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FINITE GROUPS WITH A CYCLIC NORM QUOTIENT

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Abstract. The norm $N(G)$ of a group G is the intersection of the normalizers of all the subgroups of G. In this paper, the structure of finite groups with a cyclic norm quotient is determined. As an application of the result, an interesting characteristic of cyclic groups is given, which asserts that a finite group G is cyclic if and only if $\mathrm{Aut}(G)/P(G)$ is cyclic, where $P(G)$ is the power automorphism group of G.

1. Introduction

The norm $N(G)$ of a group G, first introduced by R. Baer [1] in 1934, is the intersection of the normalizers of all the subgroups of G. It is clear that $N(G)$ is a characteristic subgroup of G and it contains the center $Z(G)$. Also $N(G)$ itself is a Dedekind group, and every element of $N(G)$ induces a power automorphism on G. Many authors have investigated both $N(G)$ and how $N(G)$ influences the structure of G (see [2], [3], [4], [9], [10] and [11]). For instance, Schenkman proved that $N(G) \leq Z_2(G)$ for any group G (see [9]), and Baer showed that a 2-group G must be a Dedekind group if the $N(G)$ is nonabelian (see [2]). In [10] and [11], the authors have determined the structure of a finite group G satisfying $|G : N(G)| = p$ or pq, where p, q are primes. In this paper, finite groups with a cyclic norm quotient are determined, see Section 2.

Based on the above result, we will establish a characteristic of cyclic groups. Recall that a power automorphism of a group G is an automorphism that leaves every subgroup of G invariant. Under such an automorphism, every element of G is mapped to one of its powers. All power automorphisms of G constitute an abelian normal subgroup, denoted by $P(G)$, of $\mathrm{Aut}(G)$. The structure of G and $P(G)$ are strictly linked (for example, see [5], [12]), especially the quotient group $\mathrm{Aut}(G)/P(G)$ strongly influences the structure of G. In [11], groups satisfying $|\mathrm{Aut}(G)/P(G)| = 1, p$ or pq are completely clarified. In Section 3,

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we shall prove that a finite group is cyclic if and only if $\mathrm{Aut}(G)/P(G)$ is cyclic. This characteristic of cyclic groups generalizes some results in [11].

All groups considered in this paper are finite. In the following, $G = B \ltimes A$ denotes G is a semidirect product of two subgroups A and B, where A is normalized by B, and $\exp(G)$ denotes the exponent of G. Other notation and terminology not mentioned here are standard, see [6] for instance.

2. Groups with cyclic norm quotient

In this section, we will determine the structure of finite groups with cyclic norm quotient.

Lemma 2.1. *A group G is a Dedekind group if and only if $G = N(G)$.*

Lemma 2.2 ([10, Lemma 2.1])**.** *Let A and B be subgroups of a finite group G such that $G = A \times B$. If $(|A|, |B|) = 1$, then $N(G) = N(A) \times N(B)$.*

Theorem 2.3. *Let G be a finite group. Then $G/N(G)$ is cyclic if and only if G is nilpotent with $P/N(P)$ being cyclic for every Sylow subgroup P of G.*

Proof. If $G/N(G)$ is cyclic, then $G/Z_2(G)$ is cyclic by [9], and so G is nilpotent. Suppose that $G = P_1 \times P_2 \times \cdots \times P_t$, where $P_i \in \mathrm{Syl}_{p_i}(G)$, $i = 1, 2, \ldots, t$. By Lemma 2.2, we have $N(G) = N(P_1) \times N(P_2) \times \cdots \times N(P_t)$. Hence, $G/N(G) = P_1/N(P_1) \times \cdots \times P_t/N(P_t)$, and each $P_i/N(P_i)$ is cyclic.

Conversely, if G is nilpotent, then $G/N(G) = P_1/N(P_1) \times \cdots \times P_t/N(P_t)$, where $P_i \in \mathrm{Syl}_{p_i}(G)$, $i = 1, 2, \ldots, t$. If each $P_i/N(P_i)$ is cyclic, then $G/N(G)$ is cyclic. □

From Lemma 2.1 and Theorem 2.3 we see that, to determine groups with cyclic norm quotient is only need to determine non-Dedekind p-groups with cyclic norm quotient.

