Bull. Korean Math. Soc. **53** (2016), No. 2, pp. 441–460
http://dx.doi.org/10.4134/BKMS.2016.53.2.441

H-V-SEMI-SLANT SUBMERSIONS FROM ALMOST QUATERNIONIC HERMITIAN MANIFOLDS

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Abstract. We introduce the notions of h-v-semi-slant submersions and almost h-v-semi-slant submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds. We obtain characterizations, investigate the integrability of distributions, the geometry of foliations, and a decomposition theorem. We find a condition for such submersions to be totally geodesic. We also obtain an inequality of a h-v-semi-slant submersion in terms of squared mean curvature, scalar curvature, and h-v-semi-slant angle. Finally, we give examples of such maps.

1. Introduction

Given a C^∞-submersion F from a (semi-)Riemannian manifold (M, g_M) onto a (semi-)Riemannian manifold (N, g_N), according to the conditions on the map $F : (M, g_M) \mapsto (N, g_N)$, we can obtain the following: a semi-Riemannian submersion and a Lorentzian submersion [8], a Riemannian submersion ([9], [16]), a slant submersion ([6], [22]), an almost Hermitian submersion [24], a contact-complex submersion [10], a quaternionic submersion [11], an almost h-slant submersion [17], a semi-invariant submersion [23], an almost h-semi-invariant submersion [18], a semi-slant submersion [21], an almost h-semi-slant submersions [19], a v-semi-slant submersions [20], etc.

As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([4], [25]), Kaluza-Klein theory ([5], [12]), Supergravity and superstring theories ([13], [15]), etc. And the quaternionic Kähler manifolds have applications in physics as the target spaces for nonlinear σ-models with supersymmetry [7].

As a generalization of v-semi-slant submersions, we will define the notions of h-v-semi-slant submersions and almost h-v-semi-slant submersions.

The paper is organized as follows. In Section 2 we recall some notions, which are needed in the following sections. In Section 3 we give the definitions of

Received December 19, 2014.
2010 *Mathematics Subject Classification.* 53C15, 53C26.
Key words and phrases. Riemannian submersion, slant angle, integrable, totally geodesic.
This research was supported by Sogang Research Team for Discrete and Geometric Structures.

h-v-semi-slant submersions and almost h-v-semi-slant submersions and we obtain some results on them: characterizations, the integrability of distributions, the equivalent conditions for distributions to be totally geodesic foliations, the equivalent conditions for such maps to be totally geodesic, etc. In Section 4 we consider an inequality of a h-v-semi-slant submersion in terms of squared mean curvature, scalar curvature, and h-v-semi-slant angle. In Section 5 we give some examples of h-v-semi-slant submersions and almost h-v-semi-slant submersions.

2. Preliminaries

Let (M, g_M) and (N, g_N) be Riemannian manifolds, where g_M and g_N are Riemannian metrics on C^∞-manifolds M and N, respectively.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a C^∞-map.

The *second fundamental form* of F is given by

$$(\nabla F_*)(X, Y) := \nabla^F_X F_* Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where ∇^F is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N [2].

Recall that F is said to be *harmonic* if $trace(\nabla F_*) = 0$ and F is called a *totally geodesic* map if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [2].

We call the map F a *C^∞-submersion* if F is surjective and the differential $(F_*)_p$ has maximal rank for any $p \in M$.

The map F is said to be a *Riemannian submersion* ([16], [8]) if F is a C^∞-submersion and

$$(F_*)_p : ((\ker(F_*)_p)^\perp, (g_M)_p) \mapsto (T_{F(p)}N, (g_N)_{F(p)})$$

is a linear isometry for any $p \in M$, where $(\ker(F_*)_p)^\perp$ is the orthogonal complement of the space $\ker(F_*)_p$ in the tangent space T_pM to M at p.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a Riemannian submersion.

For any vector field $U \in \Gamma(TM)$, we have

$$U = \mathcal{V}U + \mathcal{H}U,$$

where $\mathcal{V}U \in \Gamma(\ker F_*)$ and $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$.

Define the (O'Neill) tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F,$$
$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

for vector fields $E, F \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of g_M ([16], [8]).

Define $\widehat{\nabla}_X Y := \mathcal{V}\nabla_X Y$ for $X, Y \in \Gamma(\ker F_*)$.

Let (M, g_M, J) be an almost Hermitian manifold, where J is an almost complex structure on M.

A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *semi-slant submersion* if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$ [21].

We call the angle θ a *semi-slant angle*.

A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *v-semi-slant submersion* if there is a distribution $\mathcal{D}_1 \subset (\ker F_*)^\perp$ such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$ [20].

We call the angle θ a *v-semi-slant angle*.

Let M be a $4m$-dimensional C^∞-manifold and let E be a rank 3 subbundle of $\mathrm{End}(TM)$ such that for any point $p \in M$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call E an *almost quaternionic structure* on M and (M, E) an *almost quaternionic manifold* [1].

Moreover, let g be a Riemannian metric on M such that for any point $p \in M$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{2.1}$$

$$g(J_\alpha X, J_\alpha Y) = g(X, Y) \tag{2.2}$$

for all vector fields $X, Y \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call (M, E, g) an *almost quaternionic Hermitian manifold* [11].

Conveniently, the above basis $\{J_1, J_2, J_3\}$ satisfying (2.1) and (2.2) is said to be a *quaternionic Hermitian basis*.

Let (M, E, g) be an almost quaternionic Hermitian manifold.

We call (M, E, g) a *quaternionic Kähler manifold* if there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1, 2, 3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}$$

for any vector field $X \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3 [11].

If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on M (i.e., $\nabla J_\alpha = 0$ for $\alpha \in \{1,2,3\}$, where ∇ is the Levi-Civita connection of the metric g), then (M, E, g) is said to be a *hyperkähler manifold*. Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on M and g a *hyperkähler metric* [3].

Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds.

