PROPERTIES OF REGULAR FUNCTIONS WITH VALUES IN BICOMPLEX NUMBERS

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Abstract. In this paper, using forms of conjugations, we give some algebraic properties of bicomplex numbers. We research differential operators, elementary functions and the analogous Cauchy-Riemann system in bicomplex number systems. Also, we investigate the definition and properties of regular functions with values in bicomplex settings in Clifford analysis.

1. Introduction

The set $\mathbb{BC}$ of bicomplex numbers is defined as follows:

$$\mathbb{BC} = \{z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\},$$

where j is an imaginary unit such that

$$ij = ji, \ i^2 = j^2 = -1,$$

and $\mathbb{C}$ is the set of complex numbers with an original imaginary unit $i = \sqrt{-1}$ in the complex number system. Here i and j are hypercomplex numbers. Thus bicomplex numbers are complex numbers with complex coefficients. Many mathematicians have tried to emphasize the similarities between the properties of complex and bicomplex numbers. In 1892, bicomplex numbers were first introduced by Segre [16], and then the origin of a function theory for the bicomplex number system was studied in the Italian school of Dragoni et al. [3, 18, 19]. Also, the first theory of differentiability in $\mathbb{BC}$ was developed by Price in [12]. Price extended previous works to bicomplex number system in the hyperfunction theory, built on the space of the differentiable functions defined by one or several bicomplex variables to the multicomplex settings.

Subsequently, other authors have developed further the study of bicomplex analysis. Charak et al. [2] introduced the extended bicomplex plane and the concept of the normality of a family of bicomplex meromorphic functions on

bicomplex domains and they obtained the bicomplex analog of the Montel theorem for meromorphic functions and families of bicomplex holomorphic functions. Rochon et al. [13] presented a variety of algebraic properties of both bicomplex numbers and hyperbolic numbers. They were interested to numbers over a commutative and associative to the skew field of quaternions such that real four dimensional space, and they generalized complex numbers and quaternions. Commutativity is a powerful property in the algebra of Clifford analysis, for example, elementary calculations, multiplication of bicomplex numbers. Ryan [14, 15] was the first to understand the importance of complex Clifford algebras which $\mathbb{BC}$ has the commutative role in analysis and the ring of bicomplex numbers is not a field, since zero divisors exist in that ring. Also, they expected that bicomplex numbers can serve as scalars both in the theory of functions and in functional analysis, comparing with quaternions.

Recently, there has been the study of the properties of functions on the ring $\mathbb{BC}$ of bicomplex numbers. Elizarràs et al. [1, 11] introduced elementary functions, such as polynomials, exponentials and trigonometric functions in this algebra, as well as their inverses, something that is not possible in the case of quaternions. They showed that any pair of holomorphic functions admits derivative in the sense of bicomplex numbers. The analysis of bicomplex holomorphic functions were developed by a general theory of functional analysis with bicomplex scalars and to compare with the quaternionic scalars which were studied by Teichmüller [20] and Soukhomlinoff [17]. The initial ideas of bicomplex functional analysis have been presented by Lavoie et al. [4, 5]. They studied finite-dimensional modules defined on bicomplex numbers and proved many results, using bicomplex square matrices, linear operators, orthogonal bases and self-adjoint operators, based the spectral decomposition theorem.

We [6, 7] obtained some results for the regularity of functions on the ternary quaternion and reduced quaternion field in Clifford analysis, and for the regularity of functions on dual split quaternions in Clifford analysis. Also, we [8, 9] researched corresponding Cauchy-Riemann systems and properties of a regularity of functions with values in special quaternions such reduced quaternions, split quaternions and dual quaternions in Clifford analysis. We [10] investigated the relations between a corresponding Cauchy-Riemann system and a regularity of functions with values in hypercomplex numbers of coset algebras.

In this paper, we introduce basic definitions and notations of bicomplex number system and give some different forms of bicomplex conjugations, generalizing the usual complex conjugation which lead to different moduli of bicomplex numbers. We research properties of differential operators and the corresponding Cauchy-Riemann systems.

