

DEPTH FOR TRIANGULATED CATEGORIES

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ABSTRACT. Recently a construction of local cohomology functors for compactly generated triangulated categories admitting small coproducts is introduced and studied by Benson, Iyengar, Krause, Asadollahi and their coauthors. Following their idea, we introduce the depth of objects in such triangulated categories and get that when (R, \mathfrak{m}) is a graded-commutative Noetherian local ring, the depth of every cohomologically bounded and cohomologically finite object is not larger than its dimension.

1. Introduction

Depth is a fundamental numerical invariant of a Noetherian local ring (R, \mathfrak{m}, k) and a finite R -module M . The depth of a finite module M is classically introduced as the maximal length of an M -regular sequence in \mathfrak{m} and it can be measured by the (non)-vanishing of $\text{Ext}_R^m(k, M)$, see [6]. Later, for non-finite modules and even complexes of such, Foxby adopted the latter approach to depth, see [8].

Local cohomology functors on categories of modules are introduced in [5] and it has been shown that the depth and dimension of a module can be expressed in terms of their vanishing and non-vanishing. Furthermore, local cohomology functors of complexes also have been studied by many authors, see [5, 7, 11]. Also it is proved in [8] that for any unbounded complex X , the depth can be computed using either Koszul (co)homology, Ext , or local cohomology. Here great attention should be paid to the description

$$\text{depth}_R X = \inf\{i \in \mathbb{Z} \mid H_{\mathfrak{m}}^i(X) \neq 0\},$$

where $H_{\mathfrak{m}}^i(X)$ is the i -th local cohomology of X with support in \mathfrak{m} . Recently a construction of local cohomology functors for triangulated categories with respect to a central ring of operators is introduced in [3].

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Inspired by these, in this article we consider depth for objects in any compactly generated triangulated category admitting small coproducts and characterize it by local cohomology functors for triangulated categories. Also we study its properties. The paper is organized as follows.

In Section 2 we give some notions and terminologies needed in the paper.

After preliminaries we introduce depth for objects in a compactly generated triangulated category \mathcal{T} admitting small coproducts with the graded center $\mathcal{Z}^*(\mathcal{T})$ in Section 3 and we get the following main result.

Main Theorem. *Let (R, \mathfrak{m}) be a graded local ring. Then $\text{depth} X \leq \dim_R X$ for every cohomologically finite and cohomologically bounded object X of \mathcal{T} .*

2. Preliminaries

In this section we recall the construction and some basic notions and terminologies needed in this paper, from [3, 4, 10, 12].

Let \mathcal{T} be a triangulated category. We say that \mathcal{T} satisfies [TR5] if it has arbitrary small coproducts. An object C in \mathcal{T} admitting set-indexed coproducts is compact if the functor $\text{Hom}_{\mathcal{T}}(C, -)$ commutes with all coproducts. We write \mathcal{T}^C for the full subcategory of compact objects in \mathcal{T} . A set \mathcal{G} of objects of \mathcal{T} is called a generating set for \mathcal{T} if for each non-zero object $X \in \mathcal{T}$, there exists an object G in \mathcal{G} such that $\text{Hom}_{\mathcal{T}}(G, X) \neq 0$. The category \mathcal{T} is compactly generated if it is generated by a set of compact objects.

We write Σ for the suspension on \mathcal{T} . For objects X and Y in \mathcal{T} , let

$$\text{Hom}_{\mathcal{T}}^*(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(X, \Sigma^i Y)$$

be the graded abelian group of morphisms. Set $\text{End}_{\mathcal{T}}^*(X) = \text{Hom}_{\mathcal{T}}^*(X, X)$. This is a graded ring, and $\text{Hom}_{\mathcal{T}}^*(X, Y)$ is a right $\text{End}_{\mathcal{T}}^*(X)$ and left $\text{End}_{\mathcal{T}}^*(Y)$ -bimodule.