Lemma 2.4. *Let P be a finite p-group. If $N(P) < P$, then $N(P)$ is abelian.*

Proof. Since $N(P)$ is a Dedekind group, it follows from [8, Theorem 5.3.7] and [5, Theorem 6.5.1] that $N(P)$ is abelian. □

Lemma 2.5. *Let P be a p-group with $N(P) < P$. If $P = \langle N(P), a\rangle$ for some $a \in P$, then $o(a) > \exp(N(P))$.*

Proof. By Lemma 2.4, $N(P)$ is abelian. Let y be an element of maximal order in $N(P)$. Then $N(P) = \langle y\rangle \times B$ for some $B \leq N(P)$. If $a^y = a$, then there exists $b \in B$ such that $a^b \neq a$, and so $a^{yb} \neq a$, $o(yb) = o(y)$. This shows that $N(P)$ has an element, say g, of maximal order such that $a^g \neq a$.

Suppose that $C_{\langle a\rangle}(g) = \langle a^{p^r}\rangle$, where $r \geq 1$. Let $x = a^{p^{r-1}}$. Suppose that $o(x) = p^n$ and $x^g = x^t$. Clearly $n \geq 2$ since $x^g \neq x$. Noticing that $[x^p, g] = 1$, we have $tp \equiv p \pmod{p^n}$ and $t \equiv 1 \pmod{p^{n-1}}$, and therefore $x^g = x^{1+\mu p^{n-1}}$ for some positive integer μ with $(\mu, p) = 1$. Choose a positive integer δ such that $\mu\delta \equiv 1 \pmod p$. Then $o(g) = o(g^\delta)$, and $x^{g^\delta} = x^{1+\mu\delta p^{n-1}} = x^{1+p^{n-1}}$.

Replacing g by g^δ, we may assume that $x^g = x^{1+p^{n-1}}$. Let $Q = \langle g, x\rangle$. It is clear that $Q' = \langle x^{p^{n-1}}\rangle \le Z(Q)$ is of order p, and then for any $u, v \in Q$, $(uv)^p = u^p v^p$ if $p > 2$, and $(uv)^4 = u^4 v^4$ if $p = 2$. Take any $z \in C_P(Q)$. Since $\langle x, z\rangle$ is abelian, according to [8, Theorem 13.4.3] there exists an integer l such that $z^g = z^l$, $x^g = x^l$, from which we deduce that $l \equiv 1 \pmod{o(z)}$, $l \equiv 1 + p^{n-1} \pmod{p^n}$. If $o(z) \ge p^n$, then $l \equiv 1 \pmod{p^n}$, and therefore $p^{n-1} \equiv 0 \pmod{p^n}$, a contradiction. Hence $o(z) < p^n$ and $\exp(C_P(Q)) < p^n$, especially $\exp(Z(P)) < p^n$. Also as $g^p \in Z(Q) \le C_P(Q)$, we get that $o(g^p) < p^n$, and so $o(g) \le o(x)$.

If $o(x) < o(a)$, then $o(a) > \exp(N(P))$ by the maximality of g. The remainder of the proof is to deal with the situation $o(x) = o(a)$, namely $x = a$. Now assume by way of contradiction that $o(g) = o(x)$. Let p^k be the order of the subgroup $\langle x\rangle \cap \langle g\rangle$, where $0 \le k < n$. We consider the following two cases:

Case 1: $p = 2$ and $k = n-1$. Then $\langle g^2\rangle = \langle x^2\rangle$, $g^{2^{n-1}} = x^{2^{n-1}}$, and there exists an odd integer s such that $g^2 = x^{-2s}$. It follows that $(gx^s)^2 = x^{2^{n-1}}$. Write $x_1 = gx^s$. Then $(x_1)^2 = x^{2^{n-1}} = g^{2^{n-1}}$ and $Q = \langle g, x_1\rangle$. If $n \ge 3$, then $o(g^{2^{n-2}}x_1) = 2$ and g acts trivially on $\langle g^{2^{n-2}}x_1\rangle$, which implies that Q is abelian since $Q = \langle g, g^{2^{n-2}}x_1\rangle$, a contradiction. Hence $n = 2$, and $Q = \langle g, x \mid x^4 = 1, g^2 = x^2, x^g = x^{-1}\rangle$ is a quaternion group of order 8, and $Z(P)$ is elementary abelian. Moreover, since $N(P)/Z(P)$ acts faithfully on $\langle x\rangle$, we have $N(P)/Z(P)$ is of order 2, and $N(P) = \langle Z(P), g\rangle$. So $P = \langle g, x\rangle Z(P) = \langle g, x\rangle \times B$ for some $B \le Z(P)$. By [8, Theorem 5.3.7], we know that P is a Dedekind group and $P = N(P)$, a contradiction.

