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SURFACES OF REVOLUTION WITH POINTWISE 1-TYPE GAUSS MAP IN PSEUDO-GALILEAN SPACE

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Abstract. In this paper, we study surfaces of revolution in the three dimensional pseudo-Galilean space. We classify surfaces of revolution generated by a non-isotropic curve in terms of the Gauss map and the Laplacian of the surface. Furthermore, we give the classification of surfaces of revolution generated by an isotropic curve satisfying pointwise 1-type Gauss map equation.

1. Introduction

In late 1970's B.-Y. Chen introduced the notion of Euclidean immersions of finite type. Essentially these are submanifolds whose immersion into the m-dimensional Euclidean space $\mathbb{E}^m$ is constructed by making use of a finite number of $\mathbb{E}^m$-valued eigenfunctions of their Laplacian. The first results on this subject have been collected in the book [2]. Many works were done to characterize or classify submanifolds in terms of finite type. In a framework of the theory of finite type, B.-Y. Chen and P. Piccini [4] made a general study on submanifolds of Euclidean spaces with finite type Gauss map. Several geometers also studied submanifolds of Euclidean spaces or pseudo-Euclidean spaces with finite type Gauss map.

From the above definition one can see that a submanifold has 1-type Gauss map G if and only if G satisfies the equation

$$\Delta G = \lambda(G + C) \tag{1.1}$$

for a constant λ and a constant vector C, where Δ denotes the Laplace operator on a submanifold. A plane, a circular cylinder and a sphere are surfaces with 1-type Gauss map.

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Similarly, a submanifold is said to have pointwise 1-type Gauss map if the Laplacian of its Gauss map takes the form

$$\Delta G = F(G + C) \tag{1.2}$$

for a non-zero smooth function F and a constant vector C. More precisely, a pointwise 1-type Gauss map is said to be of the first kind if (1.2) is satisfied for $C = 0$, and of the second kind if $C \neq 0$. A helicoid, a catenoid and a right cone are the typical examples of surfaces with pointwise 1-type Gauss map. Many results of submanifolds with pointwise 1-type Gauss map were obtained in [1], [3], [5], [6], [7], [9], [12], etc, when the ambient spaces are the Euclidean space, Minkowski space and Galilean space.

In this paper, we study surfaces of revolution in the three dimensional pseudo-Galilean space G_3^1 in terms of their Gauss map. In Sections 2 and 3, we introduce pseudo-Galilean space and construct surfaces of revolution in G_3^1 by non-isotropic and isotropic rotations. In Section 4, we obtain the complete classification of surfaces of revolution generated by non-isotropic curve with pointwise 1-type Gauss map. In the last section, we focus on surfaces of revolution generated by isotropic curve with pointwise 1-type Gauss map and give the complete classification of such surfaces.

2. Pseudo-Galilean space

Let us recall the basic facts about the three dimensional pseudo-Galilean space G_3^1. The geometry of the pseudo-Galilean space has been firstly explained in [10]. The pseudo-Galilean space G_3^1 is a Cayley-Klein space with the absolute figure consisting of an ordered triple $\{\omega, f, I\}$, where ω is the ideal (absolute) plane in the three dimensional real projective space RP_3, f the line (the absolute line) in ω and I the fixed hyperbolic involution of points of f. Homogenous coordinates in G_3^1 are introduced in such a way that the absolute plane ω is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the hyperbolic involution η by $\eta : (x_0 : x_1 : x_2 : x_3) \to (0 : 0 : x_3 : x_2)$.

