

## HELICOIDAL MINIMAL SURFACES IN A CONFORMALLY FLAT 3-SPACE

KELLCIO OLIVEIRA ARAÚJO, NINGWEI CUI, AND ROMILDO DA SILVA PINA

ABSTRACT. In this work, we introduce the complete Riemannian manifold  $\mathbb{F}_3$  which is a three-dimensional real vector space endowed with a conformally flat metric that is a solution of the Einstein equation. We obtain a second order nonlinear ordinary differential equation that characterizes the helicoidal minimal surfaces in  $\mathbb{F}_3$ . We show that the helicoid is a complete minimal surface in  $\mathbb{F}_3$ . Moreover we obtain a local solution of this differential equation which is a two-parameter family of functions  $\lambda_{h,K_2}$  explicitly given by an integral and defined on an open interval. Consequently, we show that the helicoidal motion applied on the curve defined from  $\lambda_{h,K_2}$  gives a two-parameter family of helicoidal minimal surfaces in  $\mathbb{F}_3$ .

### 1. Introduction

In the monograph entitled “Ricci Curvature of Seminar on Differential Geometry” edited by Yau (see Problem Section in [13]), there is a problem which is to find the necessary and sufficient conditions for a symmetric tensor  $T_{ij}$  on a compact manifold so that one can find a metric  $g_{ij}$  satisfying

$$R_{ij} - \frac{1}{2}K g_{ij} = T_{ij},$$

where  $R_{ij}$  and  $K$  denote the Ricci tensor and the scalar curvature of  $g_{ij}$  respectively. If  $g_{ij}$  is the Lorentz metric on a 4-dimensional manifold, the above equation is the celebrated Einstein field equation which was formulated by Einstein and widely studied by many mathematicians and physicists. The left hand side of the Einstein equation describes the geometry of the space time, and the right hand side  $T_{ij}$ , called stress-energy tensor, describes the density and flux of energy which is the source of the gravitational field in general relativity. The metric  $g_{ij}$  satisfying the Einstein equation represents the gravitational field.

In the decade Pina and Tenenblat studied this problem for a special family of tensors in manifolds that are locally conformally flat (see [8] and references therein). This paper also provided some conformally flat spaces which have

---

Received March 6, 2015; Revised June 29, 2015.

2010 *Mathematics Subject Classification.* 53C21, 53C42.

*Key words and phrases.* helicoidal minimal surfaces, conformally flat space.

interesting geometries ([4]). In this paper we will introduce the complete Riemannian manifold  $\mathbb{F}_3$  which is a three-dimensional real vector space endowed with one of such conformally flat metrics. Then we will study the isometry group of  $\mathbb{F}_3$  and the helicoidal minimal surfaces in  $\mathbb{F}_3$  which are invariant under the helicoidal group.

The study of minimal surfaces is a very interesting and very hot topic in Differential Geometry. There are several recent papers which study the minimal surfaces in 3-spaces distinct of the real vector space  $\mathbb{R}^3$  endowed with the canonical Euclidean metric. In recent years, some researchers started to investigate the minimal surfaces in some spaces which are not Riemannian space forms, for instances, non-Riemannian Finsler space forms with vanishing flag curvature and homogeneous Riemannian 3-spaces with isometry group of dimension 4. For example, Souza and Tenenblat [11] studied the rotational minimal surfaces in a three-dimensional Randers space and the explicit solutions were given in [5]. In another paper, Silva and Tenenblat [10] considered a cylindrical region of  $\mathbb{R}^3$  with a Randers metric. They obtained a differential equation that characterizes the minimal surfaces in this space and they showed that a part of the catenoid and some planes intersected with this region are minimal surfaces. In another work, Benoit, Meeks and Rosenberg [6] considered the Heisenberg group  $\text{Nil}_3$  and the Lie group  $\text{Sol}_3$  endowed with their standard left invariant metrics and they studied minimal surfaces in these three-dimensional Riemannian manifolds.

