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WIENER'S LEMMA FOR INFINITE MATRICES OF GOHBERG-BASKAKOV-SJÖSTRAND CLASS

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Abstract. In this paper, we introduce a quasi-Banach algebra of infinite matrices, which is inverse-closed in the Banach algebra $\mathcal{B}(\ell^2)$ of all bounded operators on ℓ^2.

1. Introduction

N. Wiener showed that if f is a periodic function with an absolutely convergent Fourier series and it vanishes nowhere on the real line, then $1/f$ has an absolutely convergent Fourier series too [23]. This is now called the classical Wiener's lemma.

Define the *Gohberg-Baskakov-Sjöstrand class* $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ \big(a(i,j)\big)_{i,j\in\mathbb{Z}^d} : \sum_{k\in\mathbb{Z}^d} \big(\sup_{i-j=k} |a(i,j)| \big) < \infty \Big\}, \tag{1.1}$$

and the *Wiener class* $\mathcal{W}(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$\begin{aligned}\mathcal{W}(\mathbb{Z}^d, \mathbb{Z}^d) :=& \big\{ (a(i,j))_{i,j\in\mathbb{Z}^d} \in \mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d) : \\ & \quad a(i+k, j+k) = a(i,j) \text{ for all } i,j,k \in \mathbb{Z}^d \big\}.\end{aligned}$$

Then the classical Wiener's lemma can be reformulated as follows: $\mathcal{W}(\mathbb{Z}^d, \mathbb{Z}^d)$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$, the space of all bounded operators on the space ℓ^2 of square-summable sequences. Here a (quasi-)Banach algebra $\mathbb{B}$, which is a subalgebra of $\mathbb{A}$, is called *inverse-closed* if any $A \in \mathbb{B}$ with the inverse $A^{-1} \in \mathbb{A}$ implies $A^{-1} \in \mathbb{B}$.

Wiener's lemma has various extensions and applications. Define *Gröchenig-Schur class* $\mathcal{S}(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$\mathcal{S}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ \big(a(i,j)\big)_{i,j\in\mathbb{Z}^d} :$$

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$$\max\Big(\sup_{i\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}|a(i,j)|, \sup_{j\in\mathbb{Z}^d}\sum_{i\in\mathbb{Z}^d}|a(i,j)|\Big)<\infty\Big\}, \tag{1.2}$$

and the *Beurling class* $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$ by

$$\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d):=\Big\{\big(a(i,j)\big)_{i,j\in\mathbb{Z}^d}: \sum_{k\in\mathbb{Z}^d}\big(\sup_{|i-j|\geq|k|}|a(i,j)|\big)<\infty\Big\}, \tag{1.3}$$

where we set $|x|=\max(|x_1|,\ldots,|x_d|)$ for $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$. The Gröchenig-Schur class is not inverse-closed in $\mathcal{B}(\ell^2)$ but the weighted Gröchenig-Schur class is when the weight satisfies the GRS-condition [2, 6, 7, 9, 17, 19, 22]. The Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d,\mathbb{Z}^d)$ and the Beurling class $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$ are inverse-closed in $\mathcal{B}(\ell^2)$ [3, 19, 20]. The inverse-closed property has important applications in dual wavelet frames, dual Gabor frames, and algebra of pseudo-differential operators [1, 5, 8, 10, 12, 13, 18, 21]. The reader may refer to [6, 11, 16] and references therein for historical remarks, recent advances and applications.

For $0<q\leq 1$ and a weight w, Motee and Sun considered the Gröchenig-Schur class $\mathcal{S}_{q,w}(\mathbb{G})$ on a graph $\mathbb{G}$,

$$\mathcal{S}_{q,w}(\mathbb{G})=\big\{A=\big(a(i,j)\big)_{i,j\in\mathbb{G}}:\ \|A\|_{\mathcal{S}_{q,w}}<\infty\big\},$$

where

$$\|A\|_{\mathcal{S}_{q,w}}:=\max\Big\{\Big(\sup_{i\in\mathbb{G}}\sum_{j\in\mathbb{G}}|(a(i,j)|^q w(i,j)^q\Big)^{1/q},$$

$$\Big(\sup_{j\in\mathbb{G}}\sum_{i\in\mathbb{G}}|(a(i,j)|^q w(i,j)^q\Big)^{1/q}\Big\}. \tag{1.4}$$

The above class $\mathcal{S}_{q,w}(\mathbb{G})$ of matrices catches sparsity and localization of infinite matrices simultaneously. It does not form a Banach algebra, but it is a quasi-Banach algebra. More importantly, it is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$ under proper assumption on the weight w.