Let $\mathbf{x} = (x_1, y_1, z_1)$ and $\mathbf{y} = (x_2, y_2, z_2)$ be two vectors in G_3^1. A vector $\mathbf{x}$ is called isotropic if $x_1 = 0$, otherwise it is called non-isotropic. The pseudo-Galilean scalar product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{cases} x_1 x_2, & \text{if } \ x_1 \neq 0 \quad \text{or} \quad x_2 \neq 0, \\ y_1 y_2 - z_1 z_2, & \text{if } \ x_1 = 0 \quad \text{and} \quad x_2 = 0. \end{cases} \tag{2.1}$$

From this, the pseudo-Galilean norm of a vector $\mathbf{x}$ in G_3^1 is given by $||\mathbf{x}|| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$ and all unit non-isotropic vectors are the form $(1, y_1, z_1)$. There are four types of isotropic vectors: spacelike ($y_1^2 - z_1^2 > 0$), timelike ($y_1^2 - z_1^2 < 0$) and the two types of lightlike ($y_1 = \pm z_1$) vectors. A non-lightlike isotropic vector is a unit vector if $y_1^2 - z_1^2 = \pm 1$.

A plane of the form $x =$ constant is called a pseudo-Euclidean plane, otherwise it is called isotropic. An isotropic plane $ax + by + cz + d = 0$ is called light-like if $b^2 - c^2 = 0$.

The pseudo-Galilean cross product of $\mathbf{x}$ and $\mathbf{y}$ on G_3^1 is defined by

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}, \tag{2.2}$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Consider a C^r-surface M $(r \geq 1)$ in G_3^1 parameterized by

$$\mathbf{x}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).$$

Let us denote $g_i = \frac{\partial x}{\partial u_i}$, $h_{ij} = \langle \frac{\partial \tilde{\mathbf{x}}}{\partial u_i}, \frac{\partial \tilde{\mathbf{x}}}{\partial u_j} \rangle (i, j = 1, 2)$, where $\sim$ stands for the projection of a vector on the pseudo-Euclidean yz-plane. A surface M is called admissible if it does not have Euclidean tangent planes. Therefore a surface M is admissible if and only if $x_{,i} \neq 0$ for some $i = 1, 2$.

Let M be an admissible surface in G_3^1. Then, the corresponding matrix of the first fundamental form ds^2 of a surface M is given by (cf. [11])

$$ds^2 = \begin{pmatrix} ds_1^2 & 0 \\ 0 & ds_2^2 \end{pmatrix},$$

where $ds_1^2 = (g_1 du_1 + g_2 du_2)^2$ and $ds_2^2 = h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2$. Here $g_i = x_{,i}$ and $h_{ij} = \langle \tilde{\mathbf{x}}_{,i}, \tilde{\mathbf{x}}_{,j} \rangle$ $(i, j = 1, 2)$. In such case, we denote the coefficients of ds^2 by g_{ij}^*.

On the other hand, the unit normal vector field U of a surface M is defined by

$$U = \frac{1}{W}(0, x_{,1} z_{,2} - x_{,2} z_{,1}, x_{,1} y_{,2} - x_{,2} y_{,1}),$$

where

$$W = \sqrt{|(x_{,1} y_{,2} - x_{,2} y_{,1})^2 - (x_{,1} z_{,2} - x_{,2} z_{,1})^2|}.$$

The Gaussian curvature K of a surface M is defined by means of the coefficients $L_{ij}(i, j = 1, 2)$ of the second fundamental form, which are the normal components of $\mathbf{x}_{,i,j}(i, j = 1, 2)$, that is,

$$L_{ij} = \frac{1}{g_1} \langle g_1 \tilde{\mathbf{x}}_{,i,j} - g_{i,j} \tilde{\mathbf{x}}_{,1}, U \rangle = \frac{1}{g_2} \langle g_2 \tilde{\mathbf{x}}_{,i,j} - g_{i,j} \tilde{\mathbf{x}}_{,2}, U \rangle.$$

Thus, the Gaussian curvature K of M is defined by

$$K = -\epsilon \frac{L_{11} L_{22} - L_{12}^2}{W^2} \tag{2.3}$$

and the mean curvature H is given by

$$H = -\frac{\epsilon}{2W^2}(g_2^2 L_{11} - 2 g_1 g_2 L_{12} + g_1^2 L_{22}), \tag{2.4}$$

where $\epsilon (= \pm 1)$ is the sign of the unit normal vector field.