Moreover in the study of minimal surfaces in non Euclidean spaces, Hoffman and White [7] proved that if a complete, properly embedded, finite-topology minimal surface in  $\mathbb{S}^2 \times \mathbb{R}$  contains a line, then its ends are asymptotic to helicoids, and that if the surface is an annulus, it must be a helicoid. Pyo [9] constructed three kinds of complete embedded minimal surfaces in  $\mathbb{H}^2 \times \mathbb{R}$ . The first is a simply connected, singly periodic, infinite total curvature surface. The second is an annular finite total curvature surface. These two are conjugate surfaces just as the helicoid and the catenoid are in  $\mathbb{R}^3$ . The third one is a finite total curvature surface which is conformal to  $\mathbb{S}^2 - \{p_1, \dots, p_k\}$ ,  $k \geq 3$ .

Helicoidal surfaces in Euclidean spaces have been investigated by several people. It is well known that the Euclidean 3-space has isometry group of dimension 6 and the helicoidal group is a two-parameter isometry group consisting of the rotation and translation. Do Carmo and Djaczter [3] described the space  $\sum_H$  of all surfaces in  $\mathbb{R}^3$  that have constant mean curvature  $H \neq 0$  and are invariant by helicoidal motions, with a fixed axis, of  $\mathbb{R}^3$ . Baikoussis and Koufogiorgos [1] studied helicoidal surfaces with prescribed mean or Gaussian curvature, where it was also considered the Euclidean space  $\mathbb{R}^3$  endowed with the canonical Euclidean metric  $\langle, \rangle$ . We denote this space by  $(\mathbb{R}^3, \langle, \rangle)$ . In a similar case, Beneki, Kaimakamis and Papantoniou [2], studied helicoidal surfaces with prescribed Gaussian and mean curvature in Minkowski space  $\mathbb{R}_1^3$ .

In the recent work [4], Corro, Pina and Souza, began to study the surface theory for a three-dimensional complete manifold  $(\mathbb{R}^3, \langle, \rangle_g)$ , where  $\langle, \rangle_g$  is a

conformal metric to Euclidean metric  $\langle, \rangle$ , which means that there exists a positive differentiable function  $F : \mathbb{R}^3 \rightarrow \mathbb{R}$  such that

$$\langle \omega_1, \omega_2 \rangle_g = \frac{1}{F^2(x)} \langle \omega_1, \omega_2 \rangle, \forall \omega_1, \omega_2 \in T_x \mathbb{R}^3, \forall x \in \mathbb{R}^3.$$

In this case, the components of the metric  $\langle, \rangle_g$  are given by

$$g_{ij}(x) = \frac{1}{F^2(x)} \delta_{ij}, \quad x = (x_1, x_2, x_3), \quad 1 \leq i, j \leq 3.$$

Note that if  $F$  is bounded, then  $(\mathbb{R}^3, \langle, \rangle_g)$  is a complete Riemannian manifold. This manifold is a conformally flat 3-space. The study of immersed surfaces in spaces that are conformal to the Euclidean space is natural, because they include very important spaces of constant curvature as the sphere  $\mathbb{S}^3$  and the hyperbolic space  $\mathbb{H}^3$ . The authors in [4] studied surfaces of rotation with constant extrinsic curvature in  $\mathbb{E}_3 = (\mathbb{R}^3, \langle, \rangle_{g_1})$  whose the components of the metric  $\langle, \rangle_{g_1}$  are given by  $g_1 := (g_{ij}) = \frac{1}{F^2} \delta_{ij}$ , where  $F(x) = e^{-x_1^2 - x_2^2 - x_3^2}$ ,  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ .

Mathematically speaking, it is interesting to study minimal surfaces in conformally flat spaces which has interesting geometries and may have physics significance. In the present work, we decided to study the conformally flat 3-space with  $F : \mathbb{R}^3 \rightarrow \mathbb{R}$  given by  $F(x_1, x_2, x_3) = e^{-x_1^2 - x_2^2}$  and the components of metric are given by

$$g := (g_{ij}) = \frac{1}{F^2} \delta_{ij}, \quad \text{where } F(x) = e^{-x_1^2 - x_2^2}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

This metric appears as a solution of the Einstein equation obtained by Pina and Tenenblat in [8], which may have applications in physics (see [12] and references therein). We denote this Riemannian manifold by  $\mathbb{F}_3 := (\mathbb{R}^3, \langle, \rangle_g)$ . It is interesting that if we study the helicoidal minimal surfaces in  $\mathbb{F}_3$ , we can give the explicit expression. Note that  $\mathbb{F}_3$  is complete because  $F(x_1, x_2, x_3) = e^{-x_1^2 - x_2^2}$  is bounded. Moreover  $\mathbb{F}_3$  has negative sectional curvature that converges to zero when  $\|x\|^2 = \langle x, x \rangle \rightarrow +\infty$ .

