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A NOTE ON BILATERAL SEMIDIRECT PRODUCT DECOMPOSITIONS OF SOME MONOIDS OF ORDER-PRESERVING PARTIAL PERMUTATIONS

Vítor H. Fernandes* and Teresa M. Quinteiro†

Abstract. In this note we consider the monoid $\mathcal{PODI}_n$ of all monotone partial permutations on $\{1,\ldots,n\}$ and its submonoids $\mathcal{DP}_n$, $\mathcal{POI}_n$ and $\mathcal{ODP}_n$ of all partial isometries, of all order-preserving partial permutations and of all order-preserving partial isometries, respectively. We prove that both the monoids $\mathcal{POI}_n$ and $\mathcal{ODP}_n$ are quotients of bilateral semidirect products of two of their remarkable submonoids, namely of extensive and of co-extensive transformations. Moreover, we show that $\mathcal{PODI}_n$ is a quotient of a semidirect product of $\mathcal{POI}_n$ and the group $\mathcal{C}_2$ of order two and, analogously, $\mathcal{DP}_n$ is a quotient of a semidirect product of $\mathcal{ODP}_n$ and $\mathcal{C}_2$.

Introduction and preliminaries

Strongly motivated by automata theoretic ideas, in [25] Kunze studied the notion of bilateral semidirect product of two semigroups (see [26, 27] for applications in Automata Theory) and proved in [28] that the full transformation semigroup on a finite set X is a quotient of a bilateral semidirect product of the symmetric group on X and the semigroup of all order-preserving full transformations on X, for some linear order on X. Also in [28], Kunze showed that the semigroup of all order-preserving full transformations on a finite chain is a quotient of a bilateral semidirect product of two of its subsemigroups. These results as well as applications to Formal Languages were also discussed by Kunze in [29]. Bilateral semidirect products were also considered by Lavers [32] who

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†This work was developed within the FCT Project PEst-OE/MAT/UI0143/2014 of CAUL, FCUL, and of Instituto Superior de Engenharia de Lisboa.

gave conditions under which a bilateral semidirect product of two finitely presented monoids is itself finitely presented, by exhibiting explicit presentations, under some conditions.

In this note we construct bilateral semidirect decompositions, i.e., a representation of a monoid S as a quotient of a bilateral semidirect product of two proper submonoids of S, of certain monoids of partial permutations.

Denote by $\mathcal{T}(X)$ the semigroup (under composition) of all full transformations of a set X. Let S and T be two semigroups. Let

$$\begin{array}{rcccccc} \delta: & T & \longrightarrow & \mathcal{T}(S) & & & \\ & u & \longmapsto & \delta_u: & S & \longrightarrow & S \\ & & & & s & \longmapsto & u.s \end{array}$$

be an anti-homomorphism of semigroups (i.e., $(uv).s = u.(v.s)$ for $s \in S$ and $u, v \in T$) and let

$$\begin{array}{rcccccc} \varphi: & S & \longrightarrow & \mathcal{T}(T) & & & \\ & s & \longmapsto & \varphi_s: & T & \longrightarrow & T \\ & & & & u & \longmapsto & u^s \end{array}$$

be a homomorphism of semigroups (i.e., $u^{sr} = (u^s)^r$ for $s, r \in S$ and $u \in T$) such that:

(SPR) $(uv)^s = u^{v.s}v^s$ for $s \in S$ and $u, v \in T$ (*Sequential Processing Rule*); and

(SCR) $u.(sr) = (u.s)(u^s.r)$ for $s, r \in S$ and $u \in T$ (*Serial Composition Rule*).

Within these conditions, we say that δ is a *left action* of T on S and that φ is a *right action* of S on T.

In [25], Kunze proved that the set $S \times T$ is a semigroup with respect to the following multiplication:

$$(s, u)(r, v) = (s(u.r), u^r v)$$

for $s, r \in S$ and $u, v \in T$. We denote this semigroup by $S_\delta \bowtie_\varphi T$ (or, if it is not ambiguous, simply by $S \bowtie T$) and call it the *bilateral semidirect product* of S and T associated with δ and φ.

If S and T are monoids and the actions δ and φ preserve the identity (i.e., $1.s = s$ for $s \in S$, and $u^1 = u$ for $u \in T$) and are *monoidal* (i.e., $u.1 = 1$ for $u \in T$, and $1^s = 1$ for $s \in S$), then $S \bowtie T$ is a monoid with identity $(1, 1)$.

Here, we will just consider bilateral semidirect products of monoids associated to monoidal actions.