2. Notation and preliminaries

Consider a complex number $x + iy$, where x and y are real numbers. We replace x and y by $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, respectively, then $z_1 + z_2 i =$

$(x_1 - y_2) + (y_1 + x_2)i$. To obtain a new form of bicomplex numbers, we use another imaginary unit j such that

$$j^2 = -1, \ ij = ji,$$

which is outside the field of complex numbers. By the properties of j and the composition of bicomplex numbers, the set

$$\mathbb{BC} = \{Z = z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\}$$

of bicomplex numbers is isomorphic with $\mathbb{C}^2$ and it has some forms as elements:
1. The cartesian form: $Z = z_1 + z_2 j$.
2. The trigonometric form:

$$Z = z_1 + z_2 j = \sqrt{z_1^2 + z_2^2} \left(\frac{z_1}{\sqrt{z_1^2 + z_2^2}} + \frac{z_2}{\sqrt{z_1^2 + z_2^2}} j \right)$$

$$= \sqrt{z_1^2 + z_2^2} \ (\cos\theta + \sin\theta \ j),$$

where $\sqrt{z_1^2 + z_2^2}$ is called the complex modulus of $\mathbb{BC}$ and θ is the complex number which satisfies the following equations (see [11]):

$$\cos(\theta) = \frac{z_1}{\sqrt{z_1^2 + z_2^2}} \quad \text{and} \quad \sin(\theta) = \frac{z_2}{\sqrt{z_1^2 + z_2^2}}.$$

Now, we give the addition and multiplication for $Z_1 = z_{11} + z_{12} j$ and $Z_2 = z_{21} + z_{22} j$ in $\mathbb{BC}$,

$$Z_1 + Z_2 = (z_{11} + z_{21}) + (z_{12} + z_{22})j,$$
$$Z_1 Z_2 = (z_{11} z_{21} - z_{12} z_{22}) + (z_{12} z_{21} + z_{11} z_{22})j.$$

For $Z = z_1 + z_2 j$, its bicomplex conjugation is defined by $Z^\dagger := z_1 - z_2 j$. Then

$$ZZ^\dagger = Z^\dagger Z = z_1^2 + z_2^2 \in \mathbb{C}$$

is the quadratic form for bicomplex numbers. A bicomplex number $Z = z_1 + z_2 j$ is invertible if and only if $ZZ^\dagger = z_1^2 + z_2^2 \neq 0$. The inverse of Z is given by

$$Z^{-1} = \frac{Z^\dagger}{ZZ^\dagger} = \frac{Z^\dagger}{z_1^2 + z_2^2}.$$

If $z_1^2 + z_2^2 = 0$ $(z_1, z_2 \neq 0)$, then a corresponding bicomplex number $Z = z_1 + z_2 j$ is a zero divisor. Since all zero divisors $Z = z_1 + j z_2 \in \mathbb{BC}$ are characterized by $z_1^2 + z_2^2 = 0$, i.e., $z_2 = \pm i z_1$, these are of the form:

$$Z = z_1 + z_2 j = z_1 \pm z_1 ij = z_1(1 \pm ij) = (1 \pm ij)\lambda, \ \lambda \in \mathbb{C} \setminus \{0\}.$$

Consider

$$\mathbf{e} := \frac{1 + ij}{2}, \ \mathbf{e}^\dagger := \frac{1 - ij}{2},$$

satisfying

$$\mathbf{e} + \mathbf{e}^\dagger = 1, \ \mathbf{e} - \mathbf{e}^\dagger = ij, \ \mathbf{e}\mathbf{e}^\dagger = 0, \ \mathbf{e}^2 = \mathbf{e}, \ (\mathbf{e}^\dagger)^2 = \mathbf{e}^\dagger.$$

3. The idempotent representation of Z:

$$Z = \alpha \mathbf{e} + \beta \mathbf{e}^\dagger,$$

where $\alpha = z_1 - iz_2$, $\beta = z_1 + iz_2 \in \mathbb{C}$. This show that the set $\{\mathbf{e}, \mathbf{e}^\dagger\}$ is another basis for the bicomplex space $\mathbb{BC}$, and writing Z as a pair (z_1, z_2) in $\mathbb{C}^2$, one has the following formula:

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{\alpha + \beta}{2} \\ \frac{-\alpha + \beta}{2i} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2i} & -\frac{1}{2i} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