A map $F : M \mapsto N$ is called a (E_M, E_N)-*holomorphic map* if given a point $x \in M$, for any $J \in (E_M)_x$ there exists $J' \in (E_N)_{F(x)}$ such that

$$F_* \circ J = J' \circ F_*.$$

A Riemannian submersion $F : M \mapsto N$ which is a (E_M, E_N)-holomorphic map is called a *quaternionic submersion* [11].

Moreover, if (M, E_M, g_M) is a quaternionic Kähler manifold (or a hyperkähler manifold), then we say that F is a *quaternionic Kähler submersion* (or a *hyperkähler submersion*) [11].

It is well-known that any quaternionic Kähler submersion is a harmonic map [11].

Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is said to be an *almost h-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R = \theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$ [17].

We call such a basis $\{I, J, K\}$ an *almost h-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-slant angles*.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R = \theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$, and $\theta = \theta_I = \theta_J = \theta_K$ [17].

We call such a basis $\{I, J, K\}$ a *h-slant basis* and the angle θ a *h-slant angle*.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-semi-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1 \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ R(\mathcal{D}_1) = \mathcal{D}_1,\ R(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$ [18].

We call such a basis $\{I, J, K\}$ a *h-semi-invariant basis*.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-semi-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R, \ R(\mathcal{D}_2^R) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$ [18].

We call such a basis $\{I, J, K\}$ an *almost h-semi-invariant basis.*

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-semi-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1 \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ R(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in U$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$ [19].

We call such a basis $\{I, J, K\}$ a *h-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *h-semi-slant angles.*

Furthermore, if we have

$$\theta = \theta_I = \theta_J = \theta_K,$$

then we call the map $F : (M, E, g_M) \mapsto (N, g_N)$ a *strictly h-semi-slant submersion*, $\{I, J, K\}$ a *strictly h-semi-slant basis*, and the angle θ a *strictly h-semi-slant angle* [19].

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-semi-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$ [19].

We call such a basis $\{I, J, K\}$ an *almost h-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-semi-slant angles.*

Throughout this paper, we will use the above notations.

3. H-v-semi-slant submersions

Definition 3.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-v-semi-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1 \subset (\ker F_*)^\perp$ on U such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \ R(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in U$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$.

We call such a basis $\{I, J, K\}$ a *h-v-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *h-v-semi-slant angles.*

Furthermore, if we have

$$\theta = \theta_I = \theta_J = \theta_K,$$

then we call the map $F : (M, E, g_M) \mapsto (N, g_N)$ a *strictly h-v-semi-slant submersion*, $\{I, J, K\}$ a *strictly h-v-semi-slant basis*, and the angle θ a *strictly h-v-semi-slant angle.*

Definition 3.2. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-v-semi-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset (\ker F_*)^\perp$ on U such that

$$(\ker F_*)^\perp = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $(\ker F_*)^\perp$.

We call such a basis $\{I, J, K\}$ an *almost h-v-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-v-semi-slant angles.*

Remark 3.3. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an almost h-v-semi-slant submersion. Then given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset (\ker F_*)^\perp$ on U such that

$$(\ker F_*)^\perp = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $(\ker F_*)^\perp$.

If $\mathcal{D}_2^R = (\ker F_*)^\perp$ for $R \in \{I, J, K\}$, then we call the map F an *almost h-v-slant submersion* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-v-slant angles* [17]. Otherwise, if $\theta_R = \frac{\pi}{2}$ for $R \in \{I, J, K\}$, then we call the map F an *almost h-v-semi-invariant submersion* [18].

Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an almost h-v-semi-slant submersion. Then given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset (\ker F_*)^\perp$ on U such that

$$(\ker F_*)^\perp = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $(\ker F_*)^\perp$.

Then for $X \in \Gamma((\ker F_*)^\perp)$, we have

$$X = P_R X + Q_R X,$$

where $P_R X \in \Gamma(\mathcal{D}_1^R)$ and $Q_R X \in \Gamma(\mathcal{D}_2^R)$.

For $X \in \Gamma(\ker F_*)$, we get

$$RX = \phi_R X + \omega_R X,$$

where $\phi_R X \in \Gamma(\ker F_*)$ and $\omega_R X \in \Gamma((\ker F_*)^\perp)$.

For $Z \in \Gamma((\ker F_*)^\perp)$, we obtain

$$RZ = B_R Z + C_R Z,$$

where $B_R Z \in \Gamma(\ker F_*)$ and $C_R Z \in \Gamma((\ker F_*)^\perp)$.

Then

$$\ker F_* = B_R \mathcal{D}_2^R \oplus \mu_R,$$

where μ_R is the orthogonal complement of $B_R \mathcal{D}_2^R$ in $\ker F_*$ and is invariant under R.

Furthermore,

$$\begin{aligned}
&C_R \mathcal{D}_1^R = \mathcal{D}_1^R,\ B_R \mathcal{D}_1^R = 0,\ C_R \mathcal{D}_2^R \subset \mathcal{D}_2^R,\ \omega_R(\ker F_*) = \mathcal{D}_2^R,\\
&\phi_R^2 + B_R \omega_R = -id,\ C_R^2 + \omega_R B_R = -id,\\
&\omega_R \phi_R + C_R \omega_R = 0,\ B_R C_R + \phi_R B_R = 0.
\end{aligned}$$

Then it is easy to have:

Lemma 3.4. *Let F be an almost h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-v-semi-slant basis. Then we get*

(1)

$$\begin{aligned}
\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \omega_R Y &= \phi_R \widehat{\nabla}_X Y + B_R \mathcal{T}_X Y,\\
\mathcal{T}_X \phi_R Y + \mathcal{H}\nabla_X \omega_R Y &= \omega_R \widehat{\nabla}_X Y + C_R \mathcal{T}_X Y
\end{aligned}$$

for $X, Y \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$.

(2)

$$\begin{aligned}
\mathcal{V}\nabla_Z B_R W + \mathcal{A}_Z C_R W &= \phi_R \mathcal{A}_Z W + B_R \mathcal{H}\nabla_Z W,\\
\mathcal{A}_Z B_R W + \mathcal{H}\nabla_Z C_R W &= \omega_R \mathcal{A}_Z W + C_R \mathcal{H}\nabla_Z W
\end{aligned}$$

for $Z, W \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$.