Definition 2.1 (Central ring actions). Let R be a graded commutative ring. Thus R is \mathbb{Z} -graded and satisfies $rs = (-1)^{|r||s|}sr$ for each pair of homogeneous elements r, s in R . The center of \mathcal{T} , denoted by $\mathcal{Z}^*(\mathcal{T})$, is the graded-commutative ring defined via for each $n \in \mathbb{Z}$, the component in degree n being

$$\mathcal{Z}^n(\mathcal{T}) = \{\eta : \text{Id}_{\mathcal{T}} \rightarrow \Sigma^n \mid \eta \Sigma = (-1)^n \Sigma \eta\}.$$

We say that a triangulated category \mathcal{T} is R -linear, or that R acts on \mathcal{T} , if there is a homomorphism $\phi : R \rightarrow \mathcal{Z}^*(\mathcal{T})$.

This yields for each object X a homomorphism $\phi_X : R \rightarrow \text{End}_{\mathcal{T}}^*(X)$ of graded rings such that for all objects $X, Y \in \mathcal{T}$ the R -module structures on $\text{Hom}_{\mathcal{T}}^*(X, Y)$ induced by ϕ_X and ϕ_Y agree, up to the usual rule.

If $X = C$ is a compact object, then the R -module $\text{Hom}_{\mathcal{T}}^*(C, Y)$ will be denoted by $H_C^*(Y)$ and is called the cohomology of Y with respect to C .

Throughout the paper, we assume that \mathcal{T} is a [TR5] compactly generated triangulated category with the graded center $\mathcal{Z}^*(\mathcal{T})$ and R is a graded-commutative Noetherian ring with a homomorphism of graded rings $R \rightarrow$

$\mathcal{Z}^*(\mathcal{T})$. We denote $\text{Spec}(R)$ as the set of graded prime ideals of R . For any homogeneous ideal \mathfrak{a} of R , $\nu(\mathfrak{a})$ is the set $\{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$, $\mathcal{Z}(\mathfrak{a}) = \{\mathfrak{q} \in \text{Spec}R \mid \mathfrak{q} \not\subseteq \mathfrak{a}\}$.

Let \mathcal{U} be a subset of $\text{Spec}(R)$. The specialization closure of \mathcal{U} , denoted by $cl\mathcal{U}$, is defined by

$$cl\mathcal{U} = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{q} \text{ for some } \mathfrak{q} \in \mathcal{U}\}.$$

A subset \mathcal{U} of $\text{Spec}(R)$ is called specialization closed if $cl\mathcal{U} = \mathcal{U}$; equivalently, if \mathcal{U} is a union of Zariski closed subsets of $\text{Spec}(R)$.

Let ν be a specialization closed subset of $\text{Spec}(R)$. Set

$$\mathcal{T}_\nu = \{X \in \mathcal{T} \mid \text{supp}_R H_C^*(X) \subseteq \nu \text{ for any } C \in \mathcal{T}^C\}.$$

\mathcal{T}_ν is a localizing subcategory of \mathcal{T} , i.e., it is closed under direct summands and small coproducts. So we have the localization functor $L_\nu : \mathcal{T} \rightarrow \mathcal{T}$. It induces an equivalence of categories $\mathcal{T}/\text{Ker}L_\nu \cong \text{Im}L_\nu$, where $\mathcal{T}/\text{Ker}L_\nu$ denotes the Verdier quotient of \mathcal{T} with respect to $\text{Ker}L_\nu$, and $\text{Im}L_\nu$ is the essential image of L_ν .

Definition 2.2 (Local cohomology). Let ν be a specialization closed subset of $\text{Spec}(R)$ and $L_\nu : \mathcal{T} \rightarrow \mathcal{T}$ the associated localization functor. For any object X of \mathcal{T} , there exists an exact triangle

$$\Gamma_\nu X \longrightarrow X \longrightarrow L_\nu X \rightsquigarrow,$$

where $\Gamma_\nu X$ is called the local cohomology of X supported on ν .