This new basis is orthogonal with respect to the Euclidean inner product in $\mathbb{C}^2$,

$$< (z_1, z_2), (w_1, w_2) >_{\mathbb{C}^2} := z_1 \overline{w_1} + z_2 \overline{w_2},$$

where $\overline{w_r} = \lambda_r - \mu_r i$ with $\lambda_r, \mu_r \in \mathbb{R}$ $(r = 1, 2)$. Since $\mathbf{e} = \left(\frac{1}{2}, \frac{i}{2}\right)$ and $\mathbf{e}^\dagger = \left(\frac{1}{2}, -\frac{i}{2}\right)$, we have

$$< \mathbf{e}, \mathbf{e}^\dagger >= 0, \ < \mathbf{e}, \mathbf{e} >=< \mathbf{e}^\dagger, \mathbf{e}^\dagger >= \frac{1}{2}.$$

We give the addition and multiplication of bicomplex numbers for the idempotent representation, for $Z_1 = \alpha \mathbf{e} + \beta \mathbf{e}^\dagger$, $Z_2 = \gamma \mathbf{e} + \delta \mathbf{e}^\dagger$,

$$Z_1 + Z_2 = (\alpha + \gamma)\mathbf{e} + (\beta + \delta)\mathbf{e}^\dagger, \ Z_1 Z_2 = \alpha\gamma \mathbf{e} + \beta\delta \mathbf{e}^\dagger,$$

$$Z_1^n = \alpha^n \mathbf{e} + \beta^n \mathbf{e}^\dagger, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}.$$

The inverse of an invertible bicomplex number $Z = z_1 + z_2 j = \alpha \mathbf{e} + \beta \mathbf{e}^\dagger$ is given by

$$Z^{-1} = \alpha^{-1} \mathbf{e} + \beta^{-1} \mathbf{e}^\dagger,$$

where

$$\alpha^{-1} = \frac{\beta}{z_1^2 + z_2^2}, \ \beta^{-1} = \frac{\alpha}{z_1^2 + z_2^2}.$$

The bicomplex conjugation of a bicomplex number is given by $Z^\dagger = \beta \mathbf{e} + \alpha \mathbf{e}^\dagger$.

Now, using the above results, we obtain some elementary functions in bicomplex numbers. Let

$$P(Z) = \sum_{k=0}^{n} A_k Z^k = A_0 + A_1 Z + A_2 Z^2 + \cdots + A_n Z^n$$

be a bicomplex polynomial of degree n with

$$Z = \alpha \mathbf{e} + \beta \mathbf{e}^\dagger, \ A_k = \gamma_k \mathbf{e} + \delta_k \mathbf{e}^\dagger, \ Z^k = \alpha^k \mathbf{e} + \beta^k \mathbf{e}^\dagger.$$

Then

$$P(Z) = P(\alpha \mathbf{e} + \beta \mathbf{e}^\dagger) = \sum_{k=0}^{n} A_k Z^k = \sum_{k=0}^{n} (\gamma_k \alpha^k)\mathbf{e} + \sum_{k=0}^{n} (\delta_k \beta^k)\mathbf{e}^\dagger$$

$$:= \xi(\alpha)\mathbf{e} + \eta(\beta)\mathbf{e}^\dagger,$$

where ξ and η have only each variable α and β, respectively.

Let $Z = z_1 + jz_2$ be any bicomplex number. Then $Z_n := \left(1 + \frac{Z}{n}\right)^n$ is convergent. Because we have

$$
\left(1 + \frac{Z}{n}\right)^n = \left(1 + \frac{\alpha\mathbf{e} + \beta\mathbf{e}^\dagger}{n}\right)^n = \left(\mathbf{e} + \mathbf{e}^\dagger + \frac{\alpha}{n}\mathbf{e} + \frac{\beta}{n}\mathbf{e}^\dagger\right)^n
$$
$$
= \left\{\left(1 + \frac{\alpha}{n}\right)\mathbf{e} + \left(1 + \frac{\beta}{n}\right)\mathbf{e}^\dagger\right\}^n = \left(1 + \frac{\alpha}{n}\right)^n\mathbf{e} + \left(1 + \frac{\beta}{n}\right)^n\mathbf{e}^\dagger.
$$