Notice that, if the right action φ is a trivial action (i.e., $(S)\varphi = \{\mathrm{id}_T\}$), then $S \bowtie T = S \rtimes T$ is an usual semidirect product, if the left action δ is a trivial action (i.e., $(T)\delta = \{\mathrm{id}_S\}$), then $S \bowtie T$ coincides with a reverse semidirect product $S \ltimes T$ and if both actions are trivial, then $S \bowtie T$ is the usual direct product $S \times T$. Observe also that the bilateral semidirect product is quite different from the *double* semidirect product by Rhodes and Tilson [35], wherein the second components multiply always as in the direct product.

A partial transformation s on the chain $X_n = \{1 < 2 < \cdots < n\}$, $n \in \mathbb{N}$, is said to be *order-preserving* (respectively, *order-reversing*) if $i \leq j$ implies $is \leq js$ (respectively, $is \geq js$) for all $i, j \in \mathrm{Dom}(s)$. Order-preserving and order-reversing partial transformations are also called *monotone*.

Semigroups of order-preserving transformations have been considered in the literature since the 1960s. In 1962, Aĭzenštat [1] and Popova [34] exhibited presentations for $\mathcal{O}_n$, the monoid of all order-preserving full transformations on X_n, and for $\mathcal{PO}_n$, the monoid of all order-preserving partial transformations on X_n, respectively. In 1971, Howie [23] studied some combinatorial and algebraic properties of $\mathcal{O}_n$ and, in 1992, together with Gomes [20] revisited the monoids $\mathcal{O}_n$ and $\mathcal{PO}_n$. More combinatorial properties of these two monoids were presented by Laradji and Umar in [30, 31]. Certain classes of divisors of the monoid $\mathcal{O}_n$ were determined in 1995 by Higgins [21] and by Vernitskiĭ and Volkov [36], in 1997 by Fernandes [9] and in 2010 by Fernandes and Volkov [18]. In [28] Kunze proved that the monoid $\mathcal{O}_n$ is a quotient of a bilateral semidirect product of its subsemigroups $\mathcal{O}_n^- = \{s \in \mathcal{O}_n \mid is \leq i \text{ for } i \in X_n\}$ and $\mathcal{O}_n^+ = \{s \in \mathcal{O}_n \mid i \leq is \text{ for } i \in X_n\}$. See also [15, 16, 29].

The injective counterpart of $\mathcal{O}_n$, i.e., the monoid $\mathcal{POI}_n$ of all injective members of $\mathcal{PO}_n$, has been object of study by Fernandes in several papers [9, 10, 11, 12, 13], by Derech in [8], by Cowan and Reilly in [5], by Ganyushkin and Mazorchuk in [19], among other authors. Presentations for the monoid $\mathcal{POI}_n$ and for its extension $\mathcal{PODI}_n$, the monoid of all monotone partial permutations on X_n, were given by Fernandes [11] in 2001 and by Fernandes et al. [14] in 2004, respectively. The abelian kernels of $\mathcal{POI}_n$ and $\mathcal{PODI}_n$ were computed by Delgado and Fernandes [6, 7].

Next, let s be a partial permutation on X_n. We say that s is an *isometry* if $|is - js| = |i - j|$ for all $i, j \in \mathrm{Dom}(s)$.

The study of semigroups of finite partial isometries was initiated by Al-Kharousi et al. in [2, 3]. The first of these two papers was dedicated to investigate some combinatorial properties of the monoid $\mathcal{DP}_n$ of all partial isometries on X_n and of its submonoid $\mathcal{ODP}_n$ of all order-preserving partial isometries, in particular, their cardinalities. The second one presented the study of some of their algebraic properties, namely Green's structure and ranks. On the other hand, in [17] the authors exhibited presentations for both monoids $\mathcal{DP}_n$ and $\mathcal{ODP}_n$. Observe that $\mathcal{ODP}_n$, $\mathcal{POI}_n$, $\mathcal{DP}_n$ and $\mathcal{PODI}_n$ are all inverse submonoids of the symmetric inverse monoid (i.e., the monoid of all partial permutations) $\mathcal{I}_n$ on X_n (see [3, 14]). Obviously, $\mathcal{POI}_n \subseteq \mathcal{PODI}_n$ and $\mathcal{ODP}_n = \mathcal{DP}_n \cap \mathcal{POI}_n$ and, as observed by Al-Kharousi et al. [3], we also have $\mathcal{DP}_n \subseteq \mathcal{PODI}_n$. Moreover, it is easy to check that $\mathcal{ODP}_n = \{s \in \mathcal{I}_n \mid is - js = i - j \text{ for } i, j \in \mathrm{Dom}(s)\}$.