(3)

$$\begin{aligned}
\widehat{\nabla}_X B_R Z + \mathcal{T}_X C_R Z &= \phi_R \mathcal{T}_X Z + B_R \mathcal{H}\nabla_X Z,\\
\mathcal{T}_X B_R Z + \mathcal{H}\nabla_X C_R Z &= \omega_R \mathcal{T}_X Z + C_R \mathcal{H}\nabla_X Z
\end{aligned}$$

for $X \in \Gamma(\ker F_*)$, $Z \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$.

Theorem 3.5. *Let F be a h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-v-semi-slant basis. Then the following conditions are equivalent*:

a) *the slant distribution $\mathcal{D}_2$ is integrable.*
b) $\mathcal{A}_X Y = 0$ *and* $P_I((\mathcal{A}_X B_I Y - \mathcal{A}_Y B_I X) + \mathcal{H}(\nabla_X C_I Y - \nabla_Y C_I X)) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_2)$.
c) $\mathcal{A}_X Y = 0$ *and* $P_J((\mathcal{A}_X B_J Y - \mathcal{A}_Y B_J X) + \mathcal{H}(\nabla_X C_J Y - \nabla_Y C_J X)) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_2)$.
d) $\mathcal{A}_X Y = 0$ *and* $P_K((\mathcal{A}_X B_K Y - \mathcal{A}_Y B_K X) + \mathcal{H}(\nabla_X C_K Y - \nabla_Y C_K X)) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_2)$.

Proof. Given $X, Y \in \Gamma(\mathcal{D}_2)$, $Z \in \Gamma(\mathcal{D}_1)$, and $R \in \{I, J, K\}$, assume that $\mathcal{A}_X Y = 0$. Since $\mathcal{A}_X Y = \frac{1}{2}\mathcal{V}[X, Y]$, we have

$$\begin{aligned} g_M([X, Y], RZ) &= -g_M(\nabla_X RY - \nabla_Y RX, Z) \\ &= -g_M(\mathcal{V}\nabla_X B_R Y + \mathcal{A}_X B_R Y + \mathcal{A}_X C_R Y + \mathcal{H}\nabla_X C_R Y \\ &\quad - \mathcal{V}\nabla_Y B_R X - \mathcal{A}_Y B_R X - \mathcal{A}_Y C_R X - \mathcal{H}\nabla_Y C_R X, Z) \\ &= g_M(\mathcal{A}_X B_R Y + \mathcal{H}\nabla_X C_R Y - \mathcal{A}_Y B_R X - \mathcal{H}\nabla_Y C_R X, Z). \end{aligned}$$

Thus, a) $\Leftrightarrow$ $\mathcal{A}_X Y = 0$ and $P_R((\mathcal{A}_X B_R Y - \mathcal{A}_Y B_R X) + \mathcal{H}(\nabla_X C_R Y - \nabla_Y C_R X)) = 0$ for $X, Y \in \Gamma(\mathcal{D}_2)$.

Hence, we have

$$\text{a)} \Leftrightarrow \text{b)}, \quad \text{a)} \Leftrightarrow \text{c)}, \quad \text{a)} \Leftrightarrow \text{d)}.$$

Therefore, we get the result. □

In a similar way, we have:

Theorem 3.6. *Let F be a h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-v-semi-slant basis. Then the following conditions are equivalent*:

a) *the complex distribution $\mathcal{D}_1$ is integrable.*
b) $\mathcal{A}_X Y = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_1)$.

Proposition 3.7. *Let F be an almost h-v-semi-slant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Then we get*

$$C_R^2 X = -\cos^2\theta_R X \quad \text{for } X \in \Gamma(\mathcal{D}_2^R) \text{ and } R \in \{I, J, K\},$$

where $\{I, J, K\}$ is an almost h-v-semi-slant basis with the almost h-v-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$.

Proof. Since

$$\cos\theta_R = \frac{g_M(RX, C_R X)}{||RX|| \cdot ||C_R X||} = \frac{-g_M(X, C_R^2 X)}{||X|| \cdot ||C_R X||}$$

and $\cos\theta_R = \dfrac{||C_R X||}{||RX||}$, we obtain

$$\cos^2\theta_R = -\frac{g_M(X, C_R^2 X)}{||X||^2} \quad \text{for } X \in \Gamma(\mathcal{D}_2^R).$$

Hence,

$$C_R^2 X = -\cos^2\theta_R X \quad \text{for } X \in \Gamma(\mathcal{D}_2^R). \qquad \square$$

Remark 3.8. Particularly, it is easy to see that the converse of Proposition 3.7 is also true.

Remark 3.9. Let F be an almost h-v-semi-slant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N).

By Proposition 3.7, we can get

$$g_M(C_R X, C_R Y) = \cos^2\theta_R g_M(X, Y),$$
$$g_M(B_R X, B_R Y) = \sin^2\theta_R g_M(X, Y)$$

for $X, Y \in \Gamma(\mathcal{D}_2^R)$ and $R \in \{I, J, K\}$ so that for any $\theta_R \in [0, \frac{\pi}{2})$, there exists a local orthonormal frame $\{X_1, \sec\theta_R C_R X_1, \ldots, X_k, \sec\theta_R C_R X_k\}$ of $\mathcal{D}_2^R$.