Benson, Iyengar and Krause [3] proved that in the case where \mathcal{T} is the derived category of a commutative Noetherian ring A , and $A \rightarrow \mathcal{Z}^*(\mathcal{T})$ is the canonical morphism, for each specialization closed subset ν of $\text{Spec}A$ and complex of A -modules, the cohomology of $\Gamma_\nu X$ is the classical notion of local cohomology introduced in [9]. Refer to [5, 11] for details.

The proof of our main theorem involves “Koszul objects” in the following.

Definition 2.3 (Koszul object, see [2]). Let r be an element in R ; it is assumed to be homogeneous, since we are in the category of graded R -modules. Set $d = \text{degree}(r)$, the degree of r . Let C be an object in \mathcal{T} . We denote by $C//r$ any object that appears in an exact triangle

$$C \xrightarrow{r} \Sigma^d C \longrightarrow C//r \rightsquigarrow,$$

and call it a Koszul object of r on C ; it is well defined up to (nonunique) isomorphism. For any object X in \mathcal{T} , applying $\text{Hom}_{\mathcal{T}}^*(-, X)$ to the triangle above yields an exact sequence of R -modules:

$$\begin{aligned} \cdots \rightarrow \text{Hom}_{\mathcal{T}}^*(C, X)[d+1] \xrightarrow{\mp r} \text{Hom}_{\mathcal{T}}^*(C, X)[1] \rightarrow \text{Hom}_{\mathcal{T}}^*(C//r, X) \\ \rightarrow \text{Hom}_{\mathcal{T}}^*(C, X)[d] \xrightarrow{\pm r} \text{Hom}_{\mathcal{T}}^*(C, X) \rightarrow \cdots . \end{aligned}$$

Applying the functor $\text{Hom}_{\mathcal{T}}^*(X, -)$ results in a similar exact sequence. Given a sequence of elements $\mathbf{r} = r_1, \dots, r_n$ in R , consider objects C_i defined by

$C_0 = C$, $C_i = C_{i-1}/r_i$ for $i \geq 1$. Set $C/\mathbf{r} = C_n$; it is a Koszul object of \mathbf{r} on C . Finally, given an ideal \mathbf{a} in R , we write C/\mathbf{a} for any Koszul object on C with respect to some finite sequence of generators for \mathbf{a} . This object may depend on the choice of the minimal generating sequence for \mathbf{a} .

3. Depth for triangulated categories

Let \mathcal{T} be a compactly generated triangulated category which is R -linear, where R is a graded-commutative Noetherian ring.

First let us recall some definitions (see [1]). Let X be an object of \mathcal{T} and C a compact object of \mathcal{T} . Consider two invariants

$$\begin{aligned}\inf_C X &= \inf(\mathrm{H}_C^*(X)) = \inf\{n \in \mathbb{Z} \mid \mathrm{H}_C^n(X) \neq 0\}, \\ \sup_C X &= \sup(\mathrm{H}_C^*(X)) = \sup\{n \in \mathbb{Z} \mid \mathrm{H}_C^n(X) \neq 0\}.\end{aligned}$$

We say that an object X of \mathcal{T} is cohomologically bounded above (resp., bounded below) if, for any compact object C , there exists a positive integer $n(C)$ such that $\sup_C X \leq n(C)$ (resp., $\inf_C X \geq -n(C)$). X is called cohomologically bounded if it is both cohomologically bounded above and cohomologically bounded below. Let \mathcal{T}^- (resp., \mathcal{T}^+ , \mathcal{T}^b) denote the full subcategory of \mathcal{T} , consisting of cohomologically bounded above (resp., cohomologically bounded below, bounded) objects. We say that X is cohomologically finite with respect to C if $\mathrm{H}_C^*(X)$ is finitely generated as a graded R -module. An object X is called cohomologically finite if it is cohomologically finite with respect to C for any $C \in \mathcal{T}^C$.