In this paper, in Section 1, we obtain a bilateral semidirect decomposition of $\mathcal{POI}_n$ in terms of its submonoids $\mathcal{POI}_n^+ = \{s \in \mathcal{POI}_n \mid i \leq is \text{ for } i \in \mathrm{Dom}(s)\}$ and $\mathcal{POI}_n^- = \{s \in \mathcal{POI}_n \mid is \leq i \text{ for } i \in \mathrm{Dom}(s)\}$ of extensive and of co-extensive transformations, respectively. A similar decomposition is constructed for the monoid $\mathcal{ODP}_n$ by considering its submonoids $\mathcal{ODP}_n^+ = \mathcal{ODP}_n \cap \mathcal{POI}_n^+$ and $\mathcal{ODP}_n^- = \mathcal{ODP}_n \cap \mathcal{POI}_n^-$. On the other hand, in Section 2, we prove that $\mathcal{PODI}_n$ and $\mathcal{DP}_n$ are quotients of semidirect products of the form $\mathcal{POI}_n \rtimes \mathcal{C}_2$ and $\mathcal{ODP}_n \rtimes \mathcal{C}_2$, respectively, where $\mathcal{C}_2$ denotes the group of order two. In both sections we extract consequences for pseudovarieties generated by some of these families of partial permutations monoids.

Recall that a *pseudovariety* of monoids is a class of finite monoids closed under formation of finite direct products, submonoids and homomorphic images. The *semidirect product* $\mathsf{V} \rtimes \mathsf{W}$ of the pseudovarieties of monoids V and W is the pseudovariety generated by all monoidal semidirect products $M \rtimes N$, where $M \in \mathsf{V}$ and $N \in \mathsf{W}$. Similarly, we define the *reverse semidirect product* $\mathsf{V} \ltimes \mathsf{W}$ and the *bilateral semidirect product* $\mathsf{V} \bowtie \mathsf{W}$ of the pseudovarieties of monoids V and W.

Let O and J be the pseudovarieties of monoids generated by $\{\mathcal{O}_n \mid n \in \mathbb{N}\}$ and by $\{\mathcal{O}_n^+ \mid n \in \mathbb{N}\}$ (or, since $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$ are isomorphic monoids, by $\{\mathcal{O}_n^- \mid n \in \mathbb{N}\}$), respectively. It is well-known that J is the pseudovariety of $\mathcal{J}$-trivial monoids and that it also is generated by the syntactic monoids of piecewise testable languages (see e.g. [33]). Let A be the pseudovariety of all aperiodic (i.e., $\mathcal{H}$-trivial) monoids. It is easy to show that $\mathsf{J} \bowtie \mathsf{J} \subseteq \mathsf{A}$ and, as an immediate consequence of Kunze's result [28] above mentioned, we have $\mathsf{O} \subseteq \mathsf{J} \bowtie \mathsf{J}$ (see [15]). On the other hand, let Ecom be the pseudovariety of all idempotent commuting monoids (recall that a celebrated Theorem of Ash [4] states that Ecom is generated by all finite inverse monoids) and let POI and PODI be the pseudovarieties generated by $\{\mathcal{POI}_n \mid n \in \mathbb{N}\}$ and by $\{\mathcal{PODI}_n \mid n \in \mathbb{N}\}$, respectively. Notice that $\mathsf{POI} \subset \mathsf{O} \subset \mathsf{A}$ [9] and that $\mathsf{J} \cap \mathsf{Ecom}$ is the pseudovariety generated by $\{\mathcal{POI}_n^+ \mid n \in \mathbb{N}\}$ (or, since $\mathcal{POI}_n^+$ and $\mathcal{POI}_n^-$ are isomorphic monoids, by $\{\mathcal{POI}_n^- \mid n \in \mathbb{N}\}$) [22]. Finally, consider the pseudovariety of monoids Ab_2 generated by $\mathcal{C}_2$ (a pseudovariety of Abelian groups).

For basic notions on Semigroup Theory, we refer the reader to Howie's book [24].

In order to avoid trivialities, from now on we consider $n \geq 3$.

1. On the monoids $\mathcal{POI}_n$ and $\mathcal{ODP}_n$

In this section we show that $\mathcal{POI}_n$ and $\mathcal{ODP}_n$ are homomorphic images of certain bilateral semidirect products of the form $\mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+$ and $\mathcal{ODP}_n^- \bowtie \mathcal{ODP}_n^+$, respectively.

We begin by constructing a bilateral semidirect product $\mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+$.

Let $s, u \in \mathcal{POI}_n \setminus \{1\}$. Define the elements $u.s, u^s \in \mathcal{POI}_n$ by

$$\mathrm{Dom}(u.s) = \mathrm{Dom}(us) \quad \text{and} \quad \mathrm{Im}(u.s) = \{1, \ldots, |\,\mathrm{Dom}(us)|\},$$

$$\mathrm{Dom}(u^s) = \{1, \ldots, |\,\mathrm{Im}(us)|\} \quad \text{and} \quad \mathrm{Im}(u^s) = \mathrm{Im}(us).$$

Observe that any element of $\mathcal{POI}_n$ is well defined by its domain and image.