In our paper, we will prove that the isometry group of  $\mathbb{F}_3$  has dimension 2 which consists of the rotation around  $x_3$ -axis and the translation along  $x_3$ -axis. Then we given a natural definition of the helicoidal group of  $\mathbb{F}_3$ . A surface  $M^2$  invariant under the helicoid group by a curve  $(u, 0, \lambda(u))$  in  $\mathbb{F}_3$  can be parametrized by  $X : I \times \mathbb{R} \rightarrow \mathbb{F}_3$  such that

$$(1) \quad X(u, v) = (u \cos v, u \sin v, \lambda(u) + hv),$$

where  $\lambda$  is a  $C^2$ -function defined on an open interval  $I \subset \mathbb{R}$  and  $h$  is a constant, called *helicoidal surface* with  $x_3$ -axis and pitch  $h$ . If  $h = 0$ , then a helicoidal surface  $M^2$  is just a surface of revolution. If  $h \neq 0$  and  $\lambda \equiv \text{constant}$ , then  $M^2$  is just a helicoid.

Our first result shows that a helicoid is a complete minimal surface in  $\mathbb{F}_3$ :

**Theorem 1.1.** *Let  $M^2$  be a helicoid whose parametrization  $X : \mathbb{R}^2 \rightarrow \mathbb{F}_3$  is given by*

$$(2) \quad X(u, v) = (u \cos v, u \sin v, k + hv),$$

where  $h \neq 0$  and  $k$  are constants. Then,  $M^2$  is a complete minimal surface in  $\mathbb{F}_3$ .

When the  $C^2$ -function  $\lambda$  in (1) is not constant, we have the following result:

**Theorem 1.2.** *Let  $M^2$  be a helicoidal surface in  $\mathbb{F}_3$  whose parametrization  $X : U \rightarrow \mathbb{F}_3$  is given by*

$$(3) \quad X(u, v) = (u \cos v, u \sin v, \lambda(u) + hv),$$

where  $\lambda$  is a  $C^2$ -function,  $h$  is a constant and  $U$  is an open subset of  $\mathbb{R}^2$ . Assume that  $M^2$  is a minimal surface in  $\mathbb{F}_3$ . Then, for any  $K_1 \in (0, \frac{1}{2}e^{-\frac{1}{2}})$  there exists  $\delta(K_1) > 0$  and an open interval  $I_\delta = (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$  such that the function  $\lambda$  is explicitly given by

$$\lambda = \pm \int \frac{K_1 e^{2u^2} [u^2 + h^2]^{1/2}}{u [u^2 - K_1^2 e^{4u^2}]^{1/2}} du + K_2, \quad \forall u \in I_\delta,$$

where  $K_2$  is a constant.

The converse of Theorem 1.2 holds. We can find a two-parameter family of curves  $\gamma(u) = (u, 0, \lambda_{h, K_2}(u))$  defined on an open interval such that the helicoidal motion applied on  $\gamma$  gives a two-parameter family of helicoidal minimal surfaces in  $\mathbb{F}_3$ . In this context, we have the following result:

**Theorem 1.3.** *Let  $h$  and  $K_2$  be arbitrary constants. Then, for any  $K_1 \in (0, \frac{1}{2}e^{-\frac{1}{2}})$  there exists  $\delta(K_1) > 0$  and an open interval  $I_\delta = (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$  such that the function*

$$\lambda_{h, K_2} = \int \frac{K_1 e^{2u^2} [u^2 + h^2]^{1/2}}{u [u^2 - K_1^2 e^{4u^2}]^{1/2}} du + K_2$$

is defined on  $I_\delta$ . Moreover, after applying the helicoidal motion on the curve  $\gamma(u) = (u, 0, \lambda_{h, K_2}(u))$ ,  $u \in I_\delta$ , a helicoidal surface given by (3) with  $U = I_\delta \times \mathbb{R} \subset \mathbb{R}^2$  is obtained. In particular, this surface is minimal in  $\mathbb{F}_3$ .