Lemma 3.1 ([1]). *Let $X \in \mathcal{T}$ and C be a compact object of \mathcal{T} and let $r \in R$ be a homogeneous element of degree d . Then*

$$\begin{aligned}\inf_C X/r &\geq \inf_C X - 1 \text{ if } d = 0 \text{ and} \\ \inf_C X/r &= \inf_C X - d \text{ if } d > 0.\end{aligned}$$

Definition 3.2. Let (R, \mathfrak{m}) be a graded local ring (i.e., it has a unique maximal graded ideal \mathfrak{m}) and X an object of \mathcal{T}^+ . We define the depth of X to be

$$\mathrm{depth} X = \inf\{\inf_C \Gamma_{v(\mathfrak{m})} X - \inf_C X \mid C \in \mathcal{T}^C\}.$$

Obviously, for any object X of \mathcal{T}^+ , we have $\mathrm{depth} X = \mathrm{depth} \Sigma X$.

Remark 3.3. (1) For any object $0 \neq X$ of \mathcal{T}^+ , $\inf_C \Gamma_{v(\mathfrak{m})} X \geq \inf_C X$ by [1, Proposition 4.9]. So $\mathrm{depth} X \geq 0$. For an object X of \mathcal{T}^+ (not necessarily non-zero), if $\inf_C \Gamma_{v(\mathfrak{m})} X = \inf_C X = 0$ for $C \in \mathcal{T}^C$, then $\mathrm{depth} X = 0$.

(2) Let (R, \mathfrak{m}) be a commutative Noetherian local ring and $\mathcal{M} = \mathrm{Mod} R$ the category of R -modules. Let $\mathcal{T} = D(\mathcal{M})$ be the derived category of complexes of R -modules. The category \mathcal{T} is triangulated and admits coproducts. The module R , viewed as a complex concentrated in degree 0, is a compact generator for \mathcal{T} . It is also R -linear, where for each $M \in \mathcal{T}$, the homomorphism $R \rightarrow \mathrm{Hom}_{\mathcal{T}}(M, M)$ is given by scalar multiplication. When specialized to a complex over R , this definition of depth does not agree with the one introduced by

Foxby in [8], since the depth (denoted by $\text{depth}_R M$) introduced by Foxby is changed by shifts of complexes. That is, for a chain complex of R -modules M and an integer n ,

$$\text{depth}_R \Sigma^n M = \text{depth}_R M - n,$$

where $\Sigma^n M$ denotes the complex M shifted n degrees to the left.

Recall that the dimension of an object is defined in [1] as follows.

Definition 3.4. Let X be an object of \mathcal{T} . We define the dimension of X to be

$$\dim_R X = \sup\{\dim_R H_C^*(X) \mid C \text{ is a compact object of } \mathcal{T}\}.$$

Theorem 3.5. Let (R, \mathfrak{m}) be a graded local ring. Then $\text{depth} X \leq \dim_R X$ for every cohomologically finite object X of \mathcal{T}^b .

Proof. We proceed by induction on $\dim_R X$ to show that $\text{depth} X \leq \dim_R X$. If $\dim_R X = 0$, then $\text{Supp}_R(X) = \{\mathfrak{m}\}$ and so $\Gamma_{v(\mathfrak{m})} X \cong X$. Hence the result holds.

Now assume that $\dim_R X > 0$ and the result has been proved for all integers smaller than $\dim_R X$. Since $\dim_R X > 0$, $\mathfrak{m} \notin \cup_{\mathfrak{p} \in \min(R)} \mathfrak{p}$ and hence there exists a homogeneous element $r \in \mathfrak{m} \setminus \cup_{\mathfrak{p} \in \min(R)} \mathfrak{p}$. If $d = \text{degree}(r) = 0$, then consider the exact triangle

$$X \xrightarrow{r} X \longrightarrow X/r \rightsquigarrow$$

in \mathcal{T} . This induces the exact sequence of R_0 -modules

$$\cdots \longrightarrow H_C^{i-1}(X/r) \longrightarrow H_C^i(X) \xrightarrow{r} H_C^i(X) \longrightarrow H_C^i(X/r) \longrightarrow H_C^{i+1}(X) \longrightarrow \cdots$$