Notice that

$$\mathrm{Im}(u.s) = \mathrm{Dom}(u^s)$$

and, clearly,

$$u.s \in \mathcal{POI}_n^- \quad \text{and} \quad u^s \in \mathcal{POI}_n^+.$$

Define also

$$1.s = s, \quad u^1 = u, \quad u.1 = 1, \quad 1^s = 1, \quad 1.1 = 1 \quad \text{and} \quad 1^1 = 1.$$

Consider the following two (well defined) functions:

$$\begin{array}{rlcllll} \delta: & \mathcal{POI}_n^+ & \longrightarrow & \mathcal{T}(\mathcal{POI}_n^-) & & \\ & u & \longmapsto & \delta_u : \mathcal{POI}_n^- & \longrightarrow & \mathcal{POI}_n^- \\ & & & \quad s & \longmapsto & u.s \end{array}$$

and

$$\begin{array}{rlcllll} \varphi: & \mathcal{POI}_n^- & \longrightarrow & \mathcal{T}(\mathcal{POI}_n^+) & & \\ & s & \longmapsto & \varphi_s : \mathcal{POI}_n^+ & \longrightarrow & \mathcal{POI}_n^+ \\ & & & \quad u & \longmapsto & u^s. \end{array}$$

We have:

Lemma 1.1. *The above defined functions δ and φ are an anti-homomorphism of monoids and a homomorphism of monoids, respectively.*

Proof. First, notice that δ_1 and φ_1 are, clearly, the identity maps of $\mathcal{POI}_n^-$ and of $\mathcal{POI}_n^+$, respectively.

Now, let $u, v \in \mathcal{POI}_n^+$ and $s, r \in \mathcal{POI}_n^-$. Then, we must prove that

$$(uv).s = u.(v.s) \quad \text{and} \quad u^{sr} = (u^s)^r.$$

It is immediate that, if any of the elements u, v or s is the identity, then $(uv).s = u.(v.s)$, and if any of the elements u, s or r is the identity, then $u^{sr} = (u^s)^r$. So, let us suppose that none of the elements u, v, s and r is the identity.

In order to prove that $(uv).s = u.(v.s)$, it suffices to show that $\mathrm{Dom}((uv).s) = \mathrm{Dom}(u.(v.s))$. In fact, we have

$$\begin{aligned} \mathrm{Dom}((uv).s) &= \mathrm{Dom}((uv)s) \\ &= \mathrm{Dom}(u(vs)) \\ &= (\mathrm{Im}(u) \cap \mathrm{Dom}(vs))u^{-1} \\ &= (\mathrm{Im}(u) \cap \mathrm{Dom}(v.s))u^{-1} \\ &= \mathrm{Dom}(u(v.s)) \\ &= \mathrm{Dom}(u.(v.s)). \end{aligned}$$

On the other hand, in order to prove that $u^{sr} = (u^s)^r$, it suffices to show that $\mathrm{Im}(u^{sr}) = \mathrm{Im}((u^s)^r)$:

$$\begin{aligned} \mathrm{Im}(u^{sr}) &= \mathrm{Im}(u(sr)) = \mathrm{Im}((us)r) = (\mathrm{Im}(us) \cap \mathrm{Dom}\, r)r \\ &= (\mathrm{Im}(u^s) \cap \mathrm{Dom}\, r)r = \mathrm{Im}(u^s r) = \mathrm{Im}((u^s)^r), \end{aligned}$$

as required. □

Before proving that δ and φ also verify sequential processing and serial composition rules, we observe that it is easy to check the equality

$$(u.s)u^s = us \tag{1}$$

for all $u \in \mathcal{POI}_n^+$ and $s \in \mathcal{POI}_n^-$.

Lemma 1.2. *Let $s, r \in \mathcal{POI}_n^-$ and $u, v \in \mathcal{POI}_n^+$. Then*
(SPR) $(uv)^s = u^{v.s}v^s$;
(SCR) $u.(sr) = (u.s)(u^s.r)$.

Proof. (SPR) We begin by noticing that if any of the elements s, u or v is the identity, then the equality $(uv)^s = u^{v.s}v^s$ is obvious. Thus, admit that none of the elements s, u or v is the identity. Since $\operatorname{Im}(u(v.s)) \subseteq \operatorname{Im}(v.s)$ and taking in account Lemma 1.1, we obtain

$$\begin{aligned}
\operatorname{Dom}(u^{v.s}v^s) &= (\operatorname{Im}(u^{v.s}) \cap \operatorname{Dom}(v^s))(u^{v.s})^{-1} \\
&= (\operatorname{Im}(u(v.s)) \cap \operatorname{Im}(v.s))(u^{v.s})^{-1} \\
&= (\operatorname{Im}(u(v.s)))(u^{v.s})^{-1} \\
&= (\operatorname{Im}(u^{v.s}))(u^{v.s})^{-1} \\
&= \operatorname{Dom}(u^{v.s}) \\
&= \operatorname{Im}(u.(v.s)) \\
&= \operatorname{Im}((uv).s) \\
&= \operatorname{Dom}((uv)^s).
\end{aligned}$$

Then, in particular, $|\operatorname{Im}((uv)^s)| = |\operatorname{Im}(u^{v.s}v^s)|$. Hence, in order to prove that $(uv)^s = u^{v.s}v^s$, it suffices to show, for instance, the inclusion $\operatorname{Im}(u^{v.s}v^s) \subseteq \operatorname{Im}((uv)^s)$.

