\zeta) - f(z)|^2 d\mu_z(\zeta) \\ &= \int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)|^2 d\mu_z(\zeta) - |f(z)|^2.\end{aligned}$$

Since $f \in \mathcal{Q}_K$, the above estimate, together with Corollary 2.3, gives (1.3).

(iii)⇒(i). If $f \equiv 0$, the result is true. Note that $f \in H^2$. If $f \not\equiv 0$, then f must be of the form BO, where B is an inner function and O is an outer function of H^2 (see [3]). By the estimates of B and O respectively, Böe [2, p. 237] gave that for any $z \in \mathbb{D}$,

$$|f'(z)| \le \frac{4}{1-|z|}\left(\int_{\partial\mathbb{D}} \left||\widetilde{f}(\zeta)| - \widehat{|\widetilde{f}|}(z)\right| d\mu_z(\zeta) + \widehat{|\widetilde{f}|}(z) - |f(z)|\right).$$

Here we remind that

$$\widehat{|\widetilde{f}|}(z) = \int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)|d\mu_z(\zeta).$$

Thus, for any $a \in \mathbb{D}$, by Hölder's inequality, we deduce that

$$\begin{aligned}&\int_{\mathbb{D}} |f'(z)|^2 K(1-|\sigma_a(z)|^2)dA(z) \\ \lesssim &\int_{\mathbb{D}} \left(\int_{\partial\mathbb{D}} \left||\widetilde{f}(\zeta)| - \widehat{|\widetilde{f}|}(z)\right| d\mu_z(\zeta)\right)^2 \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z) \\ &+ \int_{\mathbb{D}} \left(\widehat{|\widetilde{f}|}(z) - |f(z)|\right)^2 \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z) \\ \lesssim &\int_{\mathbb{D}} \left(\int_{\partial\mathbb{D}} \left(|\widetilde{f}(\zeta)| - \widehat{|\widetilde{f}|}(z)\right)^2 d\mu_z(\zeta)\right) \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z) \\ &+ \int_{\mathbb{D}} \left(\widehat{|\widetilde{f}|}(z) - |f(z)|\right)^2 \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z) \\ \approx &\int_{\mathbb{D}} \left(\int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)|^2 d\mu_z(\zeta) - \left(\widehat{|\widetilde{f}|}(z)\right)^2\right) \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z) \\ &+ \int_{\mathbb{D}} \left(\widehat{|\widetilde{f}|}(z) - |f(z)|\right)^2 \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2}dA(z).\end{aligned}$$

By Theorem 2.1 and (1.3), we have that $f \in \mathcal{Q}_K$. The proof is complete. □

Remark. J. Xiao [12] gave an interesting characterization of $\mathcal{Q}_p$ spaces in terms of functions with absolute values. Namely, for $f \in H^2$, if $0 < p < 1$, then $f \in \mathcal{Q}_p$ if and only if $|\widetilde{f}| \in \mathcal{Q}_p(\partial\mathbb{D})$ and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\left(\int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)| d\mu_z(\zeta) \right)^2 - |f(z)|^2 \right) \frac{(1-|\sigma_a(z)|^2)^p}{(1-|z|^2)^2} dA(z) < \infty.$$

We show that our Theorem 1.1 implies Xiao's result above. In fact, set $K(t) = t^p$, $0 < p < 1$, in our Theorem 1.1 and Corollary 2.3. Note that

$$\left(\int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)| d\mu_z(\zeta) \right)^2 - |f(z)|^2 \geq \left(\int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)| d\mu_z(\zeta) - |f(z)| \right)^2$$

and

$$\left(\int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)| d\mu_z(\zeta) \right)^2 \leq \int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)|^2 d\mu_z(\zeta).$$

Thus, one can obtain Xiao's result directly.

4. An application to inner-outer factorisation of $\mathcal{Q}_K$ spaces

In this section, we will show a new criterion for inner-outer decomposition of $\mathcal{Q}_K$ spaces. In fact, an inner-outer factorisation characterization of $\mathcal{Q}_K$ spaces has been obtained in [6] as follows.

Theorem A. *Let K satisfy* (1.1) *and* (1.2) *with*

$$\widetilde{K}(|z|^2) = -\frac{\partial^2 K(1-|z|^2)}{\partial z \partial \overline{z}}, \quad z \in \mathbb{D}.$$

Let $f \in H^2$ with $f \not\equiv 0$. Then $f \in \mathcal{Q}_K$ if and only if $f = BO$, where B is an inner function and O is an outer function in $\mathcal{Q}_K$ for which

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |O(z)|^2 (1-|B(z)|^2) \widetilde{K}\left(|\sigma_a(z)|^2\right) |\sigma_a'(z)|^2 dA(z) < \infty. \tag{4.1}$$

As an application of Theorem 1.1, we obtain the following result.