for each $i \in \mathbb{Z}$, where $C \in \mathcal{T}^C$. Assume that $\inf_C X = t$. If $H_C^t(X/r) = 0$, then $H_C^t(X) = rH_C^t(X)$ and since X is cohomologically finite, Nakayama Lemma implies that $H_C^t(X) = 0$, which contradicts the assumption. Hence $H_C^t(X/r) \neq 0$. So $\inf_C X/r \leq t = \inf_C X$. By [1, Lemma 5.3], we have

$$\inf_C \Gamma_{v(\mathfrak{m})} X = \inf_C \Gamma_{v(\mathfrak{m})} X/r + 1 \leq \inf_C X/r + \dim_R X/r + 1 \leq \inf_C X + \dim_R X,$$

where the first inequality is by induction assumption. If $d = \text{degree}(r) > 0$, then consider the exact triangle

$$X \xrightarrow{r} \Sigma^d X \longrightarrow X/r \rightsquigarrow$$

in \mathcal{T} . We apply the functor $\Gamma_{v(\mathfrak{m})}(-)$ to the above exact triangle to get the following exact triangle

$$\Gamma_{v(\mathfrak{m})} X \xrightarrow{r} \Gamma_{v(\mathfrak{m})} \Sigma^d X \longrightarrow \Gamma_{v(\mathfrak{m})} X/r \rightsquigarrow .$$

By induction hypothesis, $\inf_C \Gamma_{v(\mathfrak{m})} X/r \leq \inf_C X/r + \dim_R X/r$. So by Lemma 3.1, we have

$$\begin{aligned} \inf_C \Gamma_{v(\mathfrak{m})} X &= \inf_C \Gamma_{v(\mathfrak{m})} X/r + d < \inf_C X/r + \dim_R X/r + d \\ &\leq \dim_R X + \inf_C X. \end{aligned} \quad \square$$

Proposition 3.6. *Let (R, \mathfrak{m}) be a trivially graded ring, that is concentrated in degree zero and \mathcal{T} generated by one compact object C . Then $\text{depth}X \leq \dim_R X$ for every cohomologically finite object X of \mathcal{T}^+ .*

Proof. We proceed by induction on $\dim_R X$ similar to Theorem 3.5. It remains to note that $\dim_R X/r = \dim_R X - 1$. For homogeneous element $r \in \mathfrak{m} \setminus \cup_{\mathfrak{p} \in \min(R)} \mathfrak{p}$, the exact triangle

$$X \xrightarrow{r} X \longrightarrow X/r \rightsquigarrow$$

in \mathcal{T} induces an exact sequence

$$0 \longrightarrow H_C^*(X) \xrightarrow{r} H_C^*(X) \longrightarrow H_C^*(X/r) \longrightarrow 0$$

of graded R -modules. So $H_C^*(X/r) \cong H_C^*(X)/rH_C^*(X)$. Hence

$$\dim_R H_C^*(X/r) = \dim_R H_C^*(X) - 1.$$

That is, $\dim_R X/r = \dim_R X - 1$. □

Proposition 3.7. *Let $X \longrightarrow Y \longrightarrow Z \rightsquigarrow$ be an exact triangle in \mathcal{T} . Then $\inf_C Y \geq \min\{\inf_C X, \inf_C Z\}$ for any $C \in \mathcal{T}^C$. Furthermore, for the exact triangle*

$$\Gamma_{v(\mathfrak{m})} X \longrightarrow X \longrightarrow L_{v(\mathfrak{m})} X \rightsquigarrow,$$

if $\inf_C L_{v(\mathfrak{m})} X \geq \inf_C \Gamma_{v(\mathfrak{m})} X$, then $\text{depth}X = 0$.

Proof. The first assertion follows from the long exact sequence

$$H_C^{i-1}(Z) \longrightarrow H_C^i(X) \longrightarrow H_C^i(Y) \longrightarrow H_C^i(Z)$$

of R_0 -modules. Let $g = \min\{\inf_C X, \inf_C Z\}$. Then $H_C^{g-1}(Y) = 0$. So $\inf_C Y \geq g$.