Let $y \in \operatorname{Im}(u^{v.s}v^s)$. Then there exists $x \in \operatorname{Dom}(u^{v.s}v^s)$ such that $y = x(u^{v.s}v^s)$. It follows that $xu^{v.s} \in \operatorname{Im}(u^{v.s}) = \operatorname{Im}(u(v.s))$ and so $xu^{v.s} = a(u(v.s))$, for some $a \in \operatorname{Dom}(u(v.s))$. Thus, by using (1), we have

$$\begin{aligned}
y &= (xu^{v.s})v^s = (a(u(v.s)))v^s = a(u((v.s)v^s)) \\
&= a(u(vs)) = a((uv)s) \in \operatorname{Im}((uv)s) = \operatorname{Im}((uv)^s),
\end{aligned}$$

which proves the required inclusion.

(SCR) As for (SPR), if any of the elements s, r or u is the identity, then the equality $u.(sr) = (u.s)(u^s.r)$ is trivial. Therefore, let us assume that none of these elements is the identity. In view of the inclusion $\operatorname{Dom}(u^s r) \subseteq \operatorname{Dom}(u^s)$ and of Lemma 1.1, we have

$$\begin{aligned}
\operatorname{Im}((u.s)(u^s.r)) &= (\operatorname{Im}(u.s) \cap \operatorname{Dom}(u^s.r))(u^s.r) \\
&= (\operatorname{Dom}(u^s) \cap \operatorname{Dom}(u^s r))(u^s.r) \\
&= (\operatorname{Dom}(u^s r))(u^s.r) \\
&= (\operatorname{Dom}(u^s.r))(u^s.r) \\
&= \operatorname{Im}(u^s.r) \\
&= \operatorname{Dom}((u^s)^r) \\
&= \operatorname{Dom}(u^{sr}) \\
&= \operatorname{Im}(u.(sr)).
\end{aligned}$$

It follows, in particular, that $|\operatorname{Dom}((u.s)(u^s.r))| = |\operatorname{Dom}(u.(sr))|$ and so it remains to show, for instance, that $\operatorname{Dom}((u.s)(u^s.r)) \subseteq \operatorname{Dom}(u.(sr))$. Let $x \in \operatorname{Dom}((u.s)(u^s.r))$. Then $x(u.s) \in \operatorname{Dom}(u^s.r) = \operatorname{Dom}(u^s r)$, whence $x(u.s)u^s \in \operatorname{Dom}(r)$ and so, by (1), $x(us) = x(u.s)u^s \in \operatorname{Dom}(r)$. Thus $x \in \operatorname{Dom}((us)r) = \operatorname{Dom}(u(sr)) = \operatorname{Dom}(u.(sr))$, as required. □

Now, by Lemma 1.1 and Lemma 1.2, we can consider the bilateral semidirect product $\mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+$ associated with δ and φ. Since $\mathcal{POI}_n^-$ and $\mathcal{POI}_n^+$ are monoids and the actions δ and φ preserve the identity and are monoidal, then $\mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+$ is also a monoid. Moreover, as we already observed, $\mathcal{POI}_n^-$ and $\mathcal{POI}_n^+$ are (isomorphic) $\mathcal{J}$-trivial monoids and any bilateral semidirect product of $\mathcal{J}$-trivial monoids is an aperiodic semigroup, whence $\mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+$ is an aperiodic monoid. On the other hand, $\mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+$ is not regular and is not an idempotent commuting semigroup. For instance, if $e = \binom{1}{1}$ and $f = \binom{12}{12}$, it is routine matter to show that $(e, \emptyset)$ is not regular, $(1, e)$ and (f, f) are idempotents and $(1, e)(f, f) = (e, e) \neq (f, e) = (f, f)(1, e)$.