## 2. Isometry group and helicoidal surface in $\mathbb{F}_3$

In this section, we will study the isometry group of  $\mathbb{F}_3$  and show that it has dimension 2 which consists exactly of rotation and translation corresponding to  $x_3$ -axis. It is well known that the infinitesimal isometry of a given Riemannian manifold is represented by its Killing vector field on the manifold, and the number of arbitrary constants in the expression of Killing vector field represents the dimension of isometry group.

Let  $g_{ij} := e^{2t}\delta_{ij}$ ,  $t := x_1^2 + x_2^2$ . A direct computation gives:

$$\begin{aligned} \Gamma_{jk}^i &= \frac{1}{2}g^{il}\left[\frac{\partial g_{lj}}{\partial x_k} + \frac{\partial g_{lk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_l}\right] \\ (4) \qquad &= 2\left(\sum_{a=1}^2 x_a\delta_{ak}\delta_{ij} + \sum_{a=1}^2 x_a\delta_{aj}\delta_{ik} - \sum_{a=1}^2 x_a\delta_{ai}\delta_{jk}\right), \end{aligned}$$

where  $1 \leq i, j, k \leq 3$ .

Let  $V = V^i \frac{\partial}{\partial x_i}$  be a Killing vector field on  $\mathbb{F}_3$ , i.e.,  $\mathcal{L}_V g = 0$ , where  $\mathcal{L}$  is the Lie derivative. Denote  $V_i := g_{ij}V^j$ . We have

$$(5) \qquad V_{i|j} + V_{j|i} = 0,$$

where “|” denotes the covariant derivative with respect to the metric  $g$ . By using (4), one can show that (5) is equivalent to the following system of PDEs:

$$\begin{aligned} \frac{\partial V_1}{\partial x_1} - 2(x_1V_1 - x_2V_2) &= 0, \\ \frac{\partial V_2}{\partial x_2} - 2(x_2V_2 - x_1V_1) &= 0, \\ \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - 4(x_iV_j + x_jV_i) &= 0, \quad 1 \leq i \neq j \leq 2, \\ \frac{\partial V_3}{\partial x_i} + \frac{\partial V_i}{\partial x_3} - 4x_iV_3 &= 0, \quad 1 \leq i \leq 2, \\ (6) \qquad \frac{\partial V_3}{\partial x_3} + 2(x_1V_1 + x_2V_2) &= 0. \end{aligned}$$

Since  $V_i := g_{ij}V^j$ , we have

$$V_1 = e^{2(x_1^2+x_2^2)}V^1, \quad V_2 = e^{2(x_1^2+x_2^2)}V^2, \quad V_3 = e^{2(x_1^2+x_2^2)}V^3.$$

Then the system (6) becomes

$$(7) \qquad \frac{\partial V^i}{\partial x_i} + 2(x_1V^1 + x_2V^2) = 0, \quad 1 \leq i \leq 3,$$

$$(8) \qquad \frac{\partial V^i}{\partial x_j} + \frac{\partial V^j}{\partial x_i} = 0, \quad 1 \leq i \neq j \leq 3.$$

It follows from (8) that there exists a skew-symmetric matrix  $(q_j^i)$  and a constant vector  $(b^i)$  such that  $V^i = q_j^i x_j + b^i$ . Plugging this into (7) we get  $(x_1q_3^1 + x_2q_3^2)x_3 + b^1x_1 + b^2x_2 = 0$  for any  $x_2$  and  $x_3$ , which implies  $q_3^1 = q_3^2 = b^1 = b^2 = 0$ . Therefore

$$V^1 = c_1x_2, \quad V^2 = -c_1x_1, \quad V^3 = c_2,$$

where  $c_1$  and  $c_2$  are two arbitrary constants. This means that the Killing vector field is

$$V = c_1\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right) + c_2\frac{\partial}{\partial x_3}.$$

Therefore the Lie algebra of the isometry group is spanned by two Killing vector fields  $x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$  and  $\frac{\partial}{\partial x_3}$ . The corresponding isometry group has dimension 2 consisting of the rotation around  $x_3$ -axis and the translation along  $x_3$ -axis.