By taking the limit as $n \to \infty$ and relying on the corresponding sequences $\left(1 + \frac{\alpha}{n}\right)^n$ and $\left(1 + \frac{\beta}{n}\right)^n$ which are convergent to $\exp(\alpha)$ and $\exp(\beta)$, respectively, we have

$$
\lim_{n\to\infty}\left(1 + \frac{Z}{n}\right)^n = \lim_{n\to\infty}\left(1 + \frac{\alpha}{n}\right)^n\mathbf{e} + \lim_{n\to\infty}\left(1 + \frac{\beta}{n}\right)^n\mathbf{e}^\dagger
$$
$$
= \exp(\alpha)\mathbf{e} + \exp(\beta)\mathbf{e}^\dagger = \exp(\alpha)\frac{1 + ij}{2} + \exp(\beta)\frac{1 - ij}{2}
$$
$$
= \frac{1}{2}(\exp(z_1 - iz_2) + \exp(z_1 + iz_2))
$$
$$
+ j\frac{i}{2}(\exp(z_1 - iz_2) - \exp(z_1 + iz_2))
$$
$$
= \exp(z_1)(\cos(z_2) + j\sin(z_2)),
$$

where

$$
\cos(z_2) = \frac{1}{2}(\exp(iz_2) + \exp(-iz_2)) \text{ and } \sin(z_2) = \frac{1}{2i}(\exp(iz_2) - \exp(-iz_2)).
$$

We set

$$
\exp(Z) = \exp(\alpha)\mathbf{e} + \exp(\beta)\mathbf{e}^\dagger = \exp(z_1)(\cos(z_2) + j\sin(z_2)),
$$

and then,

$$
\cos(Z) = \frac{\exp(jZ) + \exp(-jZ)}{2} = \cos(\alpha)\mathbf{e} + \cos(\beta)\mathbf{e}^\dagger
$$
$$
= \cos(z_1)\cosh(z_2) - j\sin(z_1)\sinh(z_2),
$$
$$
\sin(Z) = \frac{\exp(jZ) - \exp(-jZ)}{2j} = \sin(\alpha)\mathbf{e} + \sin(\beta)\mathbf{e}^\dagger
$$
$$
= \sin(z_1)\cosh(z_2) + j\cos(z_1)\sinh(z_2).
$$

3. bc-regular function with values in the bicomplex number system

Consider a bicomplex function $f = f_1 + jf_2 : U \subset \mathbb{BC} \to \mathbb{BC}$ such that

$$
f(Z) = f(z_1 + jz_2) = f_1(z_1, z_2) + jf_2(z_1, z_2),
$$

where f_1 and f_2 are complex-valued functions with its idempotent representation

$$
f(Z) = f(z_1 + jz_2) = \phi(z_1, z_2)\mathbf{e} + \psi(z_1, z_2)\mathbf{e}^\dagger,
$$

with $\phi(z_1, z_2) = f_1(z_1, z_2) - if_2(z_1, z_2)$ and $\psi(z_1, z_2) = f_1(z_1, z_2) + if_2(z_1, z_2)$. Also, we have

$$f(Z) = f(\alpha\mathbf{e} + \beta\mathbf{e}^\dagger) = A(\alpha, \beta)\mathbf{e} + B(\alpha, \beta)\mathbf{e}^\dagger,$$

where $A(\alpha, \beta) = g_1(\alpha, \beta) - ig_2(\alpha, \beta)$ and $B(\alpha, \beta) = g_1(\alpha, \beta) + ig_2(\alpha, \beta)$ with g_1 and g_2 are complex-valued functions of complex variables α and β.

Definition. Let $f : U \subset \mathbb{BC} \rightarrow \mathbb{BC}$ be a bicomplex function such that $f(Z) = f_1(z_1, z_2) + f(z_1, z_2)j$. If the following limit

$$\begin{aligned}
\frac{df(Z_0)}{dZ} &:= \lim_{Z \to Z_0} \frac{f(Z) - f(Z_0)}{Z - Z_0} \\
&= \lim_{Z \to Z_0} \frac{(f(Z) - f(Z_0))(Z - Z_0)^\dagger}{(Z - Z_0)(Z - Z_0)^\dagger} \\
&= \lim_{\substack{z_1 \to z_1^0 \\ z_2 \to z_2^0}} \frac{(f(Z) - f(Z_0))(z_1 - z_1^0) - (f(Z) - f(Z_0))(z_2 - z_2^0)j}{(z_1 - z_1^0)^2 + (z_2 - z_2^0)^2}
\end{aligned}$$

exists, where

$$f(Z) - f(Z_0) = (f_1(z_1, z_2) - f_1(z_1^0, z_2^0)) + (f_2(z_1, z_2) - f_2(z_1^0, z_2^0))j,$$

then f is said to be differentiable in bicomplex numbers at a point $Z_0 = z_1^0 + z_2^0 j$ for Z in the domain of f such that $Z - Z_0$ is an invertible $\mathbb{BC}$ and $\frac{df(Z_0)}{dZ}$ is said to the derivative of f at the point Z_0.