Case 2: either $p = 2$ and $k \le n-2$ or $p > 2$. Then $\langle g^{p^{n-k}}\rangle = \langle x^{p^{n-k}}\rangle$, and there exists an integer s such that $g^{p^{n-k}} = x^{-sp^{n-k}}$, where $(p, s) = 1$. It follows that $(gx^s)^{p^{n-k}} = 1$. Write $x_1 = gx^s$. Then $x_1^{p^{n-k}} = 1$ and $Q = \langle g, x_1\rangle$. Since $|Q| = p^n p^{n-k}$, we have $o(x_1) = p^{n-k}$ and $\langle g\rangle \cap \langle x_1\rangle = 1$. Thus $Q = \langle g\rangle \ltimes \langle x_1\rangle$ is a semidirect product of $\langle g\rangle$ and $\langle x_1\rangle$, and $n - k \ge 2$. Let $m = n - k$. Note that $1 \ne [x_1, g] \in Q'$, and so $o([x_1, g]) = p$. Consequently, $\langle [x_1, g]\rangle = \langle x_1^{p^{m-1}}\rangle$ and $x_1^g = x_1^{1+jp^{m-1}}$ for some positive integer j, where $(j, p) = 1$. It is no loss to assume that $x_1^g = x_1^{1+p^{m-1}}$. Let i be an integer such that $(gx_1)^g = (gx_1)^i$. Then $gx_1^{1+p^{m-1}} = g^i x_1^i [x_1, g]^{\frac{i(i-1)}{2}}$, from which we get

$$1 \equiv i \pmod{p^n} \text{ and } 1 + p^{m-1} \equiv i + \frac{i(i-1)}{2} p^{m-1} \pmod{p^m}.$$

Since $m \le n$, the former congruence implies that $1 \equiv i \pmod{p^m}$, and so $\frac{i(i-1)}{2} p^{m-1} \equiv 0 \pmod{p^m}$. It follows from the latter congruence expression that $p^{m-1} \equiv 0 \pmod{p^m}$, a contradiction. This completes our proof. □

Theorem 2.6. *Let P be a p-group with $N(P) < P$. Then $P/N(P)$ is cyclic if and only if $P = \langle B, g\rangle \ltimes \langle x\rangle$, where $\langle B, g\rangle$ is an abelian group generated by* *a subgroup B and an element g, $\langle x\rangle \trianglelefteq P$, and the subgroup B, the elements g and x satisfy the following properties*:

(i) $o(x)=p^{n+m}, o(g)=p^r$, $1\le m\le r\le n$, $\exp(B)\le p^n$, *and* $n+m\ge 3$ *if* $p=2$;

(ii) $x^g=x^{1+p^n}$, $[B,x]=1$.

Proof. " $\Rightarrow$ " By assumption, there exists an element $x\in P$ such that $x\notin N(P)$ and $P=N(P)\langle x\rangle$. Suppose that $N(P)\cap\langle x\rangle=\langle x^{p^m}\rangle$. Clearly $m\ge 1$. We claim that x^{p^m} is of maximal order in $N(P)$. It suffices to shows that $o(x^{p^{m-1}})>\exp(N(P))$. Let $Q=N(P)\langle x^{p^{m-1}}\rangle$. Then $|Q:N(P)|=p$ and $N(P)\le N(Q)$. If $N(P)=N(Q)$, then $Q=\langle N(Q),x^{p^{m-1}}\rangle$ with $x^{p^{m-1}}\notin N(Q)$, and by Lemma 2.5 we have $o(x^{p^{m-1}})>\exp(N(P))$. If $N(P)<N(Q)$, then $Q=N(Q)$ and Q is a Dedekind group. Note that $[N(P),x^{p^{m-1}}]\ne 1$, so $p=2$, and $o(x^{p^{m-1}})=4$. It follows that $o(x)=p^{m+1}$ and there exists an element $g\in N(P)$ such that $[g,x^{p^{m-1}}]\ne 1$. Suppose that $\langle[x,g]\rangle=\langle x^{p^k}\rangle$. According to [9, Theorem], $x^{p^k}\in Z(P)$ and therefore there exists an odd integer δ such that $x^{p^k}=[x,g]^\delta=[x,g^\delta]$. Hence, without loss of generality, we may assume that $[x,g]=x^{p^k}$, and then $x^g=x^{1+p^k}$. Consequently $x^{p^m}=(x^{p^m})^g=x^{p^m}x^{p^{m+k}}$, and $k\ge 1$. If $k>1$, it is easy to check that $(x^{p^{m-1}})^g=x^{p^{m-1}}$, a contradiction. Hence $x^g=x^{1+p}$, and $[x,g]=x^p\in Z(P)\cap\langle x\rangle\le N(P)\cap\langle x\rangle=\langle x^{p^m}\rangle$. From which we deduce that $m=1$ and $|P:N(P)|=p$, and therefore $P=Q$ is Dedekind, a contradiction. This shows that our claim is true.