Let  $(u, 0, \lambda(u))$  be a curve in  $x_1x_3$ -plane. Usually the helicoidal group is an isometry group which is defined by a rotation combined by a translation. Rotating this curve around  $x_3$ -axis and giving a translation along  $x_3$ -axis, we get a surface

$$(9) \quad X(u, v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 0 \\ \lambda(u) \end{pmatrix} + h \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix},$$

where  $h$  is a constant called *pitch*. The surface in the form (9) is just the surface (1) which is invariant under the helicoidal group, and will be called *helicoidal surface* in this paper.

### 3. Formulas in general conformally flat spaces

Let  $\gamma(u) = (u, 0, \lambda(u))$ ,  $u \in I$  be a  $C^2$ -curve in  $\mathbb{R}^3$  defined on any open interval  $I$  of real numbers not including zero. Now, applying a helicoidal motion on  $\gamma$  we obtain the helicoidal surface  $M^2$  in  $\mathbb{R}^3$  given by

$$X(u, v) = (u \cos v, u \sin v, \lambda(u) + hv),$$

where  $h$  is a constant.

Let us consider the Euclidean space  $(\mathbb{R}^3, \langle, \rangle)$  endowed with the canonical Euclidean metric  $g_o = \langle, \rangle$ . We denote by,  $I_{g_o}$ ,  $II_{g_o}$  and  $N$ , the *first fundamental form*, the *second fundamental form* and the *Gauss map* of  $M^2$  in  $(\mathbb{R}^3, \langle, \rangle)$ , respectively. After a straightforward computation, we have

$$\begin{aligned} I_{g_o} &= (1 + \lambda'^2)du^2 + 2h\lambda' dudv + (u^2 + h^2)dv^2, \\ II_{g_o} &= \frac{1}{\alpha}(u\lambda''du^2 - 2hdudv + u^2\lambda'dv^2), \\ N &= \frac{1}{\alpha}(h \sin v - u\lambda' \cos v, -u\lambda' \sin v - h \cos v, u), \end{aligned}$$

where  $\alpha = [u^2(1 + \lambda'^2) + h^2]^{1/2}$ . It is easy to check that the *mean curvature*  $H_{g_o}$  of  $M^2$  in  $(\mathbb{R}^3, \langle, \rangle)$  is given by

$$(10) \quad H_{g_o} = \frac{(u^2 + h^2)u\lambda'' + (1 + \lambda'^2)u^2\lambda' + 2h^2\lambda'}{2[u^2(1 + \lambda'^2) + h^2]^{3/2}}.$$

Now, we consider a conformally flat 3-space  $(\mathbb{R}^3, \langle, \rangle_g)$ . The following result (see Theorem 1 in [4]) establishes some relations between concepts defined in  $(\mathbb{R}^3, \langle, \rangle)$  and  $(\mathbb{R}^3, \langle, \rangle_g)$ , respectively.

**Theorem 3.1.** *Let  $X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a regular parametrized surface. Consider  $X(U)$  as a surface in  $(\mathbb{R}^3, \langle, \rangle)$  with the Euclidean metric  $g_o = \langle, \rangle$ , let  $N$  be the normal Gauss map,  $\lambda_i$  the principal curvatures,  $H_{g_o}$  and  $K_{g_o}$*

the mean and Gaussian curvatures, respectively. Also consider  $X(U)$  like a surface in  $(\mathbb{R}^3, \langle \cdot, \cdot \rangle_g)$ , with a metric conformal to the Euclidean metric, with the conformal factor  $F^{-2}$ , let  $\tilde{\lambda}_i$  be the principal curvatures,  $\tilde{H}$  and  $\tilde{K}_E$  the mean and the extrinsic curvatures, respectively. Then

$$(11) \quad \begin{aligned} \tilde{\lambda}_i &= F\lambda_i - \langle N, \text{grad}F \rangle, \\ \tilde{H} &= FH_{g_o} - \langle N, \text{grad}F \rangle, \\ \tilde{K}_E &= F^2K_{g_o} - 2HF\langle N, \text{grad}F \rangle + \langle N, \text{grad}F \rangle^2, \end{aligned}$$

where  $F$  denotes the evaluation of  $F$  at  $X(u, v)$ , where  $(u, v) \in U$ .