In this paper, we consider a general index set $\Lambda\subset\mathbb{R}^d$ satisfying

$$\alpha=\sup_{k\in\mathbb{Z}^d}\sum_{\lambda\in\Lambda}\chi_{k+[-2,2)^d}(\lambda)<\infty. \tag{1.5}$$

Unlike the index set $\mathbb{Z}^d$ in (1.1), our index set Λ may not form a group. The prime models are paraboloids

$$\{(x,y,z):\ z=ax^2+by^2, x,y\in\mathbb{Z}\}$$

and elliptical hyperboloids

$$\{(x,y,z):\ z^2=ax^2+by^2, x,y\in\mathbb{Z}\},$$

where $a,b>0$. For $0<q\leq 1$ and a weight w, we define the *Gohberg-Baskakov-Sjöstrand class* $\mathcal{C}_{q,w}$, GBS class for short, on Λ by

$$\mathcal{C}_{q,w}=\big\{A=\big(a(i,j)\big)_{i,j\in\Lambda}:\ \|A\|_{\mathcal{C}_{q,w}}<\infty\big\}, \tag{1.6}$$

where

$$\|A\|^q_{\mathcal{C}_{q,w}} := \alpha \sum_{k\in\mathbb{Z}^d} \sup_{\lambda,\lambda'\in\Lambda} |a(\lambda,\lambda')|^q w(\lambda,\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda').$$

In this paper, we prove that $\mathcal{C}_{q,w}, 0 < q \le 1$, are inverse-closed quasi-Banach algebras of $\mathcal{B}(\ell^2)$ under proper hypotheses on the weight w.

2. Quasi-Banach algebras

Let $\Lambda \subset \mathbb{R}^d$ satisfy (1.5). Then for any integer k

$$\max\Big\{\sup_{\lambda'\in\Lambda}\sum_{\lambda\in\Lambda}\chi_{k+(-1,1)^d}(\lambda-\lambda'),\ \sup_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda}\chi_{k+(-1,1)^d}(\lambda-\lambda')\Big\} \le \alpha. \tag{2.1}$$

We say that w is a *weight* if

$$w(\lambda,\lambda') \ge 1 \qquad \text{for any } \lambda,\lambda'\in\Lambda, \tag{2.2}$$

$$w(\lambda,\lambda') = w(\lambda',\lambda) \qquad \text{for any } \lambda,\lambda'\in\Lambda, \tag{2.3}$$

and

$$\sup_{\lambda\in\Lambda} w(\lambda,\lambda) < \infty. \tag{2.4}$$

The Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{q,w}$ of infinite matrices has the following basic properties.

Proposition 2.1. *Let $0 < q \le 1$ and w be a weight.*

(i) *If $A \in \mathcal{C}_{q,w}$, then $cA \in \mathcal{C}_{q,w}$ for any $c \in \mathbb{R}$ and $\|cA\|_{\mathcal{C}_{q,w}} = |c|\|A\|_{\mathcal{C}_{q,w}}$.*
(ii) *If $A \in \mathcal{C}_{q,w}$, then $\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{C}_{q,w}}$, so $A \in \mathcal{B}(\ell^2)$.*
(iii) *For $A, B \in \mathcal{C}_{q,w}$, $\|A+B\|^q_{\mathcal{C}_{q,w}} \le \|A\|^q_{\mathcal{C}_{q,w}} + \|B\|^q_{\mathcal{C}_{q,w}}$, so $A+B \in \mathcal{C}_{q,w}$.*
(iv) *If there exists a positive constant C_0' such that*

$$w(\lambda,\lambda') \le C_0' w(\lambda,\tilde\lambda)w(\tilde\lambda,\lambda') \quad \textit{for all} \quad \lambda,\lambda,\tilde\lambda\in\Lambda, \tag{2.5}$$

then for any $A, B \in \mathcal{C}_{q,w}$

$$\|AB\|^q_{\mathcal{C}_{q,w}} \le 2^d C_0 \|A\|^q_{\mathcal{C}_{q,w}} \|B\|^q_{\mathcal{C}_{q,w}}. \tag{2.6}$$

Proof. (i) Trivial.