For the coefficients g^*_{ij} of the first fundamental form on M we denote by (g^{*ij}) the inverse matrix of the matrix (g^*_{ij}). In terms of a local coordinate system $\{x_i\}$, the Laplacian Δf of a smooth function f is given by

$$\Delta f = -\frac{1}{\sqrt{|\mathfrak{g}|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(\sqrt{|\mathfrak{g}|} g^{*ij} \frac{\partial f}{\partial x_j}\Big), \tag{2.5}$$

where $\mathfrak{g}$ denotes the determinant of the matrix (g^*_{ij}).

3. Surfaces of revolution in G^1_3

In the three dimensional pseudo-Galilean space G^1_3, there are two types of rotations: pseudo-Euclidean rotations given by the normal form

$$\begin{aligned} \bar{x} &= x, \\ \bar{y} &= y\cosh t + z\sinh t, \\ \bar{z} &= y\sinh t + z\cosh t \end{aligned} \tag{3.1}$$

and isotropic rotations with the normal form

$$\begin{aligned} \bar{x} &= x + bt, \\ \bar{y} &= y + xt + b\frac{t^2}{2}, \\ \bar{z} &= z, \end{aligned} \tag{3.2}$$

where $t \in \mathbb{R}$ and b is a positive constant.

First of all, we consider a non-isotropic curve α parameterized by

$$\alpha(u) = (f(u), g(u), 0) \quad \text{or} \quad \alpha(u) = (f(u), 0, g(u))$$

around the x-axis by pseudo-Euclidean rotation (3.1), where g is a positive function and f is a smooth function on an open interval I. Then the surface of revolution can be written as

$$\mathbf{x}(u, v) = (f(u), g(u)\cosh v, g(u)\sinh v) \tag{3.3}$$

or

$$\mathbf{x}(u, v) = (f(u), g(u)\sinh v, g(u)\cosh v) \tag{3.4}$$

for any $v \in \mathbb{R}$.

Next, we consider the isotropic rotations. By an isotropic curve $\alpha(u) = (0, f(u), g(u))$ about the z-axis by an isotropic rotation (3.2), we obtain a surface

$$\mathbf{x}(u, v) = \left(v, f(u) + \frac{v^2}{2b}, g(u)\right), \tag{3.5}$$

where f and g are smooth functions and $b \neq 0$ [11].

4. Surfaces of revolution generated by non-isotropic curve

Let M be a surface of revolution generated by non-isotropic curve $\alpha(u) = (u, g(u), 0)$ in G_3^1. Then M is parameterized by

$$\mathbf{x}(u,v) = (u, g(u)\cosh v, g(u)\sinh v), \tag{4.1}$$

where g is a positive function.

The coefficients of the first fundamental form on M are given by

$$g_{11}^* = 1,\ g_{12}^* = 0,\ g_{22}^* = -g(u)^2.$$

We see that M is a time-like surface. By a direct computation with the help of (2.5), the Laplacian Δ on M is given by [12]

$$\Delta = -\frac{g'(u)}{g(u)}\frac{\partial}{\partial u} - \frac{\partial^2}{\partial u^2} + \frac{1}{g(u)^2}\frac{\partial^2}{\partial v^2}. \tag{4.2}$$

Also, the Gauss map G of M becomes

$$G = (0, \cosh v, \sinh v). \tag{4.3}$$

From (4.2) and (4.3), the Laplacian ΔG of the Gauss map G can be expressed as

$$\Delta G = \frac{1}{g(u)^2}G. \tag{4.4}$$

Thus, we have the following theorems.