Moreover, the function f is called holomorphic at Z_0 if f is differentiable for all points in a neighborhood of the point Z_0.

Definition. Let Ω be an open set in $\mathbb{BC}$. If a function f is holomorphic at every point of Ω, then f is a holomorphic function in Ω.

If the above limit exists, then we have

$$\lim_{Z \to Z_0} \frac{f(Z) - f(Z_0)}{Z - Z_0} = \frac{\partial f_1(z_1^0, z_2^0)}{\partial z_1} + \frac{\partial f_2(z_1^0, z_2^0)}{\partial z_1}j$$

and

$$\lim_{Z \to Z_0} \frac{f(Z) - f(Z_0)}{Z - Z_0} = -\frac{\partial f_1(z_1^0, z_2^0)}{\partial z_2}j + \frac{\partial f_2(z_1^0, z_2^0)}{\partial z_2}.$$

Let U be an open in $\mathbb{BC}$, for $Z = z_1 + jz_2$, and let $f : U \rightarrow \mathbb{BC}$ be such that $f = f_1 + jf_2 \in C^1(U)$. Then f admits bicomplex derivative $\frac{df}{dZ}$ if and only if f satisfies

$$(3.1) \qquad\qquad \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \ \frac{\partial f_1}{\partial z_2} = -\frac{\partial f_2}{\partial z_1}$$

on U, called the Cauchy-Riemann system in bicomplex numbers. We give differential operators in $\mathbb{BC}$:

$$\frac{\partial}{\partial Z} := \frac{1}{2}\Big(\frac{\partial}{\partial z_1} - j\frac{\partial}{\partial z_2}\Big), \ \frac{\partial}{\partial Z^\dagger} = \frac{1}{2}\Big(\frac{\partial}{\partial z_1} + j\frac{\partial}{\partial z_2}\Big),$$

where $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$ are usual complex differential operators in the complex analysis. Then we have the bicomplex Laplacian operator as follows:

$$\Delta^\dagger(Z) := \frac{1}{2}\Big(\frac{\partial}{\partial Z^\dagger}\frac{\partial}{\partial Z} + \frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger}\Big)$$

$$= \frac{1}{4}\Big(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} - j\frac{\partial^2}{\partial z_2 \partial z_1} + j\frac{\partial^2}{\partial z_1 \partial z_2}\Big) = \frac{1}{4}\Big(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}\Big),$$

which is operated for a C^2 function and we put $\Delta_Z^\dagger := 4\Delta^\dagger(Z)$.

Also, we have

$$\frac{\partial}{\partial Z} := D_\alpha \mathbf{e} + D_\beta \mathbf{e}^\dagger, \ \ \frac{\partial}{\partial Z^\dagger} = D_\beta \mathbf{e} + D_\alpha \mathbf{e}^\dagger,$$

where

$$D_\alpha = \frac{1}{2}\Big(\frac{\partial}{\partial z_1} + i\frac{\partial}{\partial z_2}\Big), \ D_\beta = \frac{1}{2}\Big(\frac{\partial}{\partial z_1} - i\frac{\partial}{\partial z_2}\Big).$$

For example,

$$D_\alpha \alpha = 1, \ D_\alpha \beta = 0, \ D_\beta \beta = 1, \ D_\beta \alpha = 0,$$

$$\frac{\partial}{\partial Z^\dagger}P(Z) = 0, \ \frac{\partial}{\partial Z^\dagger}\exp Z = 0, \ \frac{\partial}{\partial Z^\dagger}\cos Z = 0, \ \frac{\partial}{\partial Z^\dagger}\sin Z = 0.$$

Remark 3.1. Let Ω be a bounded open set in $\mathbb{BC}$. A function f is operated as follows cases:

If $f(Z) = f(z_1 + jz_2) = f_1(z_1, z_2) + jf_2(z_1, z_2)$, then

$$\frac{\partial f}{\partial Z} = \frac{1}{2}\Big(\frac{\partial}{\partial z_1} - j\frac{\partial}{\partial z_2}\Big)(f_1 + jf_2) = \frac{1}{2}\Big(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\Big) + j\frac{1}{2}\Big(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\Big),$$

$$\frac{\partial f}{\partial Z^\dagger} = \frac{1}{2}\Big(\frac{\partial}{\partial z_1} + j\frac{\partial}{\partial z_2}\Big)(f_1 + jf_2) = \frac{1}{2}\Big(\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2}\Big) + j\frac{1}{2}\Big(\frac{\partial f_1}{\partial z_2} + \frac{\partial f_2}{\partial z_1}\Big).$$

If $f(Z) = f(z_1 + jz_2) = \phi(z_1, z_2)\mathbf{e} + \psi(z_1, z_2)\mathbf{e}^\dagger$, then

$$\frac{\partial f}{\partial Z} = (D_\alpha \mathbf{e} + D_\beta \mathbf{e}^\dagger)(\phi(z_1, z_2)\mathbf{e} + \psi(z_1, z_2)\mathbf{e}^\dagger) = (D_\alpha\phi)\mathbf{e} + (D_\beta\psi)\mathbf{e}^\dagger,$$

$$\frac{\partial f}{\partial Z^\dagger} = (D_\beta \mathbf{e} + D_\alpha \mathbf{e}^\dagger)(\phi(z_1, z_2)\mathbf{e} + \psi(z_1, z_2)\mathbf{e}^\dagger) = (D_\beta\phi)\mathbf{e} + (D_\alpha\psi)\mathbf{e}^\dagger.$$

If $f(Z) = f(\alpha \mathbf{e} + \beta \mathbf{e}^\dagger) = A(\alpha, \beta)\mathbf{e} + B(\alpha, \beta)\mathbf{e}^\dagger$, then

$$\frac{\partial f}{\partial Z} = (D_\alpha \mathbf{e} + D_\beta \mathbf{e}^\dagger)(A(\alpha, \beta)\mathbf{e} + B(\alpha, \beta)\mathbf{e}^\dagger) = (D_\alpha A)\mathbf{e} + (D_\beta B)\mathbf{e}^\dagger,$$

$$\frac{\partial f}{\partial Z^\dagger} = (D_\beta \mathbf{e} + D_\alpha \mathbf{e}^\dagger)(A(\alpha, \beta)\mathbf{e} + B(\alpha, \beta)\mathbf{e}^\dagger) = (D_\beta A)\mathbf{e} + (D_\alpha B)\mathbf{e}^\dagger.$$

Definition. Let Ω be a bounded open set in $\mathbb{BC}$. A function $f = f_1 + jf_2$ is said to be bc-regular in Ω if f_1 and f_2 of f are continuously differential complex valued functions in Ω such that $\frac{\partial f}{\partial Z^\dagger} = 0$.

Remark 3.2. Let Ω be a bounded open set in $\mathbb{BC}$. The equation $\frac{\partial f}{\partial Z^\dagger} = 0$ is equivalent to the following equations:

$$
\begin{aligned}
\frac{\partial f}{\partial Z^\dagger} = 0 \quad &\Leftrightarrow \quad \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \ \frac{\partial f_1}{\partial z_2} = -\frac{\partial f_2}{\partial z_1} \\
&\Leftrightarrow \quad D_\beta \phi(z_1, z_2) = 0, \ D_\alpha \psi(z_1, z_2) = 0 \\
&\Leftrightarrow \quad D_\beta A(\alpha, \beta) = 0, \ D_\alpha B(\alpha, \beta) = 0.
\end{aligned}
$$

Definition. Let Ω be a bounded open set in $\mathbb{BC}$. A C^2 function $f = f_1 + j f_2$ is said to be bc-harmonic in Ω if all its components f_1 and f_2 of f are bc-harmonic in Ω, that is, f_1 and f_2 are C^2 functions on Ω such that

$$
\Delta_Z^\dagger f_1 = \frac{\partial^2 f_1}{\partial z_1^2} + \frac{\partial^2 f_1}{\partial z_2^2} = 0
$$

and

$$
\Delta_Z^\dagger f_2 = \frac{\partial^2 f_2}{\partial z_1^2} + \frac{\partial^2 f_2}{\partial z_2^2} = 0.
$$

