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SOME CHARACTERIZATIONS OF CANAL SURFACES

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Abstract. This work considers a particular type of swept surface named canal surfaces in Euclidean 3-space. For such a kind of surfaces, some interesting and important relations about the Gaussian curvature, the mean curvature and the second Gaussian curvature are found. Based on these relations, some canal surfaces are characterized.

1. Introduction

The class of surfaces formed by sweeping a sphere was first investigated by Monge in 1850, who named them canal surfaces. Canal surfaces may be generated either by sweeping a sphere along a path, or by sweeping a particular circular cross-section of the sphere along the same path (Figure 1 and Figure 2 cited from Google can show its generating process). These two methods of sweeping a sphere and sweeping a disk give rise to generate parametric formulae of canal surfaces.

Thus, a canal surface $\mathbb{M}$ can be parametrized as follows:

$$x(s,\theta) = c(s) + r(s)\{\sqrt{1-r'(s)^2}\cos\theta N + \sqrt{1-r'(s)^2}\sin\theta B - r'(s)T\}, \tag{1.1}$$

where $c(s)$ is a unit speed curve parametrized by arc-length s, $\{T, N, B\}$ is the Frenet frame of $c(s)$. In the sequel, T, N, B is the unit tangent, principal normal and binormal vector fields, respectively. Here, the curve $c(s)$ is called the spine or center curve and $r(s)$ is called the radial function of $\mathbb{M}$ (cf. [11]).

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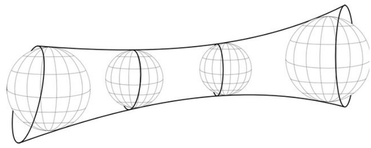

Figure 1

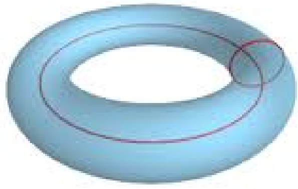

Figure 2

Canal surfaces have been applied to many fields, such as the solid and the surface modeling for CAD/CAM, construction of blending surfaces, shape reconstruction and so on. Due to their wide applications, many mathematicians and engineers studied canal surfaces from various viewpoints. Especially tubes (or pipe surfaces) are particular canal surfaces with constant radial function, which are studied in CAD. In [4] and [7], some applications of such surfaces are presented. However, to the best of our knowledge, there are very few articles which systematically discussed the geometric properties of canal surfaces.

In Section 2, some fundamental facts are prepared. The first, second and third fundamental forms, the Gaussian curvature, the second Gaussian curvature and the mean curvature are presented. About them, some interesting and important relations are obtained. For instance, the Gaussian curvature and the mean curvature of canal surfaces satisfy $2H = -(Kr + \frac{1}{r})$. Making use of these conclusions, in Section 3, the (X, Y)-Weingarten canal surfaces are discussed. Some special canal surfaces and the (X, Y)-linear Weingarten canal surfaces are studied in Section 4.

All the surfaces under consideration are assumed to be smooth, regular and topologically connected unless otherwise stated.

2. Preliminaries

From (1.1), we may assume $-r'(s) = \cos\varphi$ for some smooth function $\varphi = \varphi(s)$. Then the canal surface $\mathbb{M}$ can be written as

$$x(s,\theta) = c(s) + r(s)(\sin\varphi\cos\theta N + \sin\varphi\sin\theta B + \cos\varphi T), \tag{2.1}$$

where $s \in [0, l], \theta \in [0, 2\pi)$, $\varphi \in [0, \pi)$ and l is the total length of $c(s)$.

Review the Frenet equations of a regular space curve:

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N, \tag{2.2}$$

where the prime denotes the differentiation with respect to s. The functions κ and τ are called the curvature and torsion, respectively.

Remark 2.1. In particular, if the spine curve $c(s)$ is a straight line, then the Frenet frame $\{T, N, B\}$ can be regarded as a trivial orthogonal frame. In such case, the canal surface is nothing but a surface of revolution. All the surfaces of revolution and tubes are subclass of the canal surfaces.

For later use, some basic conclusions are given by direct calculation.
Initially, we have by (2.1) and (2.2)

$$x_s = \frac{\partial x}{\partial s} = x_s^1 T + x_s^2 N + x_s^3 B, \tag{2.3}$$

where

$$\begin{aligned} x_s^1 &= \sin^2\varphi - rr'' - r\kappa\sin\varphi\cos\theta;\\ x_s^2 &= r'\sin\varphi\cos\theta - rr'\kappa - r\tau\sin\varphi\sin\theta - rr'\varphi'\cos\theta;\\ x_s^3 &= r'\sin\varphi\sin\theta + r\tau\sin\varphi\cos\theta - rr'\varphi'\sin\theta, \end{aligned}$$

and

$$x_\theta = (-r\sin\varphi\sin\theta)N + (r\sin\varphi\cos\theta)B. \tag{2.4}$$

Then, the quantities of the first fundamental form are given by

$$\begin{aligned} E &= \langle x_s, x_s\rangle\\ &= r^2(\kappa^2\sin^2\varphi\cos^2\theta + r'^2\kappa^2 + \tau^2\sin^2\varphi + \varphi'^2 + 2\kappa\varphi'\cos\theta\\ &\quad + 2r'\kappa\tau\sin\varphi\sin\theta) + \sin^2\varphi - 2(rr'' + r\kappa\sin\varphi\cos\theta);\\ F &= \langle x_s, x_\theta\rangle = r^2\tau\sin^2\varphi + r^2 r'\kappa\sin\varphi\sin\theta;\\ G &= \langle x_\theta, x_\theta\rangle = r^2\sin^2\varphi. \end{aligned} \tag{2.5}$$

By (2.5), we get

$$EG - F^2 = r^2(rr'' + r\kappa\sin\varphi\cos\theta - \sin^2\varphi)^2.$$

From (2.3) and (2.4), the unit normal vector field to $\mathbb{M}$ is given by

$$n = \frac{x_s \times x_\theta}{\|x_s \times x_\theta\|} = (\cos\varphi)T + (\sin\varphi\cos\theta)N + (\sin\varphi\sin\theta)B,$$

which points outwards the canal surface $\mathbb{M}$.