Theorem 4.1. *There is no surfaces of revolution generated by a non-isotropic curve in G_3^1 with harmonic Gauss map.*

Proof. Let M be a surface of revolution defined by (4.1) in G_3^1. If M has harmonic Gauss map, that is, M satisfies $\Delta G = 0$, then $g^{-2}(u)G = 0$. It is impossible because $g(u)$ is a positive function and G is the unit normal vector field of M. □

Theorem 4.2. *Let M be a surface of revolution generated by a non-isotropic curve in the three dimensional pseudo-Galilean space G_3^1. Then M has pointwise 1-type Gauss map of the first kind.*

Proof. Let M be a surface of revolution generated by a non-isotropic curve in G_3^1. Suppose that M has pointwise 1-type Gauss map. Combining (1.2) and (4.4), one gets $F(u) = g^{-2}(u)$ and $C = 0$. Thus the Gauss map G of M is of pointwise 1-type of the first kind. □

Theorem 4.3. *There is no surfaces of revolution generated by a non-isotropic curve in G_3^1 with pointwise 1-type Gauss map of the second kind.*

Proof. Let M be a surface of revolution defined by (4.1) in G_3^1. By Theorem 4.2, M has only pointwise 1-type Gauss map of the first kind. Thus, the theorem is proved. □

Remark. We consider a surface defined by

$$\mathbf{x}(u,v) = (u, (a^2u + b^2)\cosh v, (a^2u + b^2)\sinh v),$$

where $a, b \in \mathbb{R}$ and $u > -\frac{b^2}{a^2}$. The surface is a Lorentzian cone satisfying the equation $(a^2x+b^2)^2 = y^2 - z^2$. From (4.4) the Laplacian ΔG of the Gauss map G of the surface is obtained by $\Delta G = \frac{1}{(a^2u+b^2)^2}G$. Thus, a Lorentzian cone in G_3^1 has pointwise 1-type Gauss map of the first kind. On the other hand, a Lorentzian cone in the three dimensional Minkowski space $\mathbb{E}_1^3$ has pointwise 1-type Gauss map of the second kind (see [8]).

5. Surfaces of revolution generated by isotropic curve

In this section, we consider the isotropic rotations. By rotating an isotropic curve $\alpha(u) = (0, f(u), g(u))$ about the z-axis, we obtain a surface of revolution M parameterized by

$$\mathbf{x}(u,v) = (v, f(u) + \frac{v^2}{2b}, g(u)), \tag{5.1}$$

where b is a non-zero constant. We assume that the isotropic curve is parameterized by arc-length, that is,

$$f'(u)^2 - g'(u)^2 = -\epsilon (= \pm 1). \tag{5.2}$$

By using (5.2), the coefficients of the first fundamental form ds^2 on M are given by

$$g_{11}^* = 1,\ g_{12}^* = 0,\ g_{22}^* = -\epsilon. \tag{5.3}$$

On the other hand, the Gauss map G and the Laplacian Δ on M, respectively, are given by

$$G = (0, -g', -f')$$

and

$$\Delta = \epsilon\frac{\partial^2}{\partial u^2} + \epsilon\frac{\partial^2}{\partial v^2}.$$

Hence the Laplacian ΔG of the Gauss map G is obtained by [12]

$$\Delta G = (0, -\epsilon g''', -\epsilon f'''). \tag{5.4}$$

In terms of the harmonic Gauss map, we have:

Theorem 5.1. *Let M be a surface of revolution generated by an isotropic curve $\alpha(u) = (0, f(u), g(u))$ in G_3^1. Then M has a harmonic Gauss map if and only if the functions f and g are quadric.*

5.1. Surfaces of revolution with pointwise 1-type Gauss map of the first kind

Let M be a surface of revolution in G_3^1 satisfying $\Delta G = FG$ for a non-zero smooth function F. Then from (5.4) we have

$$\begin{aligned} \epsilon g''' &= Fg', \\ \epsilon f''' &= Ff'. \end{aligned} \tag{5.5}$$

If $f' = 0$, from (5.2) $g(u) = \pm u + a$ with $a \in \mathbb{R}$ and from the first equation of (5.5) $F = 0$, a contradiction. Similarly, the case of $g' = 0$ is also impossible.