Proposition 3.3. *Let Ω be a bounded open set in $\mathbb{BC}$. If a function $f = f_1 + j f_2$ is bc-regular in Ω, then f is bc-harmonic in Ω.*

Proof. From the definition of $\Delta_Z^\dagger$ and a bc-regular function in Ω, we can obtain the result. $\square$

Theorem 3.4. *Let Ω be a bounded open set in $\mathbb{BC}$. If a function f is bc-regular in Ω, then for $Z_0 = \alpha_0 \mathbf{e} + \beta_0 \mathbf{e}^\dagger$, f has*

$$
\begin{aligned}
f(Z) &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f(Z_0)}{\partial Z^n} (Z - Z_0)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (D_\alpha^n A(\alpha_0, \beta_0))(\alpha - \alpha_0)^n \mathbf{e} + \sum_{n=0}^{\infty} \frac{1}{n!} (D_\beta^n B(\alpha_0, \beta_0))(\beta - \beta_0)^n \mathbf{e}^\dagger.
\end{aligned}
$$

Proof. Since we have

$$
\begin{aligned}
\frac{\partial^n f(Z_0)}{\partial Z^n} &= (D_\alpha^n \mathbf{e} + D_\beta^n \mathbf{e}^\dagger)(A(\alpha_0, \beta_0)\mathbf{e} + B(\alpha_0, \beta_0)\mathbf{e}^\dagger) \\
&= (D_\alpha^n A(\alpha_0, \beta_0))\mathbf{e} + (D_\beta^n B(\alpha_0, \beta_0))\mathbf{e}^\dagger,
\end{aligned}
$$

the result is obtained:

$$
\begin{aligned}
f(Z) = &\sum_{n=0}^{\infty} \frac{1}{n!} (D_\alpha^n A(\alpha_0, \beta_0))(\alpha - \alpha_0)^n \mathbf{e} \\
&+ \sum_{n=0}^{\infty} \frac{1}{n!} (D_\beta^n B(\alpha_0, \beta_0))(\beta - \beta_0)^n \mathbf{e}^\dagger.
\end{aligned}
$$
$\square$

Example 3.5. Let Ω be a bounded open set in $\mathbb{BC}$. A function $f(Z) = \exp(Z) = \exp(\alpha)\mathbf{e} + \exp(\beta)\mathbf{e}^\dagger$ is bc-regular in Ω, then for $Z_0 = 0$, f has

$$f(Z) = \sum_{n=0}^{\infty} \frac{1}{n!}(D_\alpha^n \exp(\alpha))|_{\alpha=0} Z^n \mathbf{e} + \sum_{n=0}^{\infty} \frac{1}{n!}(D_\beta^n \exp(\beta))|_{\beta=0} Z^n \mathbf{e}^\dagger$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\alpha^n \mathbf{e} + \sum_{n=0}^{\infty} \frac{1}{n!}\beta^n \mathbf{e}^\dagger.$$

Theorem 3.6. *Let Ω be a bounded open set in $\mathbb{BC}$ and U be any domain in Ω with smooth boundary bU such that $\overline{U} \subset \Omega$. If $f = f_1 + f_2 j$ is a bc-regular function in Ω and $\omega := dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} + j dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$, then*

$$\int_{bU} \omega f = 0,$$

where ωf is the bicomplex product of the form ω on the function $f(Z) = f_1(z_1, z_2) + f_2(z_1, z_2)j$.

Proof. Since the function $f = f_1 + f_2 j$ is bc-regular in $\mathbb{BC}$, we have

$$\omega f = f_1 dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} - f_2 dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$$
$$+ (f_2 dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} + f_1 dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2})j.$$

Then we have

$$d(\omega f) = j\frac{\partial f_1}{\partial z_1} dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} - \frac{\partial f_2}{\partial z_1} dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$$

$$- j\frac{\partial f_1}{\partial z_1} dz_2 \wedge dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} + j\frac{\partial f_2}{\partial z_2} dz_2 \wedge dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2}$$

$$= \Big(-\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\Big) dV + \Big(\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2}\Big) j \ dV,$$

where $dV = dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$ in U, and by (3.1) for $f(z)$ in $\mathbb{BC}$, $d(\omega f) = 0$. By Stokes' theorem, we obtain the result. $\square$