Furthermore, we have

$$\begin{aligned} n_s &= -(r'' + \kappa\sin\varphi\cos\theta)T - (r'\kappa + r'\varphi'\cos\theta + \tau\sin\varphi\sin\theta)N\\ &\quad + (\tau\sin\varphi\cos\theta - r'\varphi'\sin\theta)B,\\ n_\theta &= -(\sin\varphi\sin\theta)N + (\sin\varphi\cos\theta)B. \end{aligned}$$

Then, the quantities of the second fundamental form are obtained by

$$\begin{aligned} L &= -\langle x_s, n_s\rangle\\ &= -r(\kappa^2\sin^2\varphi\cos^2\theta + r'^2\kappa^2 + \tau^2\sin^2\varphi + \varphi'^2 + 2\kappa\varphi'\cos\theta\\ &\quad + 2r'\kappa\tau\sin\varphi\sin\theta) + (r'' + \kappa\sin\varphi\cos\theta);\\ M &= -\langle x_\theta, n_s\rangle = -r\tau\sin^2\varphi - rr'\kappa\sin\varphi\sin\theta;\\ N &= -\langle x_\theta, n_\theta\rangle = -r\sin^2\varphi. \end{aligned} \tag{2.6}$$

And as well, the quantities of the third fundamental form are given by

$$\begin{aligned} e &= \langle n_s, n_s \rangle \\ &= \kappa^2 \sin^2\varphi \cos^2\theta + r'^2\kappa^2 + \tau^2 \sin^2\varphi + \varphi'^2 \\ &\quad + 2\kappa\varphi' \cos\theta + 2r'\kappa\tau \sin\varphi \sin\theta; \\ f &= \langle n_\theta, n_s \rangle = \tau \sin^2\varphi + r'\kappa \sin\varphi \sin\theta; \\ g &= \langle n_\theta, n_\theta \rangle = \sin^2\varphi. \end{aligned} \tag{2.7}$$

Based on (2.5)-(2.7), we have the following lemma.

Lemma 2.2. *The first, second and third fundamental forms of canal surfaces satisfy*

$$L = \frac{E+P}{-r}, \quad M = \frac{F}{-r}, \quad N = \frac{G}{-r};$$

$$e = \frac{L-Q}{-r}, \quad f = \frac{M}{-r}, \quad g = \frac{N}{-r}$$

and

$$EG - F^2 = r^2P^2, \quad LN - M^2 = rPQ, \quad eg - f^2 = Q^2, \tag{2.8}$$

where

$$\begin{aligned} P &= rr'' + r\kappa \sin\varphi \cos\theta - \sin^2\varphi = rQ - \sin^2\varphi, \\ Q &= r'' + \kappa \sin\varphi \cos\theta. \end{aligned} \tag{2.9}$$

Remark 2.3. Due to regularity, we see that $P \neq 0$ everywhere by (2.8).

From Lemma 2.2, the Gaussian curvature K and the mean curvature H of $\mathbb{M}$ are given by, respectively

$$K = \frac{LN - M^2}{EG - F^2} = \frac{Q}{rP}, \tag{2.10}$$

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{2P + \sin^2\varphi}{-2rP}. \tag{2.11}$$

From (2.9)-(2.11), we have the following important theorem.

Theorem 2.4. *The Gaussian curvature K and the mean curvature H of canal surfaces satisfy*

$$H = -\frac{1}{2}(Kr + \frac{1}{r}). \tag{2.12}$$

Then, the principal curvatures κ_1, κ_2 are given by

$$\kappa_1 = -Kr, \quad \kappa_2 = -\frac{1}{r}. \tag{2.13}$$

In [3], one of present authors studied the second Gaussian curvature K_{II} which is analogous to the Gaussian curvature derived from the non-degenerate second fundamental form regarded as a new Riemannian (or pseudo-Riemannian) metric. The definition of the second Gaussian curvature is as follows:

$$K_{II} = \frac{1}{(LN-M^2)^2}\left\{\begin{vmatrix} -\frac{1}{2}L_{\theta\theta}+M_{s\theta}-\frac{1}{2}N_{ss} & \frac{1}{2}L_s & M_s-\frac{1}{2}L_\theta \\ M_\theta-\frac{1}{2}N_s & L & M \\ \frac{1}{2}N_\theta & M & N \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}L_\theta & \frac{1}{2}N_s \\ \frac{1}{2}L_\theta & L & M \\ \frac{1}{2}N_s & M & N \end{vmatrix}\right\}.$$