Now we suppose that $f'g' \neq 0$. If $\epsilon = -1$, then we put

$$f'(u) = \cosh\theta(u), \quad g'(u) = \sinh\theta(u), \tag{5.6}$$

where θ is a smooth function.

By substituting (5.6) into (5.5) and calculating we obtain $\theta'' = 0$, that is, $\theta(u) = a_1 u + a_2$, $a_1, a_2 \in \mathbb{R}$. From this and (5.5) we can show that $F = -a_1^2$. Therefore, the Gauss map G of M is of 1-type. In the case of $\epsilon = 1$ we have the same result.

Theorem 5.2 (The Classification Theorem)**.** *Let M be a surface of revolution generated by an isotropic curve in the three dimensional Galilean space G_3^1. If M has pointwise 1-type Gauss map of the first kind, then the Gauss map of M is of usual 1-type.*
Furthermore, M is parameterized as

$$\mathbf{x}(u,v) = \left(v, \frac{1}{a_1}\sinh(a_1 u + a_2) + a_3 + \frac{v^2}{2b}, \frac{1}{a_1}\cosh(a_1 u + a_2) + a_3\right)$$

or

$$\mathbf{x}(u,v) = \left(v, \frac{1}{d_1}\cosh(d_1 u + d_2) + d_3 + \frac{v^2}{2b}, \frac{1}{d_1}\sinh(d_1 u + d_2) + d_3\right),$$

where $a_i, d_i \in \mathbb{R}$, $i = 1, 2, 3$.

5.2. Surfaces of revolution with pointwise 1-type Gauss map of the second kind

Suppose that M has pointwise 1-type Gauss map of the second kind, that is, M satisfies $\Delta G = F(G + C)$. Then we easily see that the first component c_1 of a constant vector $C = (c_1, c_2, c_3)$ is zero and we have a system of differential equations as follows:

$$\begin{aligned} -\epsilon g''' &= F(-g' + c_2), \\ -\epsilon f''' &= F(-f' + c_3). \end{aligned} \tag{5.7}$$

If $f' = 0$, then $\epsilon = 1$ and $g(u) = \pm u + k_1$, where k_1 is constant. Hence from (5.7) we have

$$F(\pm 1 + c_2) = 0 \quad \text{and} \quad Fc_3 = 0.$$

It implies that $c_2 = \pm 1$ and $c_3 = 0$, i.e., $C = (0, \pm 1, 0)$. It follows that $C = -G$. Thus, M is parameterized by

$$\mathbf{x}(u, v) = \left(v, \frac{v^2}{2b} + k_2, \pm u + k_1\right),$$

where k_1 and k_2 are constant. The surface is a time-like parabolic cylinder and it has pointwise 1-type Gauss map of the second kind.

Similarly as above, if $g' = 0$, then $f(u) = \pm u + k_1$ for some constant k_1 and so $\epsilon = -1$. Hence, $C = (0, 0, \pm 1)$ and moreover $C = -G$. Therefore M has pointwise 1-type Gauss map of the second kind and its parametrization is given by

$$\mathbf{x}(u, v) = \left(v, \pm u + \frac{v^2}{2b} + k_1, k_2\right)$$

for some constants k_1 and k_2. We see that it is a space-like plane.

Suppose that $f'g' \neq 0$. In this case, we first consider $\epsilon = -1$ and put

$$f'(u) = \cosh\theta(u) \quad \text{and} \quad g'(u) = \sinh\theta(u).$$

Then, (5.7) can be rewritten as

$$(\theta')^2 \sinh\theta + \theta'' \cosh\theta = F(-\sinh\theta + c_2),$$
$$(\theta')^2 \cosh\theta + \theta'' \sinh\theta = F(-\cosh\theta + c_3),$$

where θ is a smooth function. It follows that

$$-\theta'^2 = F(1 - c_2 \sinh\theta + c_3 \cosh\theta) \tag{5.8}$$

and

$$\theta'' = F(c_2 \cosh\theta - c_3 \sinh\theta). \tag{5.9}$$

If $\theta' = 0$ identically, then f' and g' are constants, say a_1 and b_1. It follows that we have $\Delta G = 0$ and $C = -G$. Thus M has pointwise 1-type Gauss map of the second kind and it is parameterized by

$$\mathbf{x}(u, v) = \left(v, a_1 u + a_2 + \frac{v^2}{2b}, b_1 u + b_2\right)$$

for some constants a_1, a_2, b_1 and b_2.