By the above result, we get $N(P)=T\times(N(P)\cap\langle x\rangle)$ for some subgroup T of $N(P)$. Therefore $P=T\ltimes\langle x\rangle$. If T has an element b such that $x^b=x^{-1}$, then $(bx)^2=b^2$. However, since $P=\langle N(P),x\rangle=\langle N(P),bx\rangle$ and $bx\notin N(P)$, $o(b)<o(bx)$ by Lemma 2.5 , a contradiction. It follows from [7, Theorem 2.19] that $T/C_T(x)$ is cyclic and $o(x)\ge 8$ if $p=2$. Let $B=C_T(x)$. Then there exists $g\in T$ such that $T=\langle B,g\rangle$, where $[B,x]=1$.

It is clear that $C_{\langle x\rangle}(g)=\langle x\rangle\cap Z(P)=\langle x\rangle\cap N(P)=\langle x^{p^m}\rangle$. Suppose that $o(x^{p^m})=p^n$. Then $n\ge 1$ and $o(x)=p^{n+m}$. Suppose $\langle[x,g]\rangle=\langle x^{p^i}\rangle, i\ge 0$. Then $x^{p^i}\in\langle x\rangle\cap Z(P)=\langle x^{p^m}\rangle$ and so $i\ge m$. Also there exists an integer k prime to p such that $[x,g]^k=x^{p^i}$ or $[x,g^k]=x^{p^i}$. Replacing g by g^k, we may assume that $[x,g]=x^{p^i}$ and so $x^g=x^{1+p^i}$. Since $C_{\langle x\rangle}(g)=\langle x^{p^m}\rangle$, it follows that $i=n$, and $x^g=x^{1+p^n}$. Now suppose that $o(g)=p^r$. It is easy to verify that g induces an automorphism in $\langle x\rangle$ of order p^m. Thus $m\le r$. Moreover, $\exp(N(P))\le p^n$ by the maximality of x^{p^m} in $N(P)$, especially $r\le n$ and $\exp(B)\le p^n$.

" $\Leftarrow$ " Conversely, let $P=\langle B,g\rangle\ltimes\langle x\rangle$ be a group with the stated properties. It is easy to verify that $\langle x\rangle\cap Z(P)=\langle x^{p^m}\rangle$ and $P'=\langle[x,g]\rangle=\langle x^{p^n}\rangle\le Z(P)$. Let h be any element of P. Then h can be written as $h=bg^kx^i$, where $b\in B$, and k,i are non-negative integers. Clearly, $h^g=bg^kx^{i(1+p^n)}$. Since

$$h^{1+p^n}=(bg^kx^i)(bg^kx^i)^{p^n}$$

$$= (bg^k x^i)(bg^k)^{p^n} x^{ip^n} [x^i, bg^k]^{p^n(p^n-1)/2}$$
$$= (bg^k x^i) x^{ip^n} x^{ikp^{2n}(p^n-1)/2},$$

we see that $h^{1+p^n} = (bg^k x^i)x^{ip^n} = bg^k x^{i(1+p^n)} = h^g$ if $p > 2$ or $p = 2$ and $m < n$, or $p = 2, m = n$ and $2 \mid ik$. For the case that $p = 2, m = n$ and $2 \nmid ik$, noticing $n > 1$, we have

$$\begin{aligned} h^{1+2^n+2^{2n-1}} &= (bg^k x^i)(bg^k)^{2^n+2^{2n-1}} x^{i2^n} x^{i2^{2n-1}} [x^i, bg^k]^{\binom{2^n+2^{2n-1}}{2}} \\ &= (bg^k x^i) x^{i2^n} x^{2^{2n-1}} (x^{2^{2n-1}})^{ik(2^{n-1}+1)(2^n+2^{2n-1}-1)} \\ &= (bg^k x^i) x^{i2^n} x^{2^{2n-1}} x^{2^{2n-1}} \\ &= bg^k x^{i(1+2^n)} \\ &= h^g. \end{aligned}$$