From (2.8), the second Gaussian curvature K_{II} of $\mathbb{M}$ can be written as

$$K_{II} = \frac{1}{r^2P^2Q^2}\sum_{i=0}^{4} w_i \cos^i\theta, \tag{2.14}$$

where the coefficients w_i $(i=0,1,2,3,4)$ are given by

(2.15)
$$\begin{aligned}
w_0 =& -r^3\varphi'^4\sin^4\varphi + \frac{1}{4}r\varphi'^2\sin^4\varphi(1-3\sin^2\varphi) + \frac{3}{2}r^2\varphi'^3\sin^5\varphi - \frac{1}{4}r\kappa^2\sin^6\varphi \\
&- \frac{1}{4}r\varphi''\sin^5\varphi\cos\varphi + \frac{1}{4}\varphi'\sin^5\varphi\cos^2\varphi - \frac{1}{4}r\kappa\tau\sin^5\varphi\cos\varphi\sin\theta; \\
w_1 =& -4r^3\kappa\varphi'^3\sin^4\varphi + \frac{9}{2}r^2\kappa\varphi'^2\sin^5\varphi + \frac{1}{4}r\kappa\varphi'\sin^4\varphi(1-5\sin^2\varphi) \\
&+ \frac{1}{4}\kappa\sin^5\varphi\cos^2\varphi - \frac{1}{4}r\kappa'\sin^5\varphi\cos\varphi; \\
w_2 =& -6r^3\kappa^2\varphi'^2\sin^4\varphi + \frac{9}{2}r^2\kappa^2\varphi'\sin^5\varphi - \frac{1}{4}r\kappa^2\sin^6\varphi; \\
w_3 =& -4r^3\kappa^3\varphi'\sin^4\varphi + \frac{3}{2}r^2\kappa^3\sin^5\varphi; \\
w_4 =& -r^3\kappa^4\sin^4\varphi.
\end{aligned}$$

On the other hand, we have by (2.9)

$$r^2P^2Q^2 = \sum_{j=0}^{4} v_j\cos^j\theta, \tag{2.16}$$

where v_j $(j=0,1,2,3,4)$ are functions given by

$$\begin{aligned}
v_0 &= r^4\varphi'^4\sin^4\varphi + r^2\varphi'^2\sin^6\varphi - 2r^3\varphi'^3\sin^5\varphi; \\
v_1 &= 4r^4\kappa\varphi'^3\sin^4\varphi - 6r^3\kappa\varphi'^2\sin^5\varphi + 2r^2\kappa\varphi'\sin^6\varphi; \\
v_2 &= 6r^4\kappa^2\varphi'^2\sin^4\varphi - 6r^3\kappa^2\varphi'\sin^5\varphi + r^2\kappa^2\sin^6\varphi; \\
v_3 &= 4r^4\kappa^3\varphi'\sin^4\varphi - 2r^3\kappa^3\sin^5\varphi; \\
v_4 &= r^4\kappa^4\sin^4\varphi.
\end{aligned} \tag{2.17}$$

From (2.11) and (2.14)-(2.17), we have the following theorem which can simplify some later calculations greatly.

Theorem 2.5. *The second Gaussian curvature K_{II} and the mean curvature H of non-developable canal surfaces satisfy*

$$K_{II} = H + \frac{R}{4r^2P^2Q^2}, \tag{2.18}$$

where $R = \sum_{k=0}^{2} u_k \cos^k\theta$ and the coefficients u_k $(k = 0, 1, 2)$ are as follows:

$$\begin{aligned} u_0 &= 2rr'^2r''^2\sin^2\varphi - r\kappa^2\sin^6\varphi \\ &\quad + rr'r'''\sin^4\varphi + r'^2r''\sin^4\varphi + rr'\kappa\tau\sin^5\varphi\sin\theta; \\ u_1 &= rr'^2r''\kappa\sin^3\varphi + r'^2\kappa\sin^5\varphi + rr'\kappa'\sin^5\varphi; \\ u_2 &= r\kappa^2\sin^6\varphi. \end{aligned}$$

Next, we compute the partial derivatives of the Gaussian curvature K, the mean curvature H and the second Gaussian curvature K_{II} of $\mathbb{M}$ for later use.

In the first place, we have from (2.9) and (2.10)

(2.19)
$$\begin{aligned} K_s =& \frac{-2rr'\kappa^2\sin^2\varphi\cos^2\theta + (r'\kappa - r\kappa')\sin^3\varphi\cos\theta - 5rr'r''\kappa\sin\varphi\cos\theta}{r^2P^2} \\ &+ \frac{r'r''\sin^2\varphi - rr'''\sin^2\varphi - 4rr'r''^2}{r^2P^2}; \\ K_\theta =& \frac{\kappa\sin^3\varphi\sin\theta}{rP^2}. \end{aligned}$$

By (2.12) and (2.19), we have

(2.20)
$$\begin{aligned} H_s =& -\frac{1}{2}(K_s r + Kr' - \frac{r'}{r^2}) \\ =& \frac{2r^2r'\kappa^2\sin^2\varphi\cos^2\theta - (2rr'\kappa - r^2\kappa')\sin^3\varphi\cos\theta + 5r^2r'r''\kappa\sin\varphi\cos\theta}{2r^2P^2} \\ &+ \frac{-2rr'r''\sin^2\varphi + r^2r'''\sin^2\varphi + 4r^2r'r''^2 + r'\sin^4\varphi}{2r^2P^2}; \\ H_\theta =& -\frac{1}{2}rK_\theta \\ =& -\frac{\kappa\sin^3\varphi\sin\theta}{2P^2}. \end{aligned}$$