Assume that we have an almost h-v-semi-slant angle $\theta_R \in [0, \frac{\pi}{2})$ for some $R \in \{I, J, K\}$ and define an endomorphism $\widehat{R}$ of $(\ker F_*)^\perp$ by

$$\widehat{R} := RP_R + \frac{1}{\cos\theta_R} C_R Q_R.$$

Then,

$$\widehat{R}^2 = -id \quad \text{on } (\ker F_*)^\perp. \tag{3.1}$$

From (3.1), we get:

Theorem 3.10. *Let F be an almost h-v-semi-slant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N) such that $\{\theta_I, \theta_J, \theta_K\} \cap [0, \frac{\pi}{2}) \neq \varnothing$, where $\{\theta_I, \theta_J, \theta_K\}$ are almost h-v-semi-slant angles. Then N is an even-dimensional manifold.*

Proposition 3.11. *Let F be an almost h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-v-semi-slant basis. Then the following conditions are equivalent*:

a) *the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation.*
b) $\phi_I(\mathcal{V}\nabla_X B_I Y + \mathcal{A}_X C_I Y) + B_I(\mathcal{A}_X B_I Y + \mathcal{H}\nabla_X C_I Y) = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.
c) $\phi_J(\mathcal{V}\nabla_X B_J Y + \mathcal{A}_X C_J Y) + B_J(\mathcal{A}_X B_J Y + \mathcal{H}\nabla_X C_J Y) = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.
d) $\phi_K(\mathcal{V}\nabla_X B_K Y + \mathcal{A}_X C_K Y) + B_K(\mathcal{A}_X B_K Y + \mathcal{H}\nabla_X C_K Y) = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$,

$$\begin{aligned}\nabla_X Y = & -R\nabla_X RY \\ = & -R(\mathcal{V}\nabla_X B_R Y + \mathcal{A}_X B_R Y + \mathcal{A}_X C_R Y + \mathcal{H}\nabla_X C_R Y) \\ = & -(\phi_R \mathcal{V}\nabla_X B_R Y + \omega_R \mathcal{V}\nabla_X B_R Y + B_R \mathcal{A}_X B_R Y + C_R \mathcal{A}_X B_R Y \\ & + \phi_R \mathcal{A}_X C_R Y + \omega_R \mathcal{A}_X C_R Y + B_R \mathcal{H}\nabla_X C_R Y + C_R \mathcal{H}\nabla_X C_R Y).\end{aligned}$$

Thus,

$$\begin{aligned}& \nabla_X Y \in \Gamma((\ker F_*)^\perp) \\ \Leftrightarrow\ & \phi_R(\mathcal{V}\nabla_X B_R Y + \mathcal{A}_X C_R Y) + B_R(\mathcal{A}_X B_R Y + \mathcal{H}\nabla_X C_R Y) = 0.\end{aligned}$$

Therefore, the result follows. □

Similarly, we have:

Proposition 3.12. *Let F be an almost h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-v-semi-slant basis. Then the following conditions are equivalent*:

a) *the distribution* $\ker F_*$ *defines a totally geodesic foliation.*
b) $\omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0$ *for* $X, Y \in \Gamma(\ker F_*)$.
c) $\omega_J(\widehat{\nabla}_X \phi_J Y + \mathcal{T}_X \omega_J Y) + C_J(\mathcal{T}_X \phi_J Y + \mathcal{H}\nabla_X \omega_J Y) = 0$ *for* $X, Y \in \Gamma(\ker F_*)$.
d) $\omega_K(\widehat{\nabla}_X \phi_K Y + \mathcal{T}_X \omega_K Y) + C_K(\mathcal{T}_X \phi_K Y + \mathcal{H}\nabla_X \omega_K Y) = 0$ *for* $X, Y \in \Gamma(\ker F_*)$.

Using Proposition 3.11 and Proposition 3.12, we have:

Theorem 3.13. *Let F be an almost h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-v-semi-slant basis. Then the following conditions are equivalent*:

a) *M is locally a product Riemannian manifold.*
b) $\omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0$
for $X, Y \in \Gamma(\ker F_*)$,
$\phi_I(\mathcal{V}\nabla_Z B_I W + \mathcal{A}_Z C_I W) + B_I(\mathcal{A}_Z B_I W + \mathcal{H}\nabla_Z C_I W) = 0$
for $Z, W \in \Gamma((\ker F_*)^\perp)$.
c) $\omega_J(\widehat{\nabla}_X \phi_J Y + \mathcal{T}_X \omega_J Y) + C_J(\mathcal{T}_X \phi_J Y + \mathcal{H}\nabla_X \omega_J Y) = 0$
for $X, Y \in \Gamma(\ker F_*)$,
$\phi_J(\mathcal{V}\nabla_Z B_J W + \mathcal{A}_Z C_J W) + B_J(\mathcal{A}_Z B_J W + \mathcal{H}\nabla_Z C_J W) = 0$
for $Z, W \in \Gamma((\ker F_*)^\perp)$.
d) $\omega_K(\widehat{\nabla}_X \phi_K Y + \mathcal{T}_X \omega_K Y) + C_K(\mathcal{T}_X \phi_K Y + \mathcal{H}\nabla_X \omega_K Y) = 0$
for $X, Y \in \Gamma(\ker F_*)$,

$\phi_K(\mathcal{V}\nabla_Z B_K W + \mathcal{A}_Z C_K W) + B_K(\mathcal{A}_Z B_K W + \mathcal{H}\nabla_Z C_K W) = 0$
for $Z, W \in \Gamma((\ker F_*)^\perp)$.

Proposition 3.14. *Let F be a h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-v-semi-slant basis. Then the following conditions are equivalent*:

a) *the distribution $\mathcal{D}_2$ defines a totally geodesic foliation.*
b) $\phi_I(\mathcal{V}\nabla_X B_I Y + \mathcal{A}_X C_I Y) + B_I(\mathcal{A}_X B_I Y + \mathcal{H}\nabla_X C_I Y) = 0$,
$P_I(\omega_I(\mathcal{V}\nabla_X B_I Y + \mathcal{A}_X C_I Y) + C_I(\mathcal{A}_X B_I Y + \mathcal{H}\nabla_X C_I Y)) = 0$
for $X, Y \in \Gamma(\mathcal{D}_2)$.
c) $\phi_J(\mathcal{V}\nabla_X B_J Y + \mathcal{A}_X C_J Y) + B_J(\mathcal{A}_X B_J Y + \mathcal{H}\nabla_X C_J Y) = 0$,
$P_J(\omega_J(\mathcal{V}\nabla_X B_J Y + \mathcal{A}_X C_J Y) + C_J(\mathcal{A}_X B_J Y + \mathcal{H}\nabla_X C_J Y)) = 0$
for $X, Y \in \Gamma(\mathcal{D}_2)$.
d) $\phi_K(\mathcal{V}\nabla_X B_K Y + \mathcal{A}_X C_K Y) + B_K(\mathcal{A}_X B_K Y + \mathcal{H}\nabla_X C_K Y) = 0$,
$P_K(\omega_K(\mathcal{V}\nabla_X B_K Y + \mathcal{A}_X C_K Y) + C_K(\mathcal{A}_X B_K Y + \mathcal{H}\nabla_X C_K Y)) = 0$
for $X, Y \in \Gamma(\mathcal{D}_2)$.