(ii) It is well known that

$$\|A\|_{\mathcal{B}(\ell^2)} \le \max\Big(\sup_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda}|a(\lambda,\lambda')|,\ \sup_{\lambda'\in\Lambda}\sum_{\lambda\in\Lambda}|a(\lambda,\lambda')|\Big).$$

Since $(a+b)^q \le a^q + b^q$ for any $a, b \ge 0$, and $w(\lambda,\lambda') \ge 1$ for any $\lambda,\lambda'\in\Lambda$, we have that

$$\begin{aligned}\sup_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda}|a(\lambda,\lambda')| &\le \Big(\alpha\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}|a(\lambda,\lambda')|^q\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)^{1/q}\\ &\le \Big(\alpha\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}|a(\lambda,\lambda')|^q w(\lambda,\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)^{1/q}\end{aligned}$$

$$= \|A\|_{\mathcal{C}_{q,w}}.$$

Similarly

$$\sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} |a(\lambda, \lambda')| \le \|A\|_{\mathcal{C}_{q,w}}.$$

(iii) For $A = \big(a(\lambda, \lambda')\big)_{\lambda, \lambda' \in \Lambda}$ and $B = \big(b(\lambda, \lambda')\big)_{\lambda, \lambda' \in \Lambda}$,

$$\begin{aligned} \|A+B\|_{\mathcal{C}_{q,w}}^q &= \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda') + b(\lambda, \lambda')|^q w(\lambda, \lambda') \chi_{k+[0,1)^d}(\lambda - \lambda') \\ &\le \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')|^q w(\lambda, \lambda') \chi_{k+[0,1)^d}(\lambda - \lambda') \\ &\quad + \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |b(\lambda, \lambda')|^q w(\lambda, \lambda') \chi_{k+[0,1)^d}(\lambda - \lambda') \\ &= \|A\|_{\mathcal{C}_{q,w}}^q + \|B\|_{\mathcal{C}_{q,w}}^q. \end{aligned}$$

(iv) Let $A = \big(a(\lambda, \lambda')\big)_{\lambda, \lambda' \in \Lambda} \in \mathcal{C}_{q,w}$, $B = \big(b(\lambda, \lambda')\big)_{\lambda, \lambda' \in \Lambda} \in \mathcal{C}_{q,w}$ and $C = AB$. Then

$$C = \Big(\sum_{\tilde{\lambda} \in \Lambda} a(\lambda, \tilde{\lambda}) b(\tilde{\lambda}, \lambda') \Big)_{\lambda, \lambda' \in \Lambda}.$$

Observe that if $\lambda - \lambda' \in k + [0,1)^d$ and $\lambda - \tilde{\lambda} \in \ell + [0,1)^d$ for some $k, \ell \in \mathbb{Z}^d$, then $\tilde{\lambda} - \lambda' \in k - \ell + (-1,1)^d$. Then we have from (2.1) that

$$\begin{aligned} \|C\|_{\mathcal{C}_{q,w}}^q &= \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} \Big| \sum_{\tilde{\lambda} \in \Lambda} a(\lambda, \tilde{\lambda}) b(\tilde{\lambda}, \lambda') \Big|^q w(\lambda, \lambda') \chi_{k+[0,1)^d}(\lambda - \lambda') \\ \le\ & C_0' \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} \sum_{\tilde{\lambda} \in \Lambda} |a(\lambda, \tilde{\lambda})|^q w(\lambda, \tilde{\lambda}) |b(\tilde{\lambda}, \lambda')|^q w(\tilde{\lambda}, \lambda') \chi_{k+[0,1)^d}(\lambda - \lambda') \\ \le\ & C_0' \alpha^2 \sum_{k \in \mathbb{Z}^d} \sum_{\ell \in \mathbb{Z}^d} \big(\sup_{\lambda, \tilde{\lambda} \in \Lambda} |a(\lambda, \tilde{\lambda})|^q w(\lambda, \tilde{\lambda}) \chi_{\ell+[0,1)^d}(\lambda - \tilde{\lambda}) \big) \\ & \times \big(\sup_{\tilde{\lambda}, \lambda' \in \Lambda} |b(\tilde{\lambda}, \lambda')|^q w(\tilde{\lambda}, \lambda') \chi_{k-\ell+(-1,1)^d}(\tilde{\lambda} - \lambda') \big) \\ \le\ & 2^d C_0' \|A\|_{\mathcal{C}_{q,w}}^q \|B\|_{\mathcal{C}_{q,w}}^q. \end{aligned}$$