Next, consider the following function

$$\begin{array}{rccc} \mu: & \mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+ & \longrightarrow & \mathcal{POI}_n \\ & (s, u) & \longmapsto & su\ . \end{array}$$

Let $(s, u), (r, v) \in \mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+$. As $(u.r)u^r = ur$, we have

$$((s, u)(r, v))\mu = (s(u.r), u^r v)\mu = s(u.r)u^r v = surv = (s, u)\mu(r, v)\mu,$$

and so μ is a homomorphism. In addition, given $t \in \mathcal{POI}_n$, we may define elements $s \in \mathcal{POI}_n^-$ and $u \in \mathcal{POI}_n^+$ by

$$\operatorname{Dom}(s) = \operatorname{Dom}(t),\ \operatorname{Im}(s) = \{1, \ldots, |\operatorname{Dom}(t)|\} = \operatorname{Dom}(u) \ \text{ and } \ \operatorname{Im}(u) = \operatorname{Im}(t),$$

and we obtain $t = su = (s, u)\mu$. Hence μ is an onto homomorphism and we have:

Theorem 1.3. *The monoid $\mathcal{POI}_n$ is a homomorphic image of $\mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+$.*

As an immediate consequence of this result and the above observed fact that $\mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+ \notin \mathsf{Ecom}$, we have the following property:

Corollary 1.4. $\mathsf{POI} \subsetneq (\mathsf{J} \cap \mathsf{Ecom}) \bowtie (\mathsf{J} \cap \mathsf{Ecom}) \not\subseteq \mathsf{Ecom}$.

Next, we construct a bilateral semidirect product $\mathcal{ODP}_n^- \bowtie \mathcal{ODP}_n^+$, just by slightly modifying the definition of the previous actions. Although with different meanings, we will use the same notations in this new context.

Let $s, u \in \mathcal{ODP}_n \setminus \{1\}$ and suppose that $\operatorname{Dom}(us) = \{i_1, \ldots, i_k\}$ for some $1 \leq i_1 < \cdots < i_k \leq n$ and $0 \leq k < n$. Define the elements $u.s, u^s \in \mathcal{POI}_n$ by

$$\operatorname{Dom}(u.s) = \operatorname{Dom}(us) \quad \text{and} \quad \operatorname{Im}(u.s) = \{1, 1 + i_2 - i_1, \ldots, 1 + i_k - i_1\},$$

$$\operatorname{Dom}(u^s) = \{1, 1 + i_2 us - i_1 us, \ldots, 1 + i_k us - i_1 us\} \quad \text{and} \quad \operatorname{Im}(u^s) = \operatorname{Im}(us)$$

(considering $u.s = u^s = \emptyset$ if $us = \emptyset$).

Notice that, clearly,

$$u.s \in \mathcal{ODP}_n^- \quad \text{and} \quad u^s \in \mathcal{ODP}_n^+.$$

Moreover

$$\mathrm{Im}(u.s) = \mathrm{Dom}(u^s).$$

Define also

$$1.s = s, \quad u^1 = u, \quad u.1 = 1, \quad 1^s = 1, \quad 1.1 = 1 \quad \text{and} \quad 1^1 = 1.$$

As for the first studied case, it is easy to check the equality

$$(u.s)u^s = us \tag{2}$$

for all $u \in \mathcal{ODP}_n^+$ and $s \in \mathcal{ODP}_n^-$, and we may consider the following two functions:

$$\begin{array}{rccccc} \delta: & \mathcal{ODP}_n^+ & \longrightarrow & \mathcal{T}(\mathcal{ODP}_n^-) & & \\ & u & \longmapsto & \delta_u : \mathcal{ODP}_n^- & \longrightarrow & \mathcal{ODP}_n^- \\ & & & s & \longmapsto & u.s \end{array}$$

and

$$\begin{array}{rccccc} \varphi: & \mathcal{ODP}_n^- & \longrightarrow & \mathcal{T}(\mathcal{ODP}_n^+) & & \\ & s & \longmapsto & \varphi_s : \mathcal{ODP}_n^+ & \longrightarrow & \mathcal{ODP}_n^+ \\ & & & u & \longmapsto & u^s. \end{array}$$

By an exact replication of the proofs of Lemma 1.1 and Lemma 1.2, we prove the following lemma:

Lemma 1.5. *The functions δ and φ are a monoidal left action of $\mathcal{ODP}_n^+$ on $\mathcal{ODP}_n^-$ and a monoidal right action of $\mathcal{ODP}_n^-$ on $\mathcal{ODP}_n^+$, respectively.*