Proof. For $X, Y \in \Gamma(\mathcal{D}_2)$ and $R \in \{I, J, K\}$, we get

$$\begin{aligned}
\nabla_X Y &= -R\nabla_X RY \\
&= -R(\mathcal{V}\nabla_X B_R Y + \mathcal{A}_X B_R Y + \mathcal{A}_X C_R Y + \mathcal{H}\nabla_X C_R Y) \\
&= -(\phi_R \mathcal{V}\nabla_X B_R Y + \omega_R \mathcal{V}\nabla_X B_R Y + B_R \mathcal{A}_X B_R Y + C_R \mathcal{A}_X B_R Y \\
&\quad + \phi_R \mathcal{A}_X C_R Y + \omega_R \mathcal{A}_X C_R Y + B_R \mathcal{H}\nabla_X C_R Y + C_R \mathcal{H}\nabla_X C_R Y).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\nabla_X Y \in \Gamma(\mathcal{D}_2) \\
\Leftrightarrow\ &\begin{cases} \phi_R(\mathcal{V}\nabla_X B_R Y + \mathcal{A}_X C_R Y) + B_R(\mathcal{A}_X B_R Y + \mathcal{H}\nabla_X C_R Y) = 0, \\ P_R(\omega_R(\mathcal{V}\nabla_X B_R Y + \mathcal{A}_X C_R Y) + C_R(\mathcal{A}_X B_R Y + \mathcal{H}\nabla_X C_R Y)) = 0. \end{cases}
\end{aligned}$$

Therefore, we have the result. □

Similarly, we obtain:

Proposition 3.15. *Let F be a h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-v-semi-slant basis. Then the following conditions are equivalent*:

a) *the distribution $\mathcal{D}_1$ defines a totally geodesic foliation.*
b) $\phi_I \mathcal{A}_X IY + B_I \mathcal{H}\nabla_X IY = 0$ *and* $Q_I(\omega_I \mathcal{A}_X IY + C_I \mathcal{H}\nabla_X IY) = 0$
for $X, Y \in \Gamma(\mathcal{D}_1)$.
c) $\phi_J \mathcal{A}_X JY + B_J \mathcal{H}\nabla_X JY = 0$ *and* $Q_J(\omega_J \mathcal{A}_X JY + C_J \mathcal{H}\nabla_X JY) = 0$
for $X, Y \in \Gamma(\mathcal{D}_1)$.
d) $\phi_K \mathcal{A}_X KY + B_K \mathcal{H}\nabla_X KY = 0$ *and* $Q_K(\omega_K \mathcal{A}_X KY + C_K \mathcal{H}\nabla_X KY) = 0$
for $X, Y \in \Gamma(\mathcal{D}_1)$.

Theorem 3.16. *Let F be an almost h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such* *that (I, J, K) is an almost h-v-semi-slant basis. Then the following conditions are equivalent*:

a) *F is a totally geodesic map.*
b) $\omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0,$
$\omega_I(\widehat{\nabla}_X B_I Z + \mathcal{T}_X C_I Z) + C_I(\mathcal{T}_X B_I Z + \mathcal{H}\nabla_X C_I Z) = 0$
for $X, Y \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$.
c) $\omega_J(\widehat{\nabla}_X \phi_J Y + \mathcal{T}_X \omega_J Y) + C_J(\mathcal{T}_X \phi_J Y + \mathcal{H}\nabla_X \omega_J Y) = 0,$
$\omega_J(\widehat{\nabla}_X B_J Z + \mathcal{T}_X C_J Z) + C_J(\mathcal{T}_X B_J Z + \mathcal{H}\nabla_X C_J Z) = 0$
for $X, Y \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$.
d) $\omega_K(\widehat{\nabla}_X \phi_K Y + \mathcal{T}_X \omega_K Y) + C_K(\mathcal{T}_X \phi_K Y + \mathcal{H}\nabla_X \omega_K Y) = 0,$
$\omega_K(\widehat{\nabla}_X B_K Z + \mathcal{T}_X C_K Z) + C_K(\mathcal{T}_X B_K Z + \mathcal{H}\nabla_X C_K Z) = 0$
for $X, Y \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$.

Proof. Since F is a Riemannian submersion, we get

$$(\nabla F_*)(Z_1, Z_2) = 0 \quad \text{for } Z_1, Z_2 \in \Gamma((\ker F_*)^\perp).$$

For $X, Y \in \Gamma(\ker F_*)$, we have

$$\begin{aligned}(\nabla F_*)(X, Y) &= -F_*(\nabla_X Y) \\ &= F_*(I\nabla_X(\phi_I Y + \omega_I Y)) \\ &= F_*(\phi_I \widehat{\nabla}_X \phi_I Y + \omega_I \widehat{\nabla}_X \phi_I Y + B_I \mathcal{T}_X \phi_I Y + C_I \mathcal{T}_X \phi_I Y \\ &\quad + \phi_I \mathcal{T}_X \omega_I Y + \omega_I \mathcal{T}_X \omega_I Y + B_I \mathcal{H}\nabla_X \omega_I Y + C_I \mathcal{H}\nabla_X \omega_I Y).\end{aligned}$$

Thus,

$$(\nabla F_*)(X, Y) = 0 \Leftrightarrow \omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0.$$