This proves the conclusion (iv). □

By Proposition 2.1, there exists a positive constant K such that

$$\|A+B\|_{\mathcal{C}_{q,w}} \le K(\|A\|_{\mathcal{C}_{q,w}} + \|B\|_{\mathcal{C}_{q,w}}) \quad \text{for all } A, B \in \mathcal{C}_{q,w}. \tag{2.7}$$

So $\|\cdot\|_{\mathcal{C}_{q,w}}$ is a quasi-norm [4, 14]. Therefore $(\mathcal{C}_{q,w}, \|\cdot\|_{\mathcal{C}_{q,w}})$ forms a quasi-Banach algebra by Proposition 2.1.

Corollary 2.2. *Let* $0 \le q \le 1$. *Assume that* w *is a weight satisfying the submultiplicative condition* (2.5). *Then* $\mathcal{C}_{q,w}$ *is a quasi-Banach algebra.*

3. Wiener's lemma

In this section, we will show that $\mathcal{C}_{q,w}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$. To do it, we first establish paracompact estimate for matrices in $\mathcal{C}_{q,w}$.

Let w be a weight. A weight u is called *a companion matrix* of w if

$$w(\lambda,\lambda') \le w(\lambda,\tilde\lambda)u(\tilde\lambda,\lambda') + u(\lambda,\tilde\lambda)w(\tilde\lambda,\lambda') \quad \text{for all } \lambda,\lambda',\tilde\lambda\in\Lambda. \tag{3.1}$$

Proposition 3.1. *Let $0<q<1$, w be a weight, and u be a companion weight of w. We assume that there exist a positive constant C_1 and $0<\theta<1$ such that*

$$\inf_{\tau\ge 0}\Big\{\alpha \sum_{\substack{|k|\le\tau+1,\\ k\in\mathbb{Z}^d}} \sup_{\tilde\lambda,\lambda'\in\Lambda} u(\tilde\lambda-\lambda')\chi_{k+[0,1)^d}(\tilde\lambda-\lambda') + t\sup_{|\tilde\lambda-\lambda'|>\tau}\frac{u(\tilde\lambda,\lambda')}{w(\tilde\lambda,\lambda')}\Big\} \le C_1 t^\theta \tag{3.2}$$

for all $t\ge 1$. Then there exists a positive constant C_2 such that for any $A,B\in\mathcal{C}_{q,w}$

$$\|AB\|^q_{\mathcal{C}_{q,w}} \le C_2\|A\|^q_{\mathcal{C}_{q,w}}\|B\|^q_{\mathcal{C}_{q,w}}\Big(\Big(\frac{\|A\|_{\mathcal{B}(\ell^2)}}{\|A\|_{\mathcal{C}_{q,w}}}\Big)^{q(1-\theta)} + \Big(\frac{\|B\|_{\mathcal{B}(\ell^2)}}{\|B\|_{\mathcal{C}_{q,w}}}\Big)^{q(1-\theta)}\Big). \tag{3.3}$$

Proof. Take $A=\big(a(\lambda,\lambda')\big)_{\lambda,\lambda'\in\Lambda}$ and $B=\big(b(\lambda,\lambda')\big)_{\lambda,\lambda'\in\Lambda}$. Then

$$AB = \Big(\sum_{\tilde\lambda\in\Lambda} a(\lambda,\tilde\lambda)b(\tilde\lambda,\lambda')\Big)_{\lambda,\lambda'\in\Lambda}.$$