Hence $g \in N(P)$, and it follows that

$$N(P) = N(P) \cap (\langle B, g\rangle \ltimes \langle x\rangle) = \langle B, g\rangle \times (N(P) \cap \langle x\rangle) = \langle B, g\rangle \times \langle x^{p^m}\rangle.$$

Consequently, $P/N(P)$ is cyclic of order p^m. This completes the proof. □

Theorem 2.7. *Let P be a p-group with $N(P) < P$. If $P/N(P)$ is cyclic, then $o(h) > \exp(N(P))$ for any $h \in P - N(P)$.*

Proof. Suppose that $P = \langle N(P), x\rangle$ with $|P/N(P)| = p^m$, where $m \geq 1$. Since $P/N(P)$ is cyclic, there exists an unique subgroup series

$$N(P) = P_m < P_{m-1} < \cdots < P_1 < P_0 = P,$$

satisfying $|P_{i-1} : P_i| = p$, $i = 1, 2, \ldots, m$. Clearly $N(P) \leq N(P_i)$. If $N(P) < N(P_i)$ for some P_i, then $P_{m-1} \leq N(P_i)$ and P_{m-1} is a Dedekind group. However, from the proof of Theorem 2.6, we see that $P_{m-1} = \langle N(P), x^{p^{m-1}}\rangle$ and $|P_{m-1} : N(P_{m-1})| = p$, a contradiction. Hence $N(P) = N(P_i)$ for all P_i.

Let h be any element in $P - N(P)$. Then there must exists a subgroup P_k such that $P_k = \langle N(P), h\rangle$. Since $N(P) = N(P_k)$, it follows from Lemma 2.5 that $o(h) > \exp(N(P))$. □

Remark 2.8. Theorem 2.7 is not necessarily true if we delete the condition that $P/N(P)$ is cyclic. For example, let P be a p-group of order p^3 with the following defining relation

$$P = \langle a, b \mid a^p = b^p = c^p = 1, [a, b] = c, ac = ca, bc = cb\rangle.$$

Since $a, b \in C_P(N(P))$, we have $N(P) = Z(P)$ is of order p. Clearly $a \notin N(P)$, but $o(a) = \exp(N(P))$.

Lemma 2.9 ([11, Lemma 2.7])**.** *Let B be a maximal subgroup of an abelian p-group Q. Then there exist an element $u \in Q$ and a subgroup $A \leq B$ such that $Q = \langle u\rangle \times A$.*

Corollary 2.10 ([10, Theorem 1.4]). *A finite p-group P satisfies $|P : N(P)| = p$ if and only if $P = R \times A$, where*

$$R = \langle x, u \mid x^{p^{n+1}} = u^{p^k} = 1,\ x^u = x^{1+p^n},\ 1 \le k \le n,\ \text{and } n \ge 2 \text{ if } p = 2\rangle,$$

and A is an abelian group with $\exp(A) \le p^n$.

Proof. We only prove the necessity. Assume that $|P : N(P)| = p$. Then by Theorem 2.6, there exist $B \le P$ and $g, x \in P$ such that $P = \langle B, g\rangle \ltimes \langle x\rangle$, where $o(x) = p^{n+1}$, $\langle B, g\rangle$ is abelian with $\exp(\langle B, g\rangle) \le p^n$, $x^g = x^{1+p^n}, [B, x] = 1$, and $n \ge 2$ if $p = 2$. Noticing that $g^p \in Z(P)$, without loss of generality, we may assume that $g^p \in B$. Hence B is a maximal subgroup of $\langle B, g\rangle$. It follows from Lemma 2.9 that there exist $A \le B$ and $u \in \langle B, g\rangle$ such that $\langle B, g\rangle = A \times \langle u\rangle$. Clearly $\langle [x, u]\rangle = P' = \langle [x, g]\rangle = \langle x^{p^n}\rangle$, and so there exists an integer δ coprime to p such that $[x, u^\delta] = [x, u]^\delta = x^{p^n}$. Replacing u by u^δ, we can assume that $[x, u] = x^{p^n}$ and therefore $x^u = x^{1+p^n}$. Now let $R = \langle x, u\rangle$. Then $P = \langle B, g\rangle \ltimes \langle x\rangle = A \times R$ as desired. The proof is complete. □