At last, we calculate the partial derivative of K_{II}. By (2.18), we have

$$\begin{aligned} (K_{II})_s &= H_s + \frac{1}{4r^4P^4Q^4}\left\{\frac{\partial R}{\partial s}(r^2P^2Q^2) - R\frac{\partial(r^2P^2Q^2)}{\partial s}\right\}; \\ (K_{II})_\theta &= H_\theta + \frac{1}{4r^4P^4Q^4}\left\{\frac{\partial R}{\partial \theta}(r^2P^2Q^2) - R\frac{\partial(r^2P^2Q^2)}{\partial \theta}\right\}, \end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
\frac{\partial R}{\partial s} &= \sum_{k=0}^{2} \frac{\partial u_k}{\partial s} \cos^k \theta; \\
\frac{\partial R}{\partial \theta} &= (-2u_2 \sin\theta + rr'\kappa\tau \sin^5 \varphi)\cos\theta - u_1 \sin\theta; \\
\frac{\partial (r^2P^2Q^2)}{\partial s} &= \sum_{j=0}^{4} \frac{\partial v_j}{\partial s} \cos^j \theta; \\
\frac{\partial (r^2P^2Q^2)}{\partial \theta} &= -\sin\theta \sum_{j=1}^{4} j v_j \cos^{j-1} \theta.
\end{aligned} \tag{2.22}$$

Definition. For a pair $(X, Y), X \neq Y$, of the curvatures K, H and K_{II} of a canal surface $\mathbb{M}$, if $\mathbb{M}$ satisfies

$$\Phi(X, Y) = 0,$$

then it is said to be a (X, Y)-Weingarten canal surface, where Φ is the Jacobi function defined by $\Phi = XY - YX$.

Definition. For a pair $(X, Y), X \neq Y$, of the curvatures K, H and K_{II} of a canal surface $\mathbb{M}$, if $\mathbb{M}$ satisfies

$$aX + bY = c,$$

then it is said to be a (X, Y)-linear Weingarten canal surface, where $a, b, c \in R$ and $(a, b, c) \neq (0, 0, 0)$ (cf. [1], [5], [6], [9]).

Remark 2.6. The (X, Y)-linear Weingarten canal surfaces can be considered as a natural generalization of canal surfaces with constant Gaussian curvature, constant mean curvature or constant second Gaussian curvature.

3. (X, Y)-Weingarten canal surfaces

Theorem 3.1. *A canal surface $\mathbb{M}$ is a (K, H)-Weingarten canal surface if and only if it is a tube or a surface of revolution.*

Proof. A (K, H)-Weingarten canal surface $\mathbb{M}$ satisfies Jacobi equation

$$H_s K_\theta = H_\theta K_s. \tag{3.1}$$

From (2.20), we have

$$(Kr' - \frac{r'}{r^2})K_\theta = 0. \tag{3.2}$$

By (3.2), we consider the subset $\mathcal{O} = \{p \in \mathbb{M} \mid K_\theta(p) = 0\}$ of $\mathbb{M}$. Suppose that $\mathcal{O}$ has a non-empty interior $\mathcal{O}_1$. On $\mathcal{O}_1$, from (2.19) and $\sin\varphi \neq 0$ (or else $P = 0$, $\mathbb{M}$ is not regular), we have $\kappa = 0$. Thus, $\mathcal{O}_1$ is an open part of a surface of revolution. By continuity, $\mathcal{O}_1$ must be $\mathbb{M}$. Now, we may assume $K_\theta \neq 0$

everywhere on $\mathbb{M}$. Then, we have $Kr' = \frac{r'}{r^2}$ from (3.2). Note that $K \neq \frac{1}{r^2}$, otherwise, $\mathbb{M}$ is not regular. Hence, $r' = 0$ on $\mathbb{M}$. Thus, $\mathbb{M}$ is a tube.

Conversely, if $\mathbb{M}$ is a surface of revolution (i.e., $\kappa = 0$), then from (2.9)-(2.11), we have

$$K = \frac{r''}{r(rr'' - 1 + r'^2)}, \quad H = \frac{2rr'' - 1 + r'^2}{-2r(rr'' - 1 + r'^2)}. \tag{3.3}$$

Thus, their partial derivatives are given by

$$\begin{aligned} K_s =& \frac{1}{r^2}\{(\frac{rr''}{rr'' - 1 + r'^2})' - \frac{2r'r''}{rr'' - 1 + r'^2}\}, \\ H_s =& -\frac{1}{2r^2}\{r(\frac{rr''}{rr'' - 1 + r'^2} + 1)\}', \\ K_\theta =& H_\theta = 0. \end{aligned} \tag{3.4}$$

From (3.4), the Jacobi equation (3.1) turns into an identity, obviously.

On the other hand, if $\mathbb{M}$ is a tube (i.e., r is a constant), then from (2.9)-(2.11), we have

$$K = \frac{\kappa\cos\theta}{r(r\kappa\cos\theta - 1)}, \quad H = \frac{2r\kappa\cos\theta - 1}{-2r(r\kappa\cos\theta - 1)}. \tag{3.5}$$

Therefore, their partial derivatives are given by

$$K_s = \frac{-\kappa'\cos\theta}{rP^2}, \quad K_\theta = \frac{\kappa\sin\theta}{rP^2}; \tag{3.6}$$

$$H_s = \frac{\kappa'\cos\theta}{2P^2}, \quad H_\theta = -\frac{\kappa\sin\theta}{2P^2}. \tag{3.7}$$

By (3.6) and (3.7), the Jacobi equation (3.1) is satisfied everywhere. □

Theorem 3.2. *For a non-developable canal surface $\mathbb{M}$, the following statements are equivalent*:

- $\mathbb{M}$ *is locally a surface of revolution or a tube whose spine curve has a non-zero constant curvature*;
- $\mathbb{M}$ *is a (H, K_{II})-Weingarten canal surface*;
- $\mathbb{M}$ *is a (K, K_{II})-Weingarten canal surface.*