This lemma allows us to consider the bilateral semidirect product $\mathcal{ODP}_n^- \bowtie \mathcal{ODP}_n^+$ associated with δ and φ, which is, likewise $\mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+$, a non regular and non idempotent commuting aperiodic monoid. We may also consider the function

$$\begin{array}{rccc} \mu: & \mathcal{ODP}_n^- \bowtie \mathcal{ODP}_n^+ & \longrightarrow & \mathcal{ODP}_n \\ & (s,u) & \longmapsto & su\ , \end{array}$$

which is, by (2), clearly a homomorphism. Moreover, let $t \in \mathcal{ODP}_n$ be such that $\mathrm{Dom}(t) = \{i_1, \ldots, i_k\}$ for some $1 \le i_1 < \cdots < i_k \le n$ and $0 \le k \le n$, and define elements $s \in \mathcal{ODP}_n^-$ and $u \in \mathcal{ODP}_n^+$ by

$$\mathrm{Dom}(s) = \mathrm{Dom}(t),$$

$$\mathrm{Im}(s) = \{1, 1+i_2-i_1, \ldots, 1+i_k-i_1\} = \{1, 1+i_2t-i_1t, \ldots, 1+i_kt-i_1t\} = \mathrm{Dom}(u)$$

and

$$\mathrm{Im}(u) = \mathrm{Im}(t).$$

Then $t = su = (s,u)\mu$ and so μ is an onto homomorphism.

Hence, we have the following result, with which we finish this section.

Theorem 1.6. *The monoid $\mathcal{ODP}_n$ is a homomorphic image of $\mathcal{ODP}_n^- \bowtie \mathcal{ODP}_n^+$.*

2. On the monoids $\mathcal{PODI}_n$ and $\mathcal{DP}_n$

Let
$$h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$
Then $h \in \mathcal{DP}_n$ (and so $h \in \mathcal{PODI}_n$). Moreover, the identity (on X_n) and h are the only permutations of $\mathcal{PODI}_n$ (and so of $\mathcal{DP}_n$). On the other hand, given $\alpha \in \mathcal{PODI}_n$, it is clear that α is an order-reversing transformation if and only if $h\alpha$ (and αh) is an order-preserving transformation (see [14]). Hence, as $\alpha = h^2\alpha = h(h\alpha)$, it follows that the monoids $\mathcal{PODI}_n$ and $\mathcal{DP}_n$ are generated by $\mathcal{POI}_n \cup \{h\}$ and $\mathcal{ODP}_n \cup \{h\}$, respectively. Furthermore, we may see the cyclic group of order two $\mathcal{C}_2 = \{1, h\}$ as a submonoid of both the monoids $\mathcal{PODI}_n$ and $\mathcal{DP}_n$. Notice that, given $x, y \in \mathcal{C}_2$, we have $xy = yx$ and $x^2 = y^2 = 1$.

First, we turn our attention to the monoid $\mathcal{PODI}_n$. We obtain a semidirect decomposition of it in terms of its submonoids $\mathcal{POI}_n$ and $\mathcal{C}_2$.

For each $x \in \mathcal{C}_2$ and $s \in \mathcal{POI}_n$, define the element $x.s = xsx \in \mathcal{POI}_n$. Then, consider the function
$$\begin{array}{rcclcl} \delta: & \mathcal{C}_2 & \longrightarrow & \mathcal{T}(\mathcal{POI}_n) & & \\ & x & \longmapsto & \delta_x : \mathcal{POI}_n & \longrightarrow & \mathcal{POI}_n \\ & & & \quad s & \longmapsto & x.s\ . \end{array}$$
Since $(xy).s = xysxy = xysyx = x.(ysy) = x.(y.s)$ and $1.s = s$ for $x, y \in \mathcal{C}_2$ and $s \in \mathcal{POI}_n$, then δ is an anti-homomorphism of monoids. On the other hand, for $x \in \mathcal{C}_2$ and $s, r \in \mathcal{POI}_n$, we have $x.(sr) = xsrx = xs1rx = xsxxrx = (x.s)(x.r)$ and $x.1 = x1x = x^2 = 1$. Thus δ induces a semidirect product $\mathcal{POI}_n \rtimes \mathcal{C}_2$.

It is easy to prove that $\mathcal{POI}_n \rtimes \mathcal{C}_2$ is an inverse monoid. In fact, it is a routine matter to check that the idempotents of $\mathcal{POI}_n \rtimes \mathcal{C}_2$ commute (the idempotents of $\mathcal{POI}_n \rtimes \mathcal{C}_2$ are of the form $(e, 1)$, with e an idempotent of $\mathcal{POI}_n$) and, given $(s, x) \in \mathcal{POI}_n \rtimes \mathcal{C}_2$, the element $(xs^{-1}x, x)$ of $\mathcal{POI}_n \rtimes \mathcal{C}_2$ is an (and so *the*) inverse of (s, x). Moreover, we have:

Theorem 2.1. *The monoid $\mathcal{PODI}_n$ is a homomorphic image of $\mathcal{POI}_n \rtimes \mathcal{C}_2$.*

Proof. Consider the function
$$\begin{array}{rccc} \mu: & \mathcal{POI}_n \rtimes \mathcal{C}_2 & \longrightarrow & \mathcal{PODI}_n \\ & (s, x) & \longmapsto & sx\ . \end{array}$$
Then, for $s, r \in \mathcal{POI}_n$ and $x, y \in \mathcal{C}_2$, we have
$$((s, x)(r, y))\mu = (s(x.r), xy)\mu = (sxrx, xy)\mu = sxrx^2y = sxry = (s, x)\mu(r, y)\mu.$$
Thus μ is a homomorphism. On the other hand, let $t \in \mathcal{PODI}_n$. If $t \in \mathcal{POI}_n$, then $t = t1 = (t, 1)\mu$, otherwise $th \in \mathcal{POI}_n$ and $t = (th)h = (th, h)\mu$. Hence μ is surjective. $\square$

Observe that, clearly, μ also separates idempotents, i.e., the restriction of μ to the set of the idempotents of $\mathcal{POI}_n \rtimes \mathcal{C}_2$ is an injective function.