For $X \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$, since $(\nabla F_*)(X, Z) = (\nabla F_*)(Z, X)$, it is sufficient to consider the following case:

$$\begin{aligned}(\nabla F_*)(X, Z) &= -F_*(\nabla_X Z) \\ &= F_*(I\nabla_X(B_I Z + C_I Z)) \\ &= F_*(\phi_I \widehat{\nabla}_X B_I Z + \omega_I \widehat{\nabla}_X B_I Z + B_I \mathcal{T}_X B_I Z + C_I \mathcal{T}_X B_I Z \\ &\quad + \phi_I \mathcal{T}_X C_I Z + \omega_I \mathcal{T}_X C_I Z + B_I \mathcal{H}\nabla_X C_I Z + C_I \mathcal{H}\nabla_X C_I Z).\end{aligned}$$

Thus,

$$(\nabla F_*)(X, Z) = 0 \Leftrightarrow \omega_I(\widehat{\nabla}_X B_I Z + \mathcal{T}_X C_I Z) + C_I(\mathcal{T}_X B_I Z + \mathcal{H}\nabla_X C_I Z) = 0.$$

Hence,

$$\text{a)} \Leftrightarrow \text{b)}.$$

Similarly, we get

$$\text{a)} \Leftrightarrow \text{c)} \quad \text{and} \quad \text{a)} \Leftrightarrow \text{d)}.$$

Therefore, we obtain the result. ☐

Let $F : (M, g_M) \mapsto (N, g_N)$ be a Riemannian submersion. The map F is called a Riemannian submersion *with totally umbilical fibers* if

$$\mathcal{T}_X Y = g_M(X,Y)H \quad \text{for } X, Y \in \Gamma(\ker F_*), \tag{3.2}$$

where H is the mean curvature vector field of the fiber.

Lemma 3.17. *Let F be an almost h-v-semi-slant submersion with totally umbilical fibers from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-v-semi-slant basis. Then we obtain*

$$H \in \Gamma(\mathcal{D}_2^R) \quad \textit{for } R \in \{I, J, K\}.$$

Proof. For $X, Y \in \Gamma(\mu_R)$, $W \in \Gamma(\mathcal{D}_1^R)$, and $R \in \{I, J, K\}$, we obtain

$$\mathcal{T}_X RY + \widehat{\nabla}_X RY = \nabla_X RY = R\nabla_X Y = B_R \mathcal{T}_X Y + C_R \mathcal{T}_X Y + \phi_R \widehat{\nabla}_X Y + \omega_R \widehat{\nabla}_X Y$$

so that using (3.2), we have

$$g_M(X, RY) g_M(H, W) = -g_M(X, Y) g_M(H, RW).$$

Interchanging the role of X and Y, we get

$$g_M(Y, RX) g_M(H, W) = -g_M(Y, X) g_M(H, RW)$$

so that comparing the above two equations, we obtain

$$g_M(X, Y) g_M(H, RW) = 0,$$

which means $H \in \Gamma(\mathcal{D}_2^R)$. □

Corollary 3.18. *Let F be an almost h-v-semi-slant submersion with totally umbilical fibers from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-v-semi-slant basis and $\mathcal{D}_1^R = (\ker F_*)^\perp$ for some $R \in \{I, J, K\}$. Then each fiber is minimal.*

4. Curvature tensors

Let F be an almost h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-v-semi-slant basis. Then for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset (\ker F_*)^\perp$ on M such that

$$(\ker F_*)^\perp = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in M$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $(\ker F_*)^\perp$. Furthermore,

$$C_R \mathcal{D}_2^R \subset \mathcal{D}_2^R, \quad B_R \mathcal{D}_2^R \subset \ker F_*, \quad \ker F_* = B_R \mathcal{D}_2^R \oplus \mu_R,$$

where μ_R is the orthogonal complement of $B_R \mathcal{D}_2^R$ in $\ker F_*$ and is invariant under R. To deal with the sectional curvatures of a Kähler manifold, as we know, it is enough to calculate its holomorphic sectional curvatures.

Given a plane P being invariant by R in T_pM, $p \in M$, $R \in \{I, J, K\}$, there is an orthonormal basis $\{X, RX\}$ of P. Denote by $K(P)$, $K_*(P)$, and $\widehat{K}(P)$ the sectional curvatures of the plane P in M, N, and the fiber $F^{-1}(F(p))$, respectively, where $K_*(P)$ denotes the sectional curvature of the plane $P_* = \langle F_*X, F_*RX \rangle$ in N. Let $K(X \wedge Y)$ be the sectional curvature of the plane spanned by the tangent vectors $X, Y \in T_pM$, $p \in M$. Using both Corollary 1 of [13, p. 465] and (1.28) of [7, p. 13], we obtain the following:

(1) If $P \subset (\mu_R)_p$, then with elementary computations we have

$$K(P) = \widehat{K}(P) + ||\mathcal{T}_X X||^2 - ||\mathcal{T}_X RX||^2 - g_M(\mathcal{T}_X X, R[RX, X]).$$

(2) If $P \subset (\mathcal{D}_2^R \oplus B_R \mathcal{D}_2^R)_p$ with $X \in (\mathcal{D}_2^R)_p$, then we obtain

$$\begin{aligned} K(P) = \sin^2 \theta_R \cdot K(X \wedge B_R X) &+ 2(g_M((\nabla_X \mathcal{A})(X, C_R X), B_R X) \\ &+ g_M(\mathcal{A}_X C_R X, \mathcal{T}_{B_R X} X) - g_M(\mathcal{A}_{C_R X} X, \mathcal{T}_{B_R X} X) \\ &- g_M(\mathcal{A}_X X, \mathcal{T}_{B_R X} C_R X)) + \cos^2 \theta_R \cdot K(X \wedge C_R X). \end{aligned}$$

(3) If $P \subset (\mathcal{D}_1^R)_p$, then we get

$$K(P) = K_*(P) - 3||\mathcal{V}R\nabla_X X||^2. \tag{4.1}$$

Using (4.1), we have:

Theorem 4.1. *Let F be an almost h-v-semi-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a space $(N(c), g_N)$ of constant holomorphic sectional curvature c such that (I, J, K) is an almost h-v-semi-slant basis. Assume that the distribution $\mathcal{D}_1^R$ is a totally geodesic foliation with $\dim \mathcal{D}_1^R > 0$ for some $R \in \{I, J, K\}$. Then we have*

$$K(P) = c \quad \textit{for any } R\textit{-invariant plane } P \subset \mathcal{D}_1^R.$$

Now we will introduce an inequality of a h-v-semi-slant submersion in terms of squared mean curvature, scalar curvature, and h-v-semi-slant angle.