Proof. Suppose that $\mathbb{M}$ is a non-developable (H, K_{II})-Weingarten canal surface. Then, it satisfies

$$(K_{II})_s H_\theta = (K_{II})_\theta H_s. \tag{3.8}$$

By (2.21), we have

$$\begin{aligned} &H_s\left\{\frac{\partial R}{\partial\theta}(r^2P^2Q^2) - R\frac{\partial(r^2P^2Q^2)}{\partial\theta}\right\} \\ =& H_\theta\left\{\frac{\partial R}{\partial s}(r^2P^2Q^2) - R\frac{\partial(r^2P^2Q^2)}{\partial s}\right\}. \end{aligned} \tag{3.9}$$

Comparing the coefficients of the highest degree of (3.9) regarding $\cos\theta$ with the help of (2.20), (2.22), we have

$$2r'\kappa^8 \sin\varphi \sin\theta + r'^2\kappa^7\tau = 0. \tag{3.10}$$

Consider the open subset $\mathcal{O} = \{p \in \mathbb{M} \mid \kappa(p) \neq 0\}$ of $\mathbb{M}$. Suppose that $\mathcal{O}$ is not empty. On $\mathcal{O}$, $r' = 0$ by (3.10). Thus, we have by (2.16)-(2.18) and (2.22)

$$R = -r\kappa^2 \sin^2\theta, \quad r^2P^2Q^2 = r^2\kappa^2\cos^2\theta(r\kappa\cos\theta - 1)^2$$

and

$$\begin{aligned}
\frac{\partial R}{\partial s} &= -2r\kappa\kappa' \sin^2\theta;\\
\frac{\partial R}{\partial \theta} &= -2r\kappa^2 \sin\theta\cos\theta;\\
\frac{\partial(r^2P^2Q^2)}{\partial s} &= 4r^4\kappa^3\kappa'\cos^4\theta - 6r^3\kappa^2\kappa'\cos^3\theta + 2r^2\kappa\kappa'\cos^2\theta;\\
\frac{\partial(r^2P^2Q^2)}{\partial \theta} &= -4r^4\kappa^4\cos^3\theta\sin\theta + 6r^3\kappa^3\cos^2\theta\sin\theta - 2r^2\kappa^2\cos\theta\sin\theta.
\end{aligned}$$

Then, from (2.21), (3.7) and the above equations, we get

$$\begin{aligned}
(K_{II})_s &= \frac{\kappa'(r\kappa\cos^3\theta - 2\cos^2\theta + 1)}{2\cos\theta P^3};\\
(K_{II})_\theta &= \frac{-\sin\theta(r^2\kappa^2\cos^4\theta - 2r\kappa\cos^3\theta + 2r\kappa\cos\theta - 1)}{2r\cos^3\theta P^3}.
\end{aligned} \tag{3.11}$$

Substituting (3.7) and (3.11) into (3.8), we get

$$\frac{\kappa\kappa'(r\kappa\cos^3\theta - 2\cos^2\theta + 1)}{\cos\theta} = \frac{\kappa'(r^2\kappa^2\cos^4\theta - 2r\kappa\cos^3\theta + 2r\kappa\cos\theta - 1)}{r\cos^2\theta}.$$

From the above equation, we obtain $\kappa' = 0$. By continuity, $\mathcal{O}$ must be $\mathbb{M}$, which is an open part of a tube whose spine curve has non-zero constant curvature. Particularly, if the torsion of the spine curve is zero, $\mathbb{M}$ is an open part of a torus; it is an open part of a tube around a helix when the torsion is a non-zero constant. Now, suppose $\kappa \equiv 0$, then it is an open part of a surface of revolution. In this case, we should note that the radius cannot be a constant in order to guarantee the existence of the second Gaussian curvature.

Conversely, suppose that $\mathbb{M}$ is a surface of revolution or a tube whose spine curve has a non-zero constant curvature.

If $\mathbb{M}$ is a surface of revolution, we know from (2.16)-(2.18)

$$\begin{aligned}
R &= r'(1 - r'^2)[(rr'' - 1 + r'^2)'(rr'') - (rr'')'(rr'' - 1 + r'^2)],\\
r^2P^2Q^2 &= (rr'')^2(rr'' - 1 + r'^2)^2
\end{aligned}$$

and

$$K_{II} = H + \frac{r'}{4}\left(\frac{1}{rr''} - \frac{1}{rr'' - 1 + r'^2}\right)\left(\log\left|\frac{rr''}{rr'' - 1 + r'^2}\right|\right)'. \tag{3.12}$$

From (3.4) and (3.12), we get

$$\begin{aligned}(K_{II})_s = & -\frac{1}{2r^2}\{r(\frac{rr''}{rr''-1+r'^2}+1)\}' \\ & +\frac{r''}{4}(\frac{1}{rr''}-\frac{r''}{rr''-1+r'^2})(\log|\frac{rr''}{rr''-1+r'^2}|)' \\ & +\frac{r'}{4}(\frac{1}{rr''}-\frac{1}{rr''-1+r'^2})'(\log|\frac{rr''}{rr''-1+r'^2}|)' \\ & +\frac{r'}{4}(\frac{1}{rr''}-\frac{1}{rr''-1+r'^2})(\log|\frac{rr''}{rr''-1+r'^2}|)''; \\ (K_{II})_\theta = & \ 0.\end{aligned}$$

By (3.4) and the above equations, the Jacobi equation (3.8) turns into an identity. In case that $\mathbb{M}$ is a tube whose spine curve has a non-zero constant curvature, it obviously satisfies the Jacobi equation (3.8).