The second assertion follows from the fact $\inf_C \Gamma_{v(\mathfrak{m})} X \geq \inf_C X$ and the first assertion. □

Proposition 3.8. *Let (R, \mathfrak{m}) be a graded local ring and $r \in \mathfrak{m}$ a homogeneous element of degree d . Then for any cohomologically finite object X in \mathcal{T} ,*

$$\text{depth}X/r = \text{depth}X \text{ if } d > 0 \text{ and}$$

$$\text{depth}X/r \geq \text{depth}X - 1 \text{ if } d = 0.$$

Moreover, if (R, \mathfrak{m}) is trivially graded and \mathcal{T} is generated by one compact object C , then $\text{depth}X/r = \text{depth}X - 1$.

Proof. If $d > 0$, then $\inf_C X/r = \inf_C X - d$ and $\inf_C \Gamma_{v(\mathfrak{m})} X/r = \inf_C \Gamma_{v(\mathfrak{m})} X - d$ by Lemma 3.1. So

$$\begin{aligned} \text{depth}X/r &= \inf\{\inf_C \Gamma_{v(\mathfrak{m})} X/r - \inf_C X/r \mid C \in \mathcal{T}^C\} \\ &= \inf\{\inf_C \Gamma_{v(\mathfrak{m})} X - \inf_C X \mid C \in \mathcal{T}^C\} \\ &= \text{depth}X. \end{aligned}$$

If $d = 0$, then $\inf_C \Gamma_{v(\mathfrak{m})} X // r \geq \inf_C \Gamma_{v(\mathfrak{m})} X - 1$ by Lemma 3.1 and $\inf_C X // r \leq \inf_C X$ by the proof of Theorem 3.5. So

$$\begin{aligned} \text{depth} X // r &= \inf \{ \inf_C \Gamma_{v(\mathfrak{m})} X // r - \inf_C X // r \mid C \in \mathcal{T}^C \} \\ &\geq \inf \{ \inf_C \Gamma_{v(\mathfrak{m})} X - 1 - \inf_C X \mid C \in \mathcal{T}^C \} \\ &= \text{depth} X - 1. \end{aligned}$$

When (R, \mathfrak{m}) is trivially graded, the exact triangle

$$X \xrightarrow{r} X \longrightarrow X // r \rightsquigarrow$$

in \mathcal{T} induces an exact sequence

$$0 \longrightarrow H_C^*(X) \xrightarrow{r} H_C^*(X) \longrightarrow H_C^*(X // r) \longrightarrow 0$$

of graded R -modules. Assume that $\inf_C X = t$. If $H_C^t(X // r) = 0$, then we get the exact sequence

$$H_C^t(X) \longrightarrow H_C^t(X) \longrightarrow 0,$$

which, in view of Nakayama Lemma, is impossible. So $\inf_C X // r = \inf_C X$. By [1, Lemma 5.3], we have $\inf_C \Gamma_{v(\mathfrak{m})} X // r = \inf_C \Gamma_{v(\mathfrak{m})} X - 1$. Therefore, we get the desired equality. \square

Let \mathfrak{p} be a prime ideal of R . Since $\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \text{Spec} R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$ is a specialization closed subset of $\text{Spec}(R)$, there is a localization functor $L_{\mathcal{Z}(\mathfrak{p})} : \mathcal{T} \rightarrow \mathcal{T}$. For any $X \in \mathcal{T}$, let $X_{\mathfrak{p}}$ denote the object $L_{\mathcal{Z}(\mathfrak{p})} X$ as in [1]. In the following, we discuss the behaviors of $X_{\mathfrak{p}}$.

Let \mathfrak{p} be a prime ideal of R such that $\mathcal{Z}(\mathfrak{p}) \cap \text{supp}_R(X) = \emptyset$. Then $\text{depth}_R X = \text{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}$, where $\text{supp}_R(X) = \{\mathfrak{p} \in \text{Spec} R \mid \Gamma_{\mathfrak{p}} X = L_{\mathcal{Z}(\mathfrak{p})} \Gamma_{v(\mathfrak{p})} X \neq 0\}$ is the support of X (see [3]). This is because $\mathcal{Z}(\mathfrak{p}) \cap \text{supp}_R(X) = \emptyset$ if and only if $X_{\mathfrak{p}} \cong X$ by [3, Corollary 5.7].