Next, suppose that $\theta' \neq 0$. From (5.8), F depends only on the parameter u, i.e., $F(u, v) = F(u)$. Differentiating equation (5.8) with respect to u, we get

$$-2\theta'\theta'' = F'(1 - c_2 \sinh\theta + c_3 \cosh\theta) + F(-c_2 \cosh\theta + c_3 \sinh\theta)\theta'.$$

By using (5.9), it implies that

$$-\theta'\theta'' = F'(1 - c_2 \sinh\theta + c_3 \cosh\theta). \tag{5.10}$$

Combining (5.8) and (5.10), we have the following equation

$$\frac{\theta''}{\theta'} = \frac{F'}{F},$$

which implies

$$\theta' = kF \tag{5.11}$$

for a non-zero constant k.

Applying the composition of trigonometric function in (5.8) and (5.9), we obtain the following differential equation

$$\left(\frac{\theta'^2}{F} + 1\right)^2 - \left(\frac{\theta''}{F}\right)^2 = -c_2^2 + c_3^2.$$

With the help of (5.11), it becomes

$$(k^2 F + 1)^2 - k^2 \left(\frac{F'}{F}\right)^2 = -c_2^2 + c_3^2. \tag{5.12}$$

Without loss of generality, we assume that $k = \pm 1$. In order to solve the above equation, we put

$$p = \ln F.$$

Then, (5.12) can be rewritten as the following equation:

$$(e^p + 1)^2 - \left(\frac{dp}{du}\right)^2 = -c_2^2 + c_3^2. \tag{5.13}$$

Let us distinguish three cases according to the constant vector C.

Case 1. $C = (0, c_2, c_3)$ is null, that is, $c_2^2 - c_3^2 = 0$.

In this case, we can easily obtain a general solution given by

$$F(u) = \frac{d_1 e^{\pm u}}{1 - d_1 e^{\pm u}},$$

where d_1 is non-zero constant. Therefore, from (5.11) we have

$$\theta(u) = \mp \ln |1 - d_1 e^{\pm u}| + d_2,$$

where d_2 is constant. Thus, M is parameterized as

$$\mathbf{x}(u,v) = \Bigg(v, \int \cosh\big(\ln|1 - d_1 e^{\pm u}| + d_2\big)\, du + \frac{v^2}{2b}, \\ - \int \sinh\big(\ln|1 - d_1 e^{\pm u}| + d_2\big)\, du\Bigg).$$

Case 2. C is time-like, that is, $c_2^2 - c_3^2 < 0$.

We assume that $c_2^2 - c_3^2 = -1$. Then (5.13) becomes

$$\left(\frac{dp}{du}\right)^2 = (e^p + 1)^2 - 1$$

and its general solution is given by

$$F(u) = e^p = \frac{2}{(u \pm d_1)^2 - 1},$$

where d_1 is constant. Thus a first integration of (5.11) implies

$$\theta(u) = \ln\left|\frac{u \pm d_1 - 1}{u \pm d_1 + 1}\right| + d_2,$$

where d_2 is constant. Thus, the parametrization of M is given by

$$\mathbf{x}(u,v) = \left(v, \int \cosh\left(\ln\left|\frac{u \pm d_1 - 1}{u \pm d_1 + 1}\right| + d_2\right) du + \frac{v^2}{2b},\right.$$
$$\left.\int \sinh\left(\ln\left|\frac{u \pm d_1 - 1}{u \pm d_1 + 1}\right| + d_2\right) du\right).$$

Case 3. C is space-like, that is, $c_2^2 - c_3^2 > 0$.