Using quite similar argument developed above, we have the same results for the cases of (K, K_{II})-Weingarten canal surfaces and (H, K_{II})-Weingarten canal surfaces. This completes the proof. □

4. (X, Y)-linear Weingarten canal surfaces

In the first place, we discuss some special (X, Y)-linear Weingarten canal surfaces which include developable canal surfaces, minimal canal surfaces and the canal surfaces with vanishing second Gaussian curvature.

Theorem 4.1. *A canal surface $\mathbb{M}$ is developable if and only if it is a circular cylinder or a circular cone.*

Proof. $\mathbb{M}$ is developable if and only if its Gaussian curvature $K \equiv 0$. By (2.10), we have $Q \equiv 0$. From (2.9), we get

$$r''(s) + \kappa(s)\sin\varphi(s)\cos\theta = 0. \tag{4.1}$$

It follows that $r'' = 0$ and $\kappa = 0$. Therefore $r(s) = as + b$, where a, b are constants and $a \neq \pm 1$ (or else $\sin\varphi = 0$, a contradiction). Therefore, $\mathbb{M}$ is a circular cylinder ($a = 0$) or a circular cone ($a \neq 0$, $a \neq \pm 1$), respectively. □

Remark 4.2. From Theorem 4.1, a canal surface is non-developable if and only if $Q \neq 0$. Under this condition, the second Gaussian curvature can be defined.

Theorem 4.3. *A canal surface $\mathbb{M}$ is minimal if and only if it is a catenoid.*

Proof. Since $\mathbb{M}$ is minimal if and only if its mean curvature $H \equiv 0$, then (2.11) implies

$$2P + \sin^2\varphi = 0.$$

From (2.9), we get

$$2rr'' + 2r\kappa\sin\varphi\cos\theta - \sin^2\varphi = 0. \tag{4.2}$$

Therefore, one can obtain $2rr'' - \sin^2\varphi = 0$ and $r\kappa\sin\varphi = 0$. Since $r \neq 0$, $\sin\varphi \neq 0$, then $\kappa = 0$ and $\mathbb{M}$ is a surface of revolution. It is well known that the only minimal surface of revolution is the catenoid. □

It is well known that a minimal surface has vanishing second Gaussian curvature K_{II}. However, a surface with vanishing second Gaussian curvature need not to be minimal (cf. [9]).

Theorem 4.4. *A non-developable canal surface with vanishing second Gaussian curvature is a surface of revolution which satisfies*

$$(\log r'^2)' = \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1}(\log|\frac{\kappa_1}{\kappa_2}|)'.$$

Proof. When $K_{II} = 0$, we have from (2.18)

$$H = -\frac{R}{4r^2P^2Q^2}.$$

From (2.11), we get

$$R = 2rPQ^2(2P + \sin^2\varphi). \tag{4.3}$$

Considering the coefficient of the highest degree of (4.3) regarding $\cos\theta$, we have $\kappa = 0$. Then the canal surface is a surface of revolution.

Furthermore, by (3.3) and (3.12) we have

$$\frac{2rr'' - 1 + r'^2}{-2r(rr'' - 1 + r'^2)} + \frac{r'}{4}(\frac{1}{rr''} - \frac{1}{rr'' - 1 + r'^2})(\log|\frac{rr''}{rr'' - 1 + r'^2}|)' = 0. \tag{4.4}$$

Clearing up (4.4), we get

$$\frac{2rr'' - 1 + r'^2}{r'^2 - 1}(\log r'^2)' = (\log|\frac{rr''}{rr'' - 1 + r'^2}|)'. \tag{4.5}$$

Combining (2.13) (when $\kappa = 0$) and (4.5), we get the conclusion. □

Next, we study the traditional (X, Y)-linear Weingarten canal surfaces. Without loss of generality, we may assume $c = 1$ in $aX + bY = c$.

Theorem 4.5. *A canal surface $\mathbb{M}$ is a (K, H)-linear Weingarten canal surface if and only if it is one of the following surfaces*:

- *a tube with radius $r = -\frac{b}{2}$;*
- *a surface of revolution such as*

$$x(s, \theta) = (r(s)\sin\varphi(s)\cos\theta, r(s)\sin\varphi(s)\sin\theta, r(s)\cos\varphi(s) + s),$$

where $r(s)$ is given by (4.7).

Proof. A (K, H)-linear Weingarten canal surface implies

$$aK + bH = 1,$$

where $a, b \in R$ and $(a, b) \neq (0, 0)$.

From (2.12), we obtain

$$(2ar - br^2)K = b + 2r.$$

By (2.10), we get

$$\frac{(2ar - br^2)(r'' + \kappa \sin\varphi \cos\theta)}{r(rr'' + r\kappa \sin\varphi \cos\theta - \sin^2\varphi)} = b + 2r, \tag{4.6}$$

from which,

$$2\kappa(r^2 + br - a)\sin\varphi\cos\theta + 2(r^2 + br - a)r'' - (b + 2r)(1 - r'^2) = 0.$$

Therefore, we get

$$\kappa(r^2 + br - a)\sin\varphi = 0, \quad 2(r^2 + br - a)r'' - (b + 2r)(1 - r'^2) = 0.$$

Case 1: If $r^2 + br - a \neq 0$, then $\kappa = 0$. Thus, $\mathbb{M}$ is a surface of revolution and its radial function satisfies

$$2(r^2 + br - a)r'' = (b + 2r)(1 - r'^2).$$

Solving the above equation, we get

$$s = c_2 \pm \int \sqrt{\frac{r^2 + br - a}{r^2 + br - a - c_1}}dr, \tag{4.7}$$

where c_1, c_2 are constants (cf. [8]).