We obtain from (3.1) that

$$\begin{aligned}
\|AB\|^q_{\mathcal{C}_{q,w}} &\le \alpha\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|^q|b(\tilde\lambda,\lambda')|^q w(\lambda,\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda')\\
&\le \alpha\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\Big(\sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|^q w(\lambda,\tilde\lambda)|b(\tilde\lambda,\lambda')|^q u(\tilde\lambda,\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda')\\
&\quad + \sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|^q u(\lambda,\tilde\lambda)|b(\tilde\lambda,\lambda')|^q w(\tilde\lambda,\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)\\
&=: I_1 + I_2.
\end{aligned} \tag{3.4}$$

Let $\tau\ge 0$. Since for any $\lambda,\lambda'\in\Lambda$, $|b(\tilde\lambda,\lambda')|\le\|B\|_{\mathcal{B}(\ell^2)}$, we have that

$$\begin{aligned}
I_1 &\le \alpha\|B\|^q_{\mathcal{B}(\ell^2)}\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\Big(\sum_{\substack{\tilde\lambda\in\Lambda\\ |\tilde\lambda-\lambda'|\le\tau}}|a(\lambda,\tilde\lambda)|^q w(\lambda,\tilde\lambda)u(\tilde\lambda,\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)\\
&\quad + \alpha\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\Big(\sum_{\substack{\tilde\lambda\in\Lambda\\ |\tilde\lambda-\lambda'|>\tau}}|a(\lambda,\tilde\lambda)|^q w(\lambda,\tilde\lambda)|b(\tilde\lambda,\lambda')|^q u(\tilde\lambda,\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)\\
&\le \alpha^2\|B\|^q_{\mathcal{B}(\ell^2)}\sum_{k\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d}\sup_{\lambda,\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|^q w(\lambda,\tilde\lambda)\chi_{k-\ell+(-1,1)^d}(\lambda-\tilde\lambda)
\end{aligned}$$

$$
\begin{aligned}
&\times \sup_{\substack{\tilde\lambda,\lambda'\in\Lambda,\\ |\tilde\lambda-\lambda'|\le\tau}} u(\tilde\lambda,\lambda')\chi_{\ell+[0,1)^d}(\tilde\lambda-\lambda')\\
&+\alpha^2\sum_{k\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d}\sup_{\lambda,\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|^q w(\lambda,\tilde\lambda)\chi_{k-\ell+(-1,1)^d}(\lambda-\tilde\lambda)\\
&\times \sup_{\substack{\tilde\lambda,\lambda'\in\Lambda,\\ |\tilde\lambda-\lambda'|>\tau}} |b(\lambda,\tilde\lambda)|^q u(\tilde\lambda,\lambda')\chi_{\ell+[0,1)^d}(\tilde\lambda-\lambda')\\
\le\ & 2^d\alpha\|A\|^q_{\mathcal{C}_{q,w}}\|B\|^q_{\mathcal{B}(\ell^2)}\sum_{\ell\in\mathbb{Z}^d}\sup_{\substack{\tilde\lambda,\lambda'\in\Lambda,\\ |\tilde\lambda-\lambda'|\le\tau}} u(\tilde\lambda,\lambda')\chi_{\ell+[0,1)^d}(\tilde\lambda-\lambda')\\
&+2^d\|A\|^q_{\mathcal{C}_{q,w}}\|B\|^q_{\mathcal{C}_{q,w}}\sup_{|\tilde\lambda-\lambda'|>\tau}\frac{u(\tilde\lambda,\lambda')}{w(\tilde\lambda,\lambda')}\\
\le\ & 2^d\bigg(\alpha\|A\|^q_{\mathcal{C}_{q,w}}\|B\|^q_{\mathcal{B}(\ell^2)}\sum_{\substack{|k|\le\tau+1,\\ k\in\mathbb{Z}^d}}\sup_{\tilde\lambda,\lambda'\in\Lambda} u(\tilde\lambda,\lambda')\chi_{k+[0,1)^d}(\tilde\lambda-\lambda')\\
&+\|A\|^q_{\mathcal{C}_{q,w}}\|B\|^q_{\mathcal{C}_{q,w}}\sup_{|\tilde\lambda-\lambda'|>\tau}\frac{u(\tilde\lambda,\lambda')}{w(\tilde\lambda,\lambda')}\bigg).
\end{aligned}
$$