#### 4. Helicoidal minimal surfaces in $\mathbb{F}_3$

Consider the conformally flat 3-space  $\mathbb{F}_3 := (\mathbb{R}^3, \langle \cdot, \cdot \rangle_g)$  whose metric  $\langle \cdot, \cdot \rangle_g$  is given by

$$g := (g_{ij}) = \frac{1}{F^2}\delta_{ij}, \text{ where } F(x) = e^{-x_1^2 - x_2^2}, x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

The following lemma presents a second order nonlinear ordinary differential equation that characterizes the helicoidal minimal surfaces in  $\mathbb{F}_3$ .

**Lemma 4.1.** *Let  $M^2$  be a helicoidal surface  $X : U \rightarrow \mathbb{F}_3$  given by*

$$X(u, v) = (u \cos v, u \sin v, \lambda(u) + hv),$$

where  $h$  is a constant,  $\lambda$  is a  $C^2$ -function and  $U$  is an open subset of  $\mathbb{R}^2$ . Then,  $M^2$  is a minimal surface in  $\mathbb{F}_3$  if and only if the following differential equation holds

$$(12) \quad \frac{(u^2 + h^2)u\lambda'' + (1 + \lambda'^2)u^2\lambda' + 2h^2\lambda'}{2[u^2(1 + \lambda'^2) + h^2]^{3/2}} = 2u^2A,$$

where  $A = \lambda'[u^2(1 + \lambda'^2) + h^2]^{-1/2}$ .

*Proof.* From (11) in Theorem 3.1, the mean curvature  $\tilde{H}$  of  $M^2$  in  $\mathbb{F}_3$  is given by

$$(13) \quad \tilde{H} = FH_{g_o} - \langle N, \text{grad}F \rangle,$$

where  $F$  denotes the evaluation of  $F$  in  $X(u, v)$ ,  $H_{g_o}$  is the mean curvature and  $N$  is the Gauss map of  $M^2$  in  $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ . By definition of  $\mathbb{F}_3$  we have  $\text{grad}F = (-2x_1F, -2x_2F, 0)$ . Thus,

$$\text{grad}(F \circ X(u, v)) = (-2u \cos vF, -2u \sin vF, 0) = F(-2u \cos v, -2u \sin v, 0).$$

On the other hand, the expression for  $N$  is given by

$$N = \frac{1}{\alpha}(h \sin v - u\lambda' \cos v, -u\lambda' \sin v - h \cos v, u),$$

where  $\alpha = [u^2(1 + \lambda^2) + h^2]^{1/2}$ . Therefore we compute the canonical inner product between this relations as follows

$$\begin{aligned} \langle N, \text{grad}F \rangle &= \frac{F}{\alpha} (-2uh \sin v \cos v + 2u^2 \lambda' \cos^2 v + 2u^2 \lambda' \sin^2 v + 2uh \sin v \cos v) \\ &= \frac{F}{\alpha} (2u^2 \lambda'). \end{aligned}$$

Now, using the last equality in (13) and putting  $\tilde{H} = 0$  (by hypothesis) we obtain

$$(14) \quad H_{g_o} = \frac{2u^2 \lambda'}{\alpha},$$

where we used that  $F \neq 0$ . Let  $A$  defined by  $A = \lambda'[u^2(1 + \lambda^2) + h^2]^{-1/2} = \lambda' \alpha^{-1}$ . Thus, from (10) we can rewrite (14) as follows

$$\frac{(u^2 + h^2)u\lambda'' + (1 + \lambda'^2)u^2\lambda' + 2h^2\lambda'}{2[u^2(1 + \lambda'^2) + h^2]^{3/2}} = 2u^2 A.$$

The converse is trivial. □

As a first consequence of previous Lemma, we have the following:

*Proof of Theorem 1.1.* Note that the constant function  $\lambda(u) = k$  is a trivial solution of the differential equation (12) of Lemma 4.1. Therefore, the helicoid is a minimal surface in  $\mathbb{F}_3$ . Since  $M^2$  is closed in  $\mathbb{F}_3$  which is a complete manifold, we conclude that  $M^2$  is a complete surface in  $\mathbb{F}_3$ . □

*Remark.* Using the same arguments of the previous proof, we can conclude that the plane  $z = d$  is a complete minimal surface in  $\mathbb{F}_3$ , where  $d$  is a constant.