Since $\kappa = 0$, without loss of generality, we may assume the spine curve is $c(s) = (0, 0, s)$ and $T = (0, 0, 1), N = (1, 0, 0), B = (0, 1, 0)$, respectively. Then, $\mathbb{M}$ can be expressed by

$$x(s, \theta) = (r(s)\sin\varphi(s)\cos\theta, r(s)\sin\varphi(s)\sin\theta, r(s)\cos\varphi(s) + s),$$

where $r(s)$ is given by (4.7).

Case 2: If $\kappa \neq 0$, then $r^2 + br - a = 0$. Hence, $r = -\frac{b}{2}$ is a non-zero constant, $\mathbb{M}$ is a tube and a, b satisfy $b^2 + 4a = 0$.

Note that $\mathbb{M}$ is a circular cylinder if $\kappa = r^2 + br - a \equiv 0$. □

Corollary 4.6. *The canal surface* $\mathbb{M}$ *which has non-zero constant Gaussian curvature is a surface of revolution such as*

$$x(s, \theta) = (r(s)\sin\varphi(s)\cos\theta, r(s)\sin\varphi(s)\sin\theta, r(s)\cos\varphi(s) + s),$$

where $r(s)$ *is given by* (4.8).

Proof. By Remark 2.6 and Theorem 4.5 with $b = 0$, $\mathbb{M}$ has non-zero constant Gaussian curvature ($K = \frac{1}{a}$). Obviously, $\mathbb{M}$ cannot be a tube and it is a surface of revolution. By a similar development as was given in Theorem 4.5, it can be expressed by

$$x(s, \theta) = (r(s)\sin\varphi(s)\cos\theta, r(s)\sin\varphi(s)\sin\theta, r(s)\cos\varphi(s) + s),$$

in which, $r(s)$ is given by

$$s = c_2 \pm \int \sqrt{\frac{r^2 - a}{r^2 + c_1}} dr, \tag{4.8}$$

where c_1, c_2 are constants (cf. [8]). $\square$

Corollary 4.7. *The canal surface* $\mathbb{M}$ *which has non-zero constant mean curvature is a surface of revolution such as*

$$x(s,\theta) = (r(s)\sin\varphi(s)\cos\theta, r(s)\sin\varphi(s)\sin\theta, r(s)\cos\varphi(s) + s),$$

where $r(s)$ *is given by* (4.9).

Proof. By Remark 2.6 and Theorem 4.5 with $a = 0$, $\mathbb{M}$ has non-zero constant mean curvature ($H = \frac{1}{b}$). Similarly as Corollary 4.6, it is a surface of revolution and it can be expressed by

$$x(s,\theta) = (r(s)\sin\varphi(s)\cos\theta, r(s)\sin\varphi(s)\sin\theta, r(s)\cos\varphi(s) + s),$$

in which, $r(s)$ is given by

$$s = c_2 \pm \int \sqrt{\frac{r^2 + br}{r^2 + br - c_1}} dr, \tag{4.9}$$

where c_1, c_2 are constants (cf. [8]). $\square$

Theorem 4.8. *A non-developable canal surface* $\mathbb{M}$ *is a* (H, K_{II})*-linear Weingarten canal surface if and only if it is an open part of a surface of revolution which satisfies*

$$\frac{1}{a}(\log r'^2)' = \frac{\kappa_2 - \kappa_1}{(a+b)(\kappa_2 + \kappa_1) - 2}(\log|\frac{\kappa_1}{\kappa_2}|)',$$

where $a, b \in R$ *and* $(a, b) \neq (0, 0)$.

Proof. Suppose $\mathbb{M}$ is a (H, K_{II})-linear Weingarten canal surface. It implies

$$aK_{II} + bH = 1,$$

where $a, b \in R$ and $(a, b) \neq (0, 0)$.

From (2.11) and (2.14), we have

$$a\frac{\sum_{i=0}^{4} w_i \cos^i\theta}{r^2P^2Q^2} + b\frac{2P + \sin^2\varphi}{-2rP} = 1.$$

That is,

$$2a\sum_{i=0}^{4} w_i \cos^i\theta - brPQ^2(2P + \sin^2\varphi) = 2\sum_{j=0}^{4} v_j \cos^j\theta. \tag{4.10}$$

Comparing the coefficient of $\cos^4\theta$ in (4.10), we have

$$2aw_4 - 2br^3\kappa^4\sin^4\varphi = 2v_4.$$

By (2.15) and (2.17), we get

$$\kappa^4(r+a+b)=0.$$

Consider the open subset $\mathcal{O}=\{p\in\mathbb{M}\mid\kappa(p)\neq 0\}$ of $\mathbb{M}$. Suppose $\mathcal{O}\neq\emptyset$. Let $\mathcal{O}_1$ be a component of $\mathcal{O}$. On $\mathcal{O}_1$, we have $r+a+b=0$ which means r is a constant. By comparing the other terms in (4.10), we get $\kappa\equiv 0$ which is a contradiction. Therefore, we have $\mathcal{O}_1=\emptyset$. Thus, we have $\kappa=0$ and it is a surface of revolution. From (3.3) and (3.12), we obtain

$$\begin{aligned}&\frac{ar'}{4}\left(\frac{1}{rr''}-\frac{1}{rr''-1+r'^2}\right)(\log|\frac{rr''}{rr''-1+r'^2}|)'\\ &=1+(a+b)\{\frac{2rr''-1+r'^2}{2r(rr''-1+r'^2)}\}.\end{aligned}$$

Clearing up the above equation by (2.13) (when $\kappa=0$), we get

$$\frac{1}{a}(\log r'^2)'=\frac{\kappa_2-\kappa_1}{(a+b)(\kappa_2+\kappa_1)-2}(\log|\frac{\kappa_1}{\kappa_2}|)'.$$

□

Theorem 4.9. *A non-developable canal surface $\mathbb{M}$ is a (K,K_{II})-linear Weingarten canal surface if and only if it is an open part of a surface of revolution which satisfies*

$$\frac{1}{a}(\log r'^2)'=\frac{\kappa_2-\kappa_1}{a(\kappa_2+\kappa_1)+2b\kappa_1\kappa_2-2}(\log|\frac{\kappa_1}{\kappa_2}|)',$$

where $a,b\in R$ and $(a,b)\neq(0,0)$.