This together with (3.2) implies that

$$I_1\le 2^d\|A\|^q_{\mathcal{C}_{q,w}}\|B\|^q_{\mathcal{C}_{q,w}}\left(\frac{\|B\|_{\mathcal{B}(\ell^2)}}{\|B\|_{\mathcal{C}_{q,w}}}\right)^{q(1-\theta)}. \tag{3.5}$$

Similarly,

$$I_2\le 2^d\|A\|^q_{\mathcal{C}_{q,w}}\|B\|^q_{\mathcal{C}_{q,w}}\left(\frac{\|A\|_{\mathcal{B}(\ell^2)}}{\|A\|_{\mathcal{C}_{q,w}}}\right)^{q(1-\theta)}. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) proves (3.3). □

We remark that the assumption (3.2) on a weight u is stronger than the submultiplicative condition (2.5). In fact, putting $\tau=0$ and $t=1$ in (3.2), we get

$$u(\lambda,\lambda')\le C_1 w(\lambda,\lambda')\qquad\text{for any }\lambda,\lambda'\in\Lambda. \tag{3.7}$$

This together with (3.1) implies that

$$w(\lambda,\lambda')\le 2C_1 w(\lambda,\tilde\lambda)w(\tilde\lambda,\lambda')\qquad\text{for any }\lambda,\lambda',\tilde\lambda\in\Lambda,$$

and hence (2.5) holds with $C_0'=2C_1$. Therefore

$$\|AB\|^q_{\mathcal{C}_{q,w}}\le 2^{d+1}C_1\|A\|^q_{\mathcal{C}_{q,w}}\|B\|^q_{\mathcal{C}_{q,w}}\quad\text{for all }A,B\in\mathcal{C}_{q,w} \tag{3.8}$$

by Proposition 2.1.

To prove the inverse-closedness of $\mathcal{C}_{q,w}$ in $\mathcal{B}(\ell^2)$, we need estimate powers of a matrix A in $\mathcal{C}_{q,w}$.

Proposition 3.2. *Under the assumptions of Proposition 3.1,*

$$\|A^n\|^q_{\mathcal{C}_{q,w}} \le (2^{d+2}C_1C_2)^{\log_2 n}\Big(\frac{\|A\|_{\mathcal{C}_{q,w}}}{\|A\|_{\mathcal{B}(\ell^2)}}\Big)^{q\cdot\frac{(1+\theta)}{\theta}n^{\log_2(1+\theta)}}\|A\|^{nq}_{\mathcal{B}(\ell^2)} \tag{3.9}$$

for all $A \in \mathcal{C}_{q,w}$ *and integers* $n \ge 1$.

Proof. Let $A \in \mathcal{C}_{q,w}$ and n be a positive integer. We write $n = \sum_{j=0}^{N}\varepsilon_j 2^j$, where $\varepsilon_N = 1$ and $\varepsilon_j \in \{0,1\}$. We put

$$n_\ell = \varepsilon_\ell + 2n_{\ell+1} \text{ and } n_N = \varepsilon_N \quad \text{for } \ell = 0,\dots,N-1.$$

Without loss of generality, we assume that $\|A\|_{\mathcal{B}(\ell^2)} = 1$, otherwise replace A by $A/\|A\|_{\mathcal{B}(\ell^2)}$. Then setting $A = B$ in (3.3) gives

$$\|A^2\|^q_{\mathcal{C}_{q,w}} \le 2C_2\|A\|^{q(1+\theta)}_{\mathcal{C}_{q,w}}. \tag{3.10}$$

By (3.8), (3.10) and the observation that $N \le \log_2 n$, we have

$$\sum_{k=0}^{N}\varepsilon_k(1+\theta)^k \le 1 + (1+\theta) + \cdots + (1+\theta)^N \le \frac{1+\theta}{\theta}(1+\theta)^N,$$