Theorem 4.1. *Let K satisfy* (1.1) *and* (1.2) *with*

$$\widetilde{K}(|z|^2) = -\frac{\partial^2 K(1-|z|^2)}{\partial z \partial \overline{z}}, \quad z \in \mathbb{D}.$$

Let $f \in H^2$ with $f \not\equiv 0$. Then $f \in \mathcal{Q}_K$ if and only if $f = BO$, where B is an inner function and O is an outer function in $\mathcal{Q}_K$ for which

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |O(z)|^2 (1-|B(z)|^2)^2 \widetilde{K}\left(|\sigma_a(z)|^2\right) |\sigma_a'(z)|^2 dA(z) < \infty. \tag{4.2}$$

Remark. Theorem 4.1 shows that formula (4.1) in Theorem A can be replaced by the weaker condition (4.2), and this result is also new for $\mathcal{Q}_p$ spaces.

Proof. Necessity. This is a direct result from Theorem A.

Sufficiency. Let $f = BO$ and $O \in \mathcal{Q}_K$. Note that $O \in \mathcal{Q}_K$ is equivalent to $\widetilde{O} \in \mathcal{Q}_K(\partial\mathbb{D})$. By the triangle inequality, one gets $|\widetilde{O}| \in \mathcal{Q}_K(\partial\mathbb{D})$. Hence $|\widetilde{f}| \in \mathcal{Q}_K(\partial\mathbb{D})$. Observe that

$$\begin{aligned}&\int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)| d\mu_z(\zeta) - |f(z)| \\ =& \int_{\partial\mathbb{D}} |\widetilde{O}(\zeta)| d\mu_z(\zeta) - |O(z)| + |O(z)| - |B(z)O(z)|.\end{aligned} \tag{4.3}$$

Wulan and Ye [10] gave that if K satisfies (1.1) and (1.2), then for all $z \in \mathbb{D}$

$$\widetilde{K}(|z|^2) \approx \frac{K(1-|z|^2)}{(1-|z|^2)^2}. \tag{4.4}$$

By Hölder's inequality, $\widetilde{O} \in \mathcal{Q}_K(\partial\mathbb{D})$ and Corollary 2.3, we show that for any $a \in \mathbb{D}$,

$$\begin{aligned}&\int_{\mathbb{D}} \left(\int_{\partial\mathbb{D}} |\widetilde{O}(\zeta)| d\mu_z(\zeta) - |O(z)| \right)^2 \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2} dA(z) \\ \leq& \int_{\mathbb{D}} \left(\int_{\partial\mathbb{D}} \left|\widetilde{O}(\zeta) - O(z)\right|^2 d\mu_z(\zeta) \right) \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2} dA(z) \\ =& \int_{\mathbb{D}} \left(\int_{\partial\mathbb{D}} |\widetilde{O}(\zeta)|^2 d\mu_z(\zeta) - |O(z)|^2 \right) \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2} dA(z) < \infty.\end{aligned}$$

Combining the above inequality, (4.2), (4.3) and (4.4), we get

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)| d\mu_z(\zeta) - |f(z)| \right)^2 \frac{K(1-|\sigma_a(z)|^2)}{(1-|z|^2)^2} dA(z) < \infty.$$

Applying Theorem 1.1, we get $f \in \mathcal{Q}_K$. The proof is complete. □

For $f \in \mathcal{Q}_K \subseteq H^2$, if we ignore the choice of a constant with modulus one, then f has a unique decomposition with the form $f(z) = B(z)O(z)$, where B is an inner function and O is an outer function. Combining this with Theorem A and Theorem 4.1, we obtain an interesting result as follows.

Corollary 4.2. *Suppose that K satisfies* (1.1) *and* (1.2). *Let B be an inner function and let O be an outer function in $\mathcal{Q}_K$. Then the following conditions are equivalent.*

(i) *For some $p \in [1,2]$,*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |O(z)|^2 (1-|B(z)|^2)^p \widetilde{K}\left(|\sigma_a(z)|^2\right) |\sigma_a'(z)|^2 dA(z) < \infty.$$

(ii) *For all $p \in [1,2]$,*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |O(z)|^2 (1-|B(z)|^2)^p \widetilde{K}\left(|\sigma_a(z)|^2\right) |\sigma_a'(z)|^2 dA(z) < \infty.$$

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