In the case, we assume that $c_2^2 - c_3^2 = 1$. (5.13) becomes

$$\frac{dp}{du} = \pm\sqrt{(e^p + 1)^2 + 1}. \tag{5.14}$$

In order to solve (5.14), we put

$$h(p) = \frac{e^p + 2}{\sqrt{2}\sqrt{(e^p + 1)^2 + 1}}.$$

Then, by a direct computation, we show that

$$\frac{1}{\sqrt{(e^p + 1)^2 + 1}} = -\frac{\dot{h}(p)}{\sqrt{2}(1 - h(p)^2)}, \tag{5.15}$$

where "$\cdot$" denotes the derivative with respect to p. Thus, a direct integration of (5.14) yields

$$-\frac{1}{\sqrt{2}} \tanh^{-1} h(p) = \pm u + d_1$$

or, equivalently

$$-\frac{1}{\sqrt{2}} \tanh^{-1} \frac{F + 2}{\sqrt{2(F^2 + 2F + 2)}} = \pm u + d_1.$$

The last equation can be written as the form:

$$\left(1 - \sinh^2(\sqrt{2}u + d_1)\right) F^2 + 4F + 4 = 0, \tag{5.16}$$

where d_1 is constant.

This implies

$$F(u) = -\frac{2}{1 \pm \sinh(\sqrt{2}u + d_1)}.$$

From here, we have two values for F. First, by taking the sign + and using (5.11) with $k = \pm 1$ we get

$$\theta(u) = \pm 2 \tanh^{-1}\left(\frac{\sqrt{2}}{e^{\sqrt{2}u + d_1} + 1}\right) + d_2. \tag{5.17}$$

Finally, by taking the sign $-$ we get

$$\theta(u) = \pm 2\tanh^{-1}\left(\frac{\sqrt{2}e^{\sqrt{2}u+d_1}}{e^{\sqrt{2}u+d_1}+1}\right) + d_2, \tag{5.18}$$

where d_2 is constant. In this case, the parametrization of M is given by

$$\mathbf{x}(u,v) = \left(v, \int \cosh\theta(u)du + \frac{v^2}{2b}, \int \sinh\theta(u)du\right),$$

where $\theta(u)$ is given by (5.17) or (5.18). Consequently, we have

Theorem 5.3 (The Classification Theorem)**.** *Let M be a surface of revolution generated by isotropic curve in the three dimensional Galilean space G_3^1. If M has pointwise 1-type Gauss map of the second kind, then M is one of the following surfaces*:

(1) $\mathbf{x}(u,v) = \left(v, \frac{v^2}{2b} + d_2, \pm u + d_1\right)$.

(2) $\mathbf{x}(u,v) = \left(v, \pm u + \frac{v^2}{2b} + d_1, d_2\right)$.

(3) $\mathbf{x}(u,v) = \left(v, d_1u + d_2 + \frac{v^2}{2b}, d_3u + d_4\right)$.

(4) $\mathbf{x}(u,v) = \Big(v, \int \cosh(\ln|1 - d_1e^{\pm u}| + d_2)du + \frac{v^2}{2b},$
$-\int \sinh(\ln|1 - d_1e^{\pm u}| + d_2)du\Big)$.

(5) $\mathbf{x}(u,v) = \left(v, \int \cosh(\ln\left|\frac{u\pm d_1-1}{u\pm d_1+1}\right| + d_2)du + \frac{v^2}{2b}, \int \sinh(\ln\left|\frac{u\pm d_1-1}{u\pm d_1+1}\right| + d_2)du\right)$.

(6) $\mathbf{x}(u,v) = \left(v, \int \cosh\theta(u)du + \frac{v^2}{2b}, \int \sinh\theta(u)du\right)$, *where $\theta(u)$ is given by* (5.17) *or* (5.18).

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