and

$$\begin{aligned}
\|A^n\|^q_{\mathcal{C}_{q,w}} &\le 2^{d+1}C_1\|A\|^{q\varepsilon_0}_{\mathcal{C}_{q,w}}\|A^{2n_1}\|^q_{\mathcal{C}_{q,w}}\\
&\le 2^{d+1}C_1(2C_2)\|A\|^{q\varepsilon_0}_{\mathcal{C}_{q,w}}\|A^{n_1}\|^{q(1+\theta)}_{\mathcal{C}_{q,w}}\\
&\le (2^{d+1}C_1)^2(2C_2)\|A\|^{q\varepsilon_0+q\varepsilon_1(1+\theta)}_{\mathcal{C}_{q,w}}\|A^{2n_2}\|^{q(1+\theta)}_{\mathcal{C}_{q,w}}\\
&\le (2^{d+1}C_1)^2(2C_2)^2\|A\|^{q\varepsilon_0+q\varepsilon_1(1+\theta)}_{\mathcal{C}_{q,w}}\|A^{n_2}\|^{q(1+\theta)^2}_{\mathcal{C}_{q,w}}\\
&\dots\\
&\le (2^{d+1}C_1)^N(2C_2)^N\|A\|^{q(\varepsilon_0+\varepsilon_1(1+\theta)+\cdots+\varepsilon_N(1+\theta)^N)}_{\mathcal{C}_{q,w}}.\\
&\le (2^{d+2}C_1C_2)^{\log_2 n}\|A\|^{q\frac{1+\theta}{\theta}\cdot n^{\log_2(1+\theta)}}_{\mathcal{C}_{q,w}}.
\end{aligned} \tag{3.11}$$

This proves (3.9). □

Finally, we prove inverse-closedness of the subalgebra $\mathcal{C}_{q,w}$ in $\mathcal{B}(\ell^2)$.

Theorem 3.3. *Let* $0 < q < 1$. *Under the assumptions of Proposition 3.1, the quasi-Banach algebra* $\mathcal{C}_{q,w}$ *is inverse-closed in* $\mathcal{B}(\ell^2)$, *that is, if* $A \in \mathcal{C}_{q,w}$ *and* $A^{-1} \in \mathcal{B}(\ell^2)$, *then* $A^{-1} \in \mathcal{C}_{q,w}$.

Proof. Let $A \in \mathcal{C}_{q,w}$ and $A^{-1} \in \mathcal{B}(\ell^2)$. We put $B = I - \|A\|^{-2}_{\mathcal{C}_{q,w}}A^TA$. Then $\|B\|_{\mathcal{B}(\ell^2)} \le 1 - \|A\|^{-2}_{\mathcal{C}_{q,w}}\|A\|^2_{\mathcal{B}(\ell^2)} \le r_0$, where $r_0 = 1 - \|A\|^{-2}_{\mathcal{C}_{q,w}}\|A^{-1}\|^{-2}_{\mathcal{B}(\ell^2)} \in [0,1)$. Since from (3.9) $\lim_{n\to\infty}\|B^n\|^{q/n}_{\mathcal{C}_{q,w}} \le r_0^q < 1$, $\sum_{n=1}^\infty \|B^n\|^q_{\mathcal{C}_{q,w}} < \infty$. Observing that $A^{-1} = (A^TA)^{-1}A^T = \|A\|^{-2}_{\mathcal{C}_{q,w}}(I-B)^{-1}A^T$, we have that

$$\|A^{-1}\|^q_{\mathcal{C}_{q,w}} \le \|A\|^{-q}_{\mathcal{C}_{q,w}}(\|I\|^q_{\mathcal{C}_{q,w}} + \sum_{n=1}^{\infty}\|B^n\|^q_{\mathcal{C}_{q,w}}) < \infty,$$

where $\|I\|_{\mathcal{C}_{q,w}} = \sup_{\lambda\in\Lambda} w(\lambda,\lambda)$. Hence $A^{-1} \in \mathcal{C}_{q,w}$. □

We conclude this section with remarks on polynomial weights and subexponential weights that satisfy (3.1) and (3.2).