Proof. Suppose that $\mathbb{M}$ is a (K,K_{II})-linear Weingarten canal surface. Then, we have

$$aK_{II}+bK=1,$$

where $a,b\in R$ and $(a,b)\neq(0,0)$.

From (2.10) and (2.14), we have

$$a\frac{\sum_{i=0}^{4}w_i\cos^i\theta}{r^2P^2Q^2}+b\frac{Q}{rP}=1,$$

from which,

$$a\sum_{i=0}^{4}w_i\cos^i\theta+brPQ^3=\sum_{j=0}^{4}v_j\cos^j\theta. \tag{4.11}$$

Comparing the coefficient of $\cos^4\theta$ in (4.11), we have

$$aw_4+br^2\kappa^4\sin^4\varphi=v_4.$$

With the help of (2.15) and (2.17), we have

$$\kappa^4(r^2+ar-b)=0.$$

By a similar argument as in Theorem 4.8, we get the conclusion. □

In Theorem 4.8 (or Theorem 4.9), we put $b=0$, then the canal surface has non-zero constant second Gaussian curvature ($K_{II}=\frac{1}{a}$).

Corollary 4.10. *The canal surface $\mathbb{M}$ which has non-zero constant second Gaussian curvature is an open part of a surface of revolution which satisfies*

$$\frac{1}{a}(\log r'^2)' = \frac{\kappa_2-\kappa_1}{a(\kappa_2+\kappa_1)-2}(\log|\frac{\kappa_1}{\kappa_2}|)',$$

where $a(a\neq 0)\in R$.

At last, we study the (X,Y)-linear Weingarten canal surfaces which satisfy $K_{II}=K$ and $K_{II}=H$, respectively.

Theorem 4.11. *A non-developable canal surface satisfying $K_{II}=K$ is a surface of revolution which satisfies*

$$(\log r'^2)' = \frac{\kappa_2-\kappa_1}{(\kappa_2+\kappa_1)-2\kappa_1\kappa_2}(\log|\frac{\kappa_1}{\kappa_2}|)'.$$

Proof. When $K_{II}=K$, we have by (2.10) and (2.18)

$$\frac{2P+\sin^2\varphi}{-2rP}+\frac{R}{4r^2P^2Q^2}=\frac{Q}{rP}. \tag{4.12}$$

Comparing the coefficient of the highest degree of (4.12) regarding $\cos\theta$, we have $\kappa=0$. Then the canal surface is a surface of revolution.

By (3.3) and (3.12), we have

$$\frac{2rr''-1+r'^2}{-2r(rr''-1+r'^2)}+\frac{r'}{4}(\frac{1}{rr''}-\frac{1}{rr''-1+r'^2})(\log|\frac{rr''}{rr''-1+r'^2}|)'$$
$$=\frac{r''}{r(rr''-1+r'^2)}.$$

From (2.13) (when $\kappa=0$), we obtain the conclusion. □

Theorem 4.12. *For a non-developable canal surface satisfying $K_{II}=H$, the following conclusions are equivalent*:

- *the ratio of the principal curvatures is constant*;
- *the canal surface is a surface of revolution parametrized by*

$$x(s,\theta)=(r(s)\sin\varphi\cos\theta, r(s)\sin\varphi\sin\theta, r(s)\cos\varphi+s),$$

where $r(s)$ is given by (4.16).

Proof. When $K_{II}=H$, we have $R=0$ by (2.18). Considering the coefficient of $\cos^2\theta$ in R, we get $\kappa=0$. Then the canal surface is a surface of revolution.

Since $\kappa=0$, we have from (3.12)

$$\frac{r'}{4}(\frac{1}{rr''}-\frac{1}{rr''-1+r'^2})(\log|\frac{rr''}{rr''-1+r'^2}|)'=0. \tag{4.13}$$

By (4.13) and the canal surface is non-developable, we get

$$(\log|\frac{rr''}{rr''-1+r'^2}|)'=0. \tag{4.14}$$

Equation (4.14) implies

$$\frac{rr''}{rr''-1+r'^2}=c, \tag{4.15}$$

where c ($c\neq 1$) is a constant. From (2.13) (when $\kappa=0$), the ratio of the principal curvatures is a constant.

Solving (4.15), we get

$$s=c_2+\int\frac{(c-1)dv}{c(1-v^2)|1-v^2|^{\frac{c_1(c-1)}{2c}}},\quad r=\frac{1}{|1-v^2|^{\frac{c_1(c-1)}{2c}}}, \tag{4.16}$$

where c_1 and c_2 are constants (cf. [8]).

Similarly as Theorem 4.5, the canal surface is parametrized by

$$x(s,\theta)=(r(s)\sin\varphi(s)\cos\theta,r(s)\sin\varphi(s)\sin\theta,r(s)\cos\varphi(s)+s),$$

where $r(s)$ is given by (4.16). □

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