Remark 2.11. By Theorem 2.6, a finite p-group P satisfying $P/N(P)$ is cyclic is a semidirect product of an abelian subgroup $\langle B, g\rangle$ and a cyclic normal subgroup $\langle x\rangle$, where B, g and x satisfy the properties stated in Theorem 2.6. In general, if $|P : N(P)| > p$ or $m \ge 2$, then the subgroup $\langle B, g\rangle$ of P may not necessarily be decomposed as $\langle B, g\rangle = A \times \langle u\rangle$ with $A \le Z(P)$. For example, let $B = \langle b\rangle, o(b) = p^m$, $o(g) = p^{m+1}$, and $\langle b\rangle \cap \langle g\rangle$ be of order p, where $m \ge 2$. Let $K = \langle b, g\rangle$. It is clear that $C_K(x) = \langle b\rangle$. Assume that there exist $A \le K$ and $u \in K$ such that $K = A \times \langle u\rangle$ and $A \le Z(P)$. Then $P' = \langle [x, u]\rangle = \langle x^{p^n}\rangle$. Similar to the proof of Corollary 2.10, we may assume that $x^u = x^{1+p^n}$. Hence $x^u = x^g$ and $u = cg$, where $c \in C_K(x)$. It follows that $u^{p^m} = g^{p^m} \ne 1$, and $u^{p^m} \in C_K(x)$. Since $C_K(x)$ is cyclic, we obtain that $A = 1$ and K is cyclic, a contradiction.

3. A characteristic of finite cyclic groups

In this section, as an application of Theorem 2.6, we shall prove a characteristic of cyclic groups: A finite group G is cyclic if and only if $\mathrm{Aut}(G)/P(G)$ is cyclic. For convenience of statement, in the following, we will call a group G a C-group if $\mathrm{Aut}(G)/P(G)$ is cyclic. Since $G/N(G) \lesssim \mathrm{Aut}(G)/P(G)$, we see that if G is a C-group, then $G/N(G)$ is cyclic, and so G is nilpotent and every Sylow subgroup of G is either a Dedekind group or a group with a non-trivial cyclic norm quotient whose structure has been determined by Theorem 2.6. The proof will be given following several lammas.

Lemma 3.1. *A non-cyclic abelian p-group R can not be a C-group.*

Proof. Without loss of generality, we may assume that $R = \langle x_1\rangle \times \langle x_2\rangle$, where $o(x_1) = p^{k_1}$, $o(x_2) = p^{k_2}$. Then two automorphisms α, β of R are given by

$$x_1^\alpha = x_1, x_2^\alpha = x_2 x_1^{p^{k_1-1}}\ ;\quad x_1^\beta = x_1 x_2^{p^{k_2-1}}, x_2^\beta = x_2.$$

Clearly, $o(\alpha) = o(\beta) = p$, and $\alpha, \beta \notin P(R)$. If R is a C-group, then there exist an integer i and some $\gamma \in P(R)$ such that $\alpha = \beta^i \gamma$. However, $x_2^\alpha = x_2 x_1^{p^{k_1-1}} \notin \langle x_2 \rangle$, $x_2^{\beta^i \gamma} = x_2^\gamma \in \langle x_2 \rangle$, a contradiction. □

Lemma 3.2. *A non-abelian Dedekind 2-group R can not be a C-group.*

Proof. By [8, Theorem 5.3.7], $R = Q_8 \times A$, where Q_8 is the quaternion group of order 8, and A is elementary abelian. Let $Q_8 = \langle a, b \rangle$. Define two automorphisms α and β of R by

$$a^\alpha = b,\ b^\alpha = a,\ [A, \alpha] = 1;\ a^\beta = b,\ b^\beta = ab,\ [A, \beta] = 1.$$

Clearly, $\alpha, \beta \notin P(R)$, and $o(\alpha) = 2, o(\beta) = 3$. If R is a C-group, then $6 \mid o(\alpha\beta)$. However, $o(\alpha\beta) = 4$ since $a^{\alpha\beta} = ab,\ b^\beta = b$, a contradiction. □

Lemma 3.3. *A p-group $R = \langle B, g \rangle \ltimes \langle x \rangle$ with B, g and x being the same as in Theorem 2.6 can not be a C-group.*

Proof. Consider the automorphism α of R induced by $x^{p^{m-1}}$. It is easy to see that $o(\alpha) = p,\ x^\alpha = x,\ g^\alpha = gx^{-p^{n+m-1}}$, and $\alpha \notin P(R)$.