Remark 3.4. For $\alpha > 0$, consider polynomial weights $w_\alpha := ((1+|i-j|)^\alpha)_{i,j\in\mathbb{Z}^d}$. The constant weight u_α with $u_\alpha(i,j) = 2^\alpha$ for any $i,j \in \mathbb{Z}^d$ satisfies the companion weight condition (3.1). Also,

$$\begin{aligned}
&\inf_{\tau\ge 0}\Big\{\sum_{|k|\le\tau+1}\sup_{\lambda,\lambda'\in\Lambda} u(\lambda-\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda') + t\sup_{|\lambda-\lambda'|>\tau}\frac{u(\lambda,\lambda')}{w(\lambda,\lambda')}\Big\}\\
&\le \inf_{\tau\ge 0}\Big\{\sum_{|k|\le\tau+1} 2^\alpha + t\cdot 2^\alpha(1+\tau)^{-\alpha}\Big\}\\
&\le \inf_{\tau\ge 0}\Big\{2^\alpha\big((2\tau+3)^d + t(1+\tau)^{-\alpha}\big)\Big\}\\
&\le \inf_{\tau\ge 0}\Big\{2^{\alpha+2d}\big((\tau+1)^d + t(1+\tau)^{-\alpha}\big)\Big\}\\
&\le 2^{\alpha+2d+1}t^{\frac{d}{d+\alpha}} \quad \text{for all } t\ge 1,
\end{aligned}$$

where in the last inequality τ satisfies the equation $(\tau+1)^d = t(\tau+1)^{-\alpha}$. Hence the polynomial weights $w_\alpha, \alpha > 0$, satisfy (3.2).

Next, for $D > 0$ and $0 < \delta < 1$, we consider the subexponential weight $e_{D,\delta} = (e^{D|i-j|^\delta})_{i,j\in\mathbb{Z}^d}$. The weight $e_{D(2^\delta-1),\delta}(i,j) := u(i,j) = e^{D(2^\delta-1)|i-j|^\delta}$ satisfies the companion weight condition (3.1). Choose $C' > 1$ and $\tau' > 0$ such that $C'(2^\delta-1) < 1$ and $\big(\frac{\tau+1}{\tau}\big)^\delta < C'$ for $\tau \ge \tau'$. If $\tau' > (\ln t/D)^{1/\delta}$, that is, t is bounded, then for $0<\theta<1$, there exists $C_1 > 0$ such that for $1 \le t < e^{D(\tau')^\delta}$

$$\begin{aligned}
&\inf_{\tau\ge 0}\Big\{\sum_{|k|\le\tau+1}\sup_{\lambda,\lambda'\in\Lambda} u(\lambda-\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda') + t\sup_{|\lambda-\lambda'|>\tau}\frac{u(\lambda,\lambda')}{w(\lambda,\lambda')}\Big\}\\
&\le \inf_{\tau\ge 0}\Big\{\sum_{|k|\le\tau+1} e^{D(2^\delta-1)|k|^\delta} + t\cdot e^{D(2^\delta-2)\tau^\delta}\Big\} \le C_1 t^\theta, \qquad (3.12)
\end{aligned}$$

where we let $\tau = 0$ in the third equality.

For $t\ge 1$, $\tau' \le \big(\frac{\ln t}{D}\big)^{1/\delta}$ and

$$\begin{aligned}
&\inf_{\tau\ge 0}\Big\{\sum_{|k|\le\tau+1}\sup_{\lambda,\lambda'\in\Lambda} u(\lambda-\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda') + t\sup_{|\lambda-\lambda'|>\tau}\frac{u(\lambda,\lambda')}{w(\lambda,\lambda')}\Big\}\\
&\le \inf_{\tau\ge 0}\Big\{\sum_{|k|\le\tau+1} e^{D(2^\delta-1)|k|^\delta} + t\cdot e^{D(2^\delta-2)\tau^\delta}\Big\}\\
&\le e^{C'D(2^\delta-1)\tau^\delta}(2\tau+3)^d + t^{2^\delta-1} \qquad (3.13)\\
&\le t^{C'(2^\delta-1)}(2\tau+3)^d + t^{2^\delta-1}\\
&\le t^{C'(2^\delta-1)}\big((2\tau+3)^d+1\big)
\end{aligned}$$

$$(3.14) \quad = t^{C'(2^\delta-1)}\Big(\big(3+2\big(\frac{\ln t}{D}\big)^{1/\delta}\big)^d+1\Big),$$

where in the third inequality $\tau \geq \tau'$ satisfies the equation $\tau^\delta = \frac{\ln t}{D}$. Hence for any θ with $C'(2^\delta - 1) < \theta < 1$ there exists $C_1 > 0$ such that (3.2) holds. Combining (3.12) and (3.14) proves (3.2).

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