Consider the following form of a conjugation of bicomplex numbers:

$$Z^\ddagger = z_2 + z_1 j,$$

called the anti-conjugation of bicomplex numbers. Then we have

$$Z - Z^\ddagger j = 2z_1, \ Z^\ddagger - Zj = 2z_2, \ ZZ^\ddagger = (z_1^2 + z_2^2)j.$$

By the definition of the differential operator $\frac{\partial}{\partial Z}$, we use

$$\frac{\partial}{\partial Z^\ddagger} = \frac{1}{2}\Big(\frac{\partial}{\partial z_2} - j\frac{\partial}{\partial z_1}\Big)$$

which satisfies the analogous Laplacian operator which is operated for a C^2 function such that

$$(3.2) \qquad \Delta_Z^\ddagger := \frac{1}{2}\Big(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\ddagger} + \frac{\partial}{\partial Z^\ddagger}\frac{\partial}{\partial Z}\Big) = -\frac{1}{4}j\Big(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}\Big).$$

For the function f, defined on Ω in $\mathbb{BC}$, since the equation $\Delta_Z^{\ddagger} f = 0$ is equivalent to the equation $\Delta_Z^{\dagger} f = 0$, the function f is bc-harmonic in Ω. We have the following proposition.

Proposition 3.7. *Let Ω be a bounded open set in $\mathbb{BC}$. If a function $f \in \mathcal{C}^2(\Omega)$ satisfies the equation*

$$\Delta_Z^{\ddagger} f = 0,$$

where $\Delta_Z^{\ddagger}$ is the analogous Laplacian operator, then f is a bc-harmonic function in Ω.

Theorem 3.8. *Let Ω be a bounded open set in $\mathbb{BC}$. If the function f has the derivative in Ω, then the following statements are equivalent:*

(i) *f satisfies the Cauchy-Riemann equations in bicomplex numbers, that is, f satisfies*

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_1}{\partial z_1}, \ \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}.$$

(ii) *f is a bc-regular function in Ω, that is, f satisfies the equation*

$$\frac{\partial f}{\partial Z^{\dagger}} = 0.$$

(iii) *f satisfies the equation*

$$\frac{\partial f}{\partial Z^{\ddagger}} = 0.$$

Proof. We already see that the statement (i) is equivalent to (ii) from Remarks 3.1 and 3.2. It is sufficient to prove that (i) is equivalent to (iii). Suppose that (i), then

$$\begin{aligned}
\frac{\partial f}{\partial Z^{\ddagger}} &= \frac{1}{2}\Big(\frac{\partial}{\partial z_2} - j\frac{\partial}{\partial z_1}\Big)(f_1 + f_2 j) \\
&= \frac{1}{2}\Big(\frac{\partial f_1}{\partial z_2} - j\frac{\partial f_1}{\partial z_1}\Big) + \frac{1}{2}\Big(\frac{\partial f_2}{\partial z_2}j + \frac{\partial f_2}{\partial z_1}\Big) \\
&= \frac{1}{2}\Big(\frac{\partial f_1}{\partial z_2} + \frac{\partial f_2}{\partial z_1}\Big) + \frac{1}{2}\Big(\frac{\partial f_2}{\partial z_2} - \frac{\partial f_1}{\partial z_1}\Big)j = 0.
\end{aligned}$$

Conversely, suppose that f satisfies the equation

$$\begin{aligned}
0 = \frac{\partial f}{\partial Z^{\ddagger}} &= \frac{1}{2}\Big(\frac{\partial f_1}{\partial z_2} - j\frac{\partial f_1}{\partial z_1}\Big) + \frac{1}{2}\Big(\frac{\partial f_2}{\partial z_2}j + \frac{\partial f_2}{\partial z_1}\Big) \\
&= \frac{1}{2}\Big(\frac{\partial f_1}{\partial z_2} + \frac{\partial f_2}{\partial z_1}\Big) + \frac{1}{2}\Big(\frac{\partial f_2}{\partial z_2} - \frac{\partial f_1}{\partial z_1}\Big)j.
\end{aligned}$$

Then we have

$$\frac{\partial f_1}{\partial z_2} = -\frac{\partial f_2}{\partial z_1}, \ \frac{\partial f_2}{\partial z_2} = \frac{\partial f_1}{\partial z_1}.$$

Therefore, we can obtain the equations in (i). $\qquad\qquad\square$

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