The next result follows immediately from Theorem 2.1.

Corollary 2.2. $\mathsf{PODI} \subseteq \mathsf{POI} \rtimes \mathsf{Ab}_2$.

On the other hand, we also have:

Lemma 2.3. $\mathcal{POI}_n \rtimes \mathcal{C}_2 \in \mathsf{PODI}$.

Proof. It is easy to show that the function

$$\begin{array}{ccc} \mathcal{POI}_n \rtimes \mathcal{C}_2 & \longrightarrow & \mathcal{PODI}_n \times \mathcal{C}_2 \\ (s,x) & \longmapsto & (sx,x) \end{array}$$

is an injective homomorphism. □

Supported by this result, we formulate the following conjecture:

Conjecture 2.4. $\mathsf{PODI} = \mathsf{POI} \rtimes \mathsf{Ab}_2$.

Notice that, since $\mathcal{C}_2$ is a commutative monoid, the left action of $\mathcal{C}_2$ on $\mathcal{POI}_n$ may also be considered as a right action. Furthermore, similar results to Theorem 2.1 and Corollary 2.2 (and Lemma 2.3) also hold for reverse semidirect products.

We finish this section by establishing the analogous result to Theorem 2.1 for the monoid $\mathcal{DP}_n$. This aim will be accomplished by noticing that $\mathcal{DP}_n$ is a submonoid of $\mathcal{PODI}_n$ that fits in the general framework described below.

Let S be a monoid and let S_1 and S_2 be two submonoids of S. Let δ be a left action of S_2 on S_1 such that the function

$$\begin{array}{cccc} \mu : & S_1 \rtimes S_2 & \longrightarrow & S \\ & (s,u) & \mapsto & su \end{array}$$

is a homomorphism. Let T be a submonoid of S, T_1 a submonoid of S_1 and T_2 a submonoid of S_2. It is a routine matter to check that, if $(s)(u)\delta \in T_1$ for all $s \in T_1$ and $u \in T_2$, then δ induces a (restriction) left action of T_2 on T_1 and the corresponding semidirect product $T_1 \rtimes T_2$ is a submonoid of $S_1 \rtimes S_2$. If, in addition, $T = T_1T_2$, then

$$\begin{array}{cccc} \mu|_{T_1 \rtimes T_2} : & T_1 \rtimes T_2 & \longrightarrow & T \\ & (s,u) & \mapsto & su \end{array}$$

is a surjective homomorphism.

For $s \in \mathcal{ODP}_n$ and $x \in \mathcal{C}_2$, it is clear that $x.s = xsx \in \mathcal{ODP}_n$. Thus, we may consider the semidirect product $\mathcal{ODP}_n \rtimes \mathcal{C}_2$ induced by the left action δ of $\mathcal{C}_2$ on $\mathcal{POI}_n$. Moreover, since $\mathcal{DP}_n = \mathcal{ODP}_n\mathcal{C}_2$, then $\mu|_{\mathcal{ODP}_n \rtimes \mathcal{C}_2} : \mathcal{ODP}_n \rtimes \mathcal{C}_2 \longrightarrow \mathcal{DP}_n$ is a surjective homomorphism and so we have:

Theorem 2.5. *The monoid $\mathcal{DP}_n$ is a homomorphic image of $\mathcal{ODP}_n \rtimes \mathcal{C}_2$.*

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Vítor H. Fernandes
Departamento de Matemática
Faculdade de Ciências e Tecnologia
Universidade NOVA de Lisboa
Monte da Caparica
2829-516 Caparica, Portugal
and
Centro de Álgebra da Universidade de Lisboa
Av. Prof. Gama Pinto 2
1649-003 Lisboa, Portugal
E-mail address: `vhf@fct.unl.pt`

Teresa M. Quinteiro
Instituto Superior de Engenharia de Lisboa
Rua Conselheiro Emídio Navarro 1
1950-062 Lisboa, Portugal
and
Centro de Álgebra da Universidade de Lisboa
Av. Prof. Gama Pinto 2
1649-003 Lisboa, Portugal
E-mail address: `tmelo@adm.isel.pt`