Proposition 3.9. *Let \mathfrak{p} be a prime ideal of R . For the exact triangle*

$$\Gamma_{v(\mathfrak{m})} X \longrightarrow X \longrightarrow L_{v(\mathfrak{m})} X \rightsquigarrow,$$

if $\inf_C L_{v(\mathfrak{m})} X \geq \inf_C \Gamma_{v(\mathfrak{m})} X$ for any $C \in \mathcal{T}^C$, then $\text{depth}_R X \leq \text{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} + \dim R/\mathfrak{p}$.

Proof. If $\mathfrak{p} \notin \text{Supp}_R(X)$, then $X_{\mathfrak{p}} = 0$. So $\text{depth}_R X = \text{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} = 0$.

If $\mathfrak{p} \in \text{Supp}_R(X)$, then $X_{\mathfrak{p}} \neq 0$. So

$$\begin{aligned} \text{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} &= \inf \{ \inf_C \Gamma_{v(\mathfrak{m})} L_{\mathcal{Z}(\mathfrak{p})} X - \inf_C L_{\mathcal{Z}(\mathfrak{p})} X \mid C \in \mathcal{T}^C \} \\ &= \inf \{ \inf_C L_{\mathcal{Z}(\mathfrak{p})} \Gamma_{v(\mathfrak{m})} X - \inf_C L_{\mathcal{Z}(\mathfrak{p})} X \mid C \in \mathcal{T}^C \} \geq 0. \end{aligned}$$

Hence the result follows from the fact $\dim R/\mathfrak{p} \geq 0$ and Proposition 3.7. \square

Note that for an R -module M , $\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim_R M - \dim R/\mathfrak{p}$ ([6]). For $X \in \mathcal{T}$, we have:

Lemma 3.10. *Let \mathfrak{p} be a prime ideal of R . Then $\dim_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \leq \dim_R X - \dim R/\mathfrak{p}$.*

Proof. It follows from [3, Theorem 4.7] that $H_C^*(X_{\mathfrak{p}}) \cong H_C^*(X)_{\mathfrak{p}}$ for any $C \in \mathcal{T}^C$. Then

$$\dim_{R_{\mathfrak{p}}} H_C^*(X_{\mathfrak{p}}) = \dim_{R_{\mathfrak{p}}} H_C^*(X)_{\mathfrak{p}} \leq \dim_R H_C^*(X) - \dim R/\mathfrak{p}.$$

The inequality is from the result for modules. So $\dim_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \leq \dim_R X - \dim R/\mathfrak{p}$. \square

Definition 3.11. A cohomologically finite object X of \mathcal{T}^b is called Cohen-Macaulay if $\text{depth} X = \dim_R X$.

Proposition 3.12. *Let (R, \mathfrak{m}) be a trivially graded local ring and $r \in \mathfrak{m}$ a homogeneous element. Suppose X belongs to \mathcal{T}^b .*

- (1) *X is Cohen-Macaulay if and if X/r is so;*
- (2) *If $\inf_C L_{v(\mathfrak{m})} X \geq \inf_C \Gamma_{v(\mathfrak{m})} X$ for any $C \in \mathcal{T}^C$ and X is Cohen-Macaulay, then $X_{\mathfrak{p}}$ is Cohen-Macaulay for every $\mathfrak{p} \in \text{Spec}(R)$.*

Proof. (1) It follows from the fact $\dim_R X/r = \dim_R X - 1$ by the proof of Propositions 3.6 and 3.8.

(2) We have

$$\dim_{R_{\mathfrak{p}}} X_{\mathfrak{p}} + \dim R/\mathfrak{p} \leq \dim_R X = \text{depth} X \leq \text{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} + \dim R/\mathfrak{p}$$

by Lemma 3.10 and Proposition 3.9. So $\dim_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \leq \text{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}$. Since $\dim_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \geq \text{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}$, the result follows. \square

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