In what follows, we consider the case where the function  $\lambda$  is not constant. In this case, we find a nontrivial solution  $\lambda$  for the differential equation (12).

*Proof of Theorem 1.2.* At beginning, it is easy to compute that

$$2A + uA' = \frac{(u^2 + h^2)u\lambda'' + (1 + \lambda'^2)u^2\lambda' + 2h^2\lambda'}{[u^2(1 + \lambda'^2) + h^2]^{3/2}},$$

where  $A = \lambda'[u^2(1 + \lambda'^2) + h^2]^{-1/2}$ . Since  $M^2$  is a minimal surface in  $\mathbb{F}_3$ , it follows from Lemma 4.1 that

$$(15) \quad A + \frac{u}{2}A' = 2u^2A,$$

which means that

$$\frac{A'}{A} = 4u - \frac{2}{u}.$$

So integrating this equation, we obtain

$$\ln |A| = 2u^2 - \ln u^2 + K_o,$$

where  $K_o$  is an integration constant. Thus,

$$|A| = e^{(2u^2 - \ln u^2 + K_o)} = K_1 e^{2u^2 - \ln u^2},$$



where  $K_1 = e^{K_o}$  is a positive number. Since  $A = \lambda'[u^2(1 + \lambda'^2) + h^2]^{-1/2}$ , it follows that

$$\lambda'^2 [u^2(1 + \lambda'^2) + h^2]^{-1} = K_1^2 e^{2(2u^2 - \ln u^2)}.$$

We note that the last equation can be rewritten as

$$\lambda'^2 = K_1^2 e^{2(2u^2 - \ln u^2)} [u^2 + u^2 \lambda'^2 + h^2],$$

which means that

$$\lambda'^2 \left( 1 - K_1^2 u^2 e^{2(2u^2 - \ln u^2)} \right) = K_1^2 e^{2(2u^2 - \ln u^2)} [u^2 + h^2].$$

Now, we note that  $1 - K_1^2 u^2 e^{2(2u^2 - \ln u^2)} > 0$  if and only if  $K_1 < \frac{u}{e^{2u^2}}$ . On the other hand, for any  $K_1 > 0$  it is possible to choose  $K_o$  such that  $e^{K_o} = K_1$ . Also, we observe that the function  $f(u) := \frac{u}{e^{2u^2}}$  is continuous and attains its maximum  $f(\frac{1}{2}) = \frac{1}{2}e^{-\frac{1}{2}}$ . Therefore, if  $K_1 \in (0, \frac{1}{2}e^{-\frac{1}{2}})$ , then there exists  $\delta(K_1) > 0$  such that  $K_1 < f(u), \forall u \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ . Hence, the following equation holds

$$\begin{aligned} \lambda'^2 &= \frac{K_1^2 e^{2(2u^2 - \ln u^2)} [u^2 + h^2]}{1 - K_1^2 u^2 e^{2(2u^2 - \ln u^2)}} \\ &= \frac{K_1^2 e^{4u^2} [u^2 + h^2]}{u^2 [u^2 - K_1^2 e^{4u^2}]} \end{aligned}$$

for all  $u \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ . So integrating the last equality we get the general solution

$$\lambda = \pm \int \frac{K_1 e^{2u^2} [u^2 + h^2]^{1/2}}{u [u^2 - K_1^2 e^{4u^2}]^{1/2}} du + K_2,$$

which is defined on an open interval  $I_\delta := (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ , where  $K_2$  is an integration constant. □

The converse of previous proof holds and gives the last proof this paper.

*Proof of Theorem 1.3.* For each  $K_1 \in (0, \frac{1}{2}e^{-\frac{1}{2}})$  there is a number  $K_o$  such that  $K_1 = e^{K_o}$ . Since the function  $f(u) := \frac{u}{e^{2u^2}}$  is continuous and attains the maximum  $f(\frac{1}{2}) = \frac{1}{2}e^{-\frac{1}{2}}$ , there exists  $\delta(K_1) > 0$  such that  $K_1 < f(u), \forall u \in I_\delta := (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ . Therefore,

$$u^2 - K_1^2 e^{4u^2} > 0, \forall u \in I_\delta,$$

and the following function is defined on  $I_\delta$

$$(16) \quad \lambda_{h,K_2} = \int \frac{K_1 e^{2u^2} [u^2 + h^2]^{1/2}}{u [u^2 - K_1^2 e^{4u^2}]^{1/2}} du + K_2.$$