Let (M, E, g_M) be a quaternionic Kähler manifold with a local quaternionic Hermitian basis $\{J_1, J_2, J_3\}$.

Given a nonzero tangent vector $X \in T_pM$, $p \in M$, we have a 4-dimensional subspace

$$Q(X) := \{a_0 X + a_1 J_1 X + a_2 J_2 X + a_3 J_3 X \in T_pM \mid a_i \in \mathbb{R}, 0 \leq i \leq 3\}.$$

Let $\overline{S}(X)$ be the set of all 2-dimensional subspaces P in $Q(X)$.

Denote by $K(P)$ the sectional curvature of the plane $P \subset T_pM$, $p \in M$, in (M, g_M).

Then we define a function $\rho_X : \overline{S}(X) \mapsto \mathbb{R}$ by

$$\rho_X(P) := K(P) \quad \text{for } P \in \overline{S}(X).$$

If the function ρ_X is constant, then we call ρ_X the *Q-sectional curvature*.

A quaternionic Kähler manifold (M, E, g_M) is said to be of *constant Q-sectional curvature c* if ρ_X is the Q-sectional curvature for any nonzero $X \in$

T_pM, $p \in M$ and the function $\rho : \overline{S} \mapsto \mathbb{R}$, given by $\rho(P) := K(P)$, is also constant and is equal to c, where $\overline{S} = \bigcup_{X \in TM} \overline{S}(X)$.

It is well-known that a quaternionic Kähler manifold (M, E, g_M) is of constant Q-sectional curvature c if and only if its curvature tensor is given by [14]

$$\begin{aligned}(4.2)\qquad R(X,Y)Z = \frac{c}{4}\{g_M(Z,Y)X - g_M(X,Z)Y + \sum_{i=1}^{3}(g_M(Z,J_iY)J_iX \\ - g_M(Z,J_iX)J_iY + 2g_M(X,J_iY)J_iZ)\}\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$.

Let $(M^n(c), E, g_M)$ be a $4n$-dimensional quaternionic Kähler manifold of constant Q-sectional curvature c and (N^{4n-2}, g_N) a $(4n-2)$-dimensional Riemannian manifold.

Let $F : (M^n(c), E, g_M) \mapsto (N^{4n-2}, g_N)$ be a h-v-semi-slant submersion.

Given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on U such that for any $R \in \{J_1, J_2, J_3\}$, there is a distribution $\mathcal{D}_1 \subset (\ker F_*)^\perp$ on U such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2,\ R(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in U$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$.

Assume that $\theta_R \in (0, \frac{\pi}{2})$ for some $R \in \{J_1, J_2, J_3\}$.

Using Remark 3.9, we can obtain a local orthonormal frame

$$\{e_1, J_1e_1, J_2e_1, J_3e_1, \ldots, e_{n-1}, J_1e_{n-1}, J_2e_{n-1}, J_3e_{n-1}, v, \sec\theta_R C_R v\}$$

of $(\ker F_*)^\perp$ such that $\{e_1, J_1e_1, J_2e_1, J_3e_1, \ldots, e_{n-1}, J_1e_{n-1}, J_2e_{n-1}, J_3e_{n-1}\} \subset \Gamma(\mathcal{D}_1)$, $\{v, \sec\theta_R C_R v\} \subset \Gamma(\mathcal{D}_2)$, and $\{\csc\theta_R B_R v, \csc\theta_R \sec\theta_R B_R C_R v\}$ is a local orthonormal frame of $\ker F_*$.

Denote by $\hat{\tau}$ and H the scalar curvature of any fiber and the mean curvature vector field of any fiber, respectively.

On U, we have

$$\hat{\tau} = \widehat{K}(\ker F_*) = \csc^4\theta_R \sec^2\theta_R g_M(\widehat{R}(B_Rv, B_RC_Rv)B_RC_Rv, B_Rv)$$

and

$$H = \frac{1}{2}\csc^2\theta_R(\mathcal{T}_{B_Rv}B_Rv + \sec^2\theta_R\mathcal{T}_{B_RC_Rv}B_RC_Rv),$$

where $\widehat{R}$ is the Riemannian curvature tensor of any fiber.

Denote by $||H||^2$ the squared mean curvature, i.e., $||H||^2 = g_M(H, H)$.

With the above notations, we get:

Theorem 4.2. *Let $(M^n(c), E, g_M)$ be a $4n$-dimensional quaternionic Kähler manifold of constant Q-sectional curvature c and (N^{4n-2}, g_N) a $(4n-2)$-dimensional Riemannian manifold.*

Let $F : (M^n(c), E, g_M) \mapsto (N^{4n-2}, g_N)$ *be a h-v-semi-slant submersion. Assume that* $\theta_R \in (0, \frac{\pi}{2})$ *for some* $R \in \{J_1, J_2, J_3\}$*.*
Then we have

$$||H||^2 \geq \frac{1}{2}\hat{\tau} - \frac{c}{8}(1 + 3\cos^2\theta_R) \tag{4.3}$$

with equality holding if and only if all the fibers are totally geodesic and $\theta_{J_i} = \frac{\pi}{2}$ *for* $J_i \in \{J_1, J_2, J_3\} - \{R\}$*.*