If $p > 2$, or $p = 2$ and $r \geq 2$, let β be the automorphism of R of order 2 defined by $x^\beta = x^{-1},\ y^\beta = y$ for all $y \in \langle B, g \rangle$. Assume that $\langle gx^{p^m} \rangle^\beta = \langle gx^{p^m} \rangle$. Then there exists an integer k such that $gx^{-p^m} = g^k x^{kp^m}$, and so $1 \equiv k \pmod{p^r}$ and $-p^m \equiv kp^m \pmod{p^{n+m}}$. Therefore $1 \equiv k \pmod{p^r}$ and $-1 \equiv k \pmod{p^n}$. Consequently, $1 \equiv -1 \pmod{p^r}$, a contradiction. Hence we have proved that $\beta \notin P(R)$. Moreover, since $x^{\alpha\beta} = x^{-1},\ g^{\alpha\beta} = gx^{p^{n+m-1}}$, it is clear that $o(\alpha\beta) = 2$ and $\alpha\beta \notin P(R)$. If R is a C-group, then $2p \mid o(\alpha\beta)$ when $p > 2$, and $\alpha\beta \in P(R)$ when $p = 2$, a contradiction.

For the case $p = 2$ and $r = 1$, it must be $m = 1$ and $n \geq 2$. Note that $(gx)^{2^n} = x^{2^n}$, and so $o(gx) = 2^{n+1}$, $R = \langle B, g \rangle \ltimes \langle gx \rangle$ and $(gx)^g = (gx)^{1+2^n}$. It follows that the map: $x \mapsto gx,\ y \mapsto y$ for all $y \in \langle B, g \rangle$ determines an automorphism β of R of order 2, and $\beta, \alpha\beta \notin P(R)$ since $x^{\alpha\beta} = gx$. Similar to the above proof, we still have that R can not be a C-group. □

Lemma 3.4. *Let G be a nilpotent group. If $G = R_1 \times R_2 \times \cdots \times R_t$, where $R_i \in \mathrm{Syl}_{p_i}(G)$, $i = 1, 2, \ldots, t$, then*

$$\mathrm{Aut}(G)/P(G) \simeq \mathrm{Aut}(R_1)/P(R_1) \times \cdots \times \mathrm{Aut}(R_t)/P(R_t).$$

Proof. Clearly $\mathrm{Aut}(G) \simeq \mathrm{Aut}(R_1) \times \cdots \times \mathrm{Aut}(R_t)$. Also by [11, Lemma 3.11], we have $P(G) \simeq P(R_1) \times \cdots \times P(R_t)$. Hence

$$\mathrm{Aut}(G)/P(G) \simeq \mathrm{Aut}(R_1)/P(R_1) \times \cdots \times \mathrm{Aut}(R_t)/P(R_t).$$

□

Theorem 3.5. *Let G be a finite group. Then G is cyclic if and only if G is a C-group.*

Proof. The necessity is clear, we only prove the sufficiency. Let G be a C-group. Then G is nilpotent. Suppose that $G = R_1 \times R_2 \times \cdots \times R_t$, where $R_i \in \mathrm{Syl}_{p_i}(G)$, $i = 1, 2, \ldots, t$. By Lemma 3.4, we see that every R_i is a C-group, and so every R_i is either a Dedekind group or a group of prime power order with a non-trivial cyclic norm quotient. It follows from Lemma 3.1, Lemma 3.2 and Lemma 3.3 that every R_i is cyclic, and therefore G is cyclic. □

Corollary 3.6 ([11, Theorem 3.2])**.** *A group G is cyclic if and only if* $\mathrm{Aut}(G) = P(G)$.

Proof. If G is cyclic, then every automorphism of G must be a power automorphism, and so $\mathrm{Aut}(G) = P(G)$. Conversely, if $\mathrm{Aut}(G) = P(G)$, then $\mathrm{Aut}(G)/P(G)$ is cyclic, and G is cyclic by Theorem 3.5. □

Corollary 3.7 ([11, Theorem 3.8])**.** *There exists no finite group G such that $|\mathrm{Aut}(G)/P(G)|$ is a prime.*

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