By the proof of Theorem 1.1 it follows that (16) is a solution of (15). Applying a helicoidal motion on the curve  $\gamma(u) = (u, 0, \lambda_{h,K_2}(u))$ ,  $u \in I_\delta$ , we get a helicoidal surface of pitch  $h$  given by (3) where  $U = I_\delta \times \mathbb{R} \subset \mathbb{R}^2$ . From Lemma 4.1 we conclude that this helicoidal surface is minimal in  $\mathbb{F}_3$ . □

## References

- [1] C. Baikoussis and T. Koufogiorgos, *Helicoidal surfaces with prescribed mean or gaussian curvature*, J. Geom. **63** (1998), no. 1-2, 25–29.
- [2] Chr. C. Beneki, G. Kaimakamis, and B. J. Papantoniou, *Helicoidal surfaces in three-dimensional Minkowski space*, J. Math. Anal. Appl. **275** (2002), no. 2, 586–614.
- [3] M. P. do Carmo and M. Dajczer, *Helicoidal surfaces with constant mean curvature*, Tohoku Math. J. **34** (1982), no. 3, 425–435.
- [4] A. V. Corro, R. Pina, and M. Souza, *Surfaces of rotation with constant extrinsic curvature in a conformally flat 3-space*, Results. Math. **60** (2011), no. 1-4, 225–234.
- [5] N. Cui and Y.-B. Shen, *Minimal rotational hypersurfaces in Minkowski  $(\alpha, \beta)$ -space*, Geom. Dedicata **151** (2011), 27–39.
- [6] B. Daniel, W. H. Meeks, and H. Rosenberg, *Half-space theorems for minimal surfaces in  $Nil_3$  and  $Sol_3$* , J. Differential Geom. **88** (2011), no. 1, 41–59.
- [7] D. Hoffman and B. White, *Axial minimal surfaces in  $S^2 \times R$  are helicoidal*, J. Differential Geom. **87** (2011), no. 3, 515–523.
- [8] R. Pina and K. Tenenblat, *On solutions of the Ricci curvature equation and the Einstein equation*, Israel J. Math. **171** (2009), 61–76.
- [9] J. Pyo, *New complete embedded minimal surfaces in  $\mathbb{H}^2 \times \mathbb{R}$* , Ann. Global Anal. Geom. **40** (2011), no. 2, 167–176.
- [10] R. M. da Silva and K. Tenenblat, *Minimal surfaces in a cylindrical region of  $\mathbb{R}^3$  with a Randers metric*, Houston J. Math. **37** (2011), no. 3, 745–771.
- [11] M. A. Souza and K. Tenenblat, *Minimal surfaces of rotation in Finsler space with a Randers metric*, Math. Ann. **325** (2003), no. 4, 625–642.
- [12] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions Of Einstein Field Equations*, Cambridge University Press, Cambridge, 2003.
- [13] S. T. Yau, *Seminar on Differential Geometry*, Princeton University Press, Princeton, NJ, 1982.

KELLCIO OLIVEIRA ARAÚJO  
 DEPARTAMENTO DE MATEMATICA  
 UNIVERSIDADE DE BRASILIA - UNB  
 70910-900, BRASILIA, DF, BRAZIL  
*E-mail address:* [kellcioaraujo@gmail.com](mailto:kellcioaraujo@gmail.com)

NINGWEI CUI  
 DEPARTAMENTO DE MATEMATICA  
 UNIVERSIDADE DE BRASILIA - UNB  
 70910-900, BRASILIA, DF, BRAZIL  
*E-mail address:* [ningweicui@gmail.com](mailto:ningweicui@gmail.com)

ROMILDO DA SILVA PINA  
 INSTITUTO DE MATEMATICA E ESTATISTICA  
 UNIVERSIDADE FEDERAL DE GOIAS - UFG  
 74001-970, GOIANIA, GO, BRAZIL  
*E-mail address:* [romildo@mat.ufg.br](mailto:romildo@mat.ufg.br)