Proof. For convenience, let $v_1 := \csc\theta_R B_R v$ and $v_2 := \csc\theta_R \sec\theta_R B_R C_R v$.
Then we have

$$||H||^2 = \frac{1}{4}\{g_M(\mathcal{T}_{v_1}v_1, \mathcal{T}_{v_1}v_1) + g_M(\mathcal{T}_{v_2}v_2, \mathcal{T}_{v_2}v_2) + 2g_M(\mathcal{T}_{v_1}v_1, \mathcal{T}_{v_2}v_2)\}$$

and

$$\begin{aligned}\hat{\tau} &= g_M(\widehat{R}(v_1, v_2)v_2, v_1)\\ &= \frac{c}{4}\left(1 + 3\sum_{i=1}^{3} g_M(v_1, J_i v_2)^2\right) + g_M(\mathcal{T}_{v_1}v_1, \mathcal{T}_{v_2}v_2) - g_M(\mathcal{T}_{v_1}v_2, \mathcal{T}_{v_1}v_2).\end{aligned}$$

Since $B_R C_R + \phi_R B_R = 0$ on $(\ker F_*)^\perp$, by Remark 3.9, we obtain

$$\begin{aligned}g_M(v_1, Rv_2)^2 &= \csc^4\theta_R \cdot \sec^2\theta_R g_M(RB_R v, B_R C_R v)^2\\ &= \csc^4\theta_R \cdot \sec^2\theta_R g_M(B_R C_R v, B_R C_R v)^2\\ &= \cos^2\theta_R.\end{aligned}$$

Using the above results, we have

$$\begin{aligned}||H||^2 =& \frac{1}{2}\hat{\tau} - \frac{c}{8}\left(1 + 3\cos^2\theta_R + 3\sum_{J_i \neq R} g_M(v_1, J_i v_2)^2\right)\\ &+ \frac{1}{4}||\mathcal{T}_{v_1}v_1||^2 + \frac{1}{4}||\mathcal{T}_{v_2}v_2||^2 + \frac{1}{2}||\mathcal{T}_{v_1}v_2||^2.\end{aligned}$$

For any $J_i \in \{J_1, J_2, J_3\} - \{R\}$, since $g_M(v_1, \phi_{J_i}v_1) = g_M(v_2, \phi_{J_i}v_2) = 0$ and $g_M(v_1, \phi_{J_i}v_2) = -g_M(\phi_{J_i}v_1, v_2)$, we get

$$\begin{aligned}g_M(v_1, J_i v_2) = 0 &\Leftrightarrow g_M(v_1, \phi_{J_i}v_2) = 0\\ &\Leftrightarrow \phi_{J_i} = 0 \text{ on } \ker F_*\\ &\Leftrightarrow \theta_{J_i} = \frac{\pi}{2}.\end{aligned}$$

Hence,

$$||H||^2 \geq \frac{1}{2}\hat{\tau} - \frac{c}{8}(1 + 3\cos^2\theta_R)$$

with equality holding if and only if $\mathcal{T} = 0$ and $\theta_{J_i} = \frac{\pi}{2}$ for $J_i \in \{J_1, J_2, J_3\} - \{R\}$.

Therefore, we obtain the result. □

5. Examples

Note that given an Euclidean space $\mathbb{R}^{4m}$ with coordinates $(x_1, x_2, \ldots, x_{4m})$, we can canonically choose complex structures I, J, K on $\mathbb{R}^{4m}$ as follows:

$$I(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+2}},\ I(\tfrac{\partial}{\partial x_{4k+2}}) = -\tfrac{\partial}{\partial x_{4k+1}},\ I(\tfrac{\partial}{\partial x_{4k+3}}) = \tfrac{\partial}{\partial x_{4k+4}},$$
$$I(\tfrac{\partial}{\partial x_{4k+4}}) = -\tfrac{\partial}{\partial x_{4k+3}},\ J(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+3}},\ J(\tfrac{\partial}{\partial x_{4k+2}}) = -\tfrac{\partial}{\partial x_{4k+4}},$$
$$J(\tfrac{\partial}{\partial x_{4k+3}}) = -\tfrac{\partial}{\partial x_{4k+1}},\ J(\tfrac{\partial}{\partial x_{4k+4}}) = \tfrac{\partial}{\partial x_{4k+2}},\ K(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+4}},$$
$$K(\tfrac{\partial}{\partial x_{4k+2}}) = \tfrac{\partial}{\partial x_{4k+3}},\ K(\tfrac{\partial}{\partial x_{4k+3}}) = -\tfrac{\partial}{\partial x_{4k+2}},\ K(\tfrac{\partial}{\partial x_{4k+4}}) = -\tfrac{\partial}{\partial x_{4k+1}}$$

for $k \in \{0, 1, \ldots, m-1\}$.

Then we can check that $(I, J, K, \langle\ ,\ \rangle)$ is a hyperkähler structure on $\mathbb{R}^{4m}$, where $\langle\ ,\ \rangle$ denotes the Euclidean metric on $\mathbb{R}^{4m}$. Throughout this section, we will use these notations.

Example 5.1. Let (M, E, g) be an almost quaternionic Hermitian manifold. Let $\pi : TM \mapsto M$ be the natural projection. Then the map π is a strictly h-v-semi-slant submersion such that $\mathcal{D}_1 = (\ker \pi_*)^\perp$ [11].

Example 5.2. Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds. Let $F : M \mapsto N$ be a quaternionic submersion. Then the map F is a strictly h-v-semi-slant submersion such that $\mathcal{D}_1 = (\ker F_*)^\perp$ [11].

Example 5.3. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^3$ by

$$F(x_1, \ldots, x_8) = (x_5 \sin\alpha - x_7 \cos\alpha, x_1, x_2),$$

where α is constant. Then the map F is a strictly h-v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle \text{ and } \mathcal{D}_2 = \langle \sin\alpha \frac{\partial}{\partial x_5} - \cos\alpha \frac{\partial}{\partial x_7} \rangle$$

with the strictly h-v-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 5.4. Let (M, E, g_M) be a $4m$-dimensional almost quaternionic Hermitian manifold and (N, g_N) a $(4m-1)$-dimensional Riemannian manifold. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be a Riemannian submersion. Then the map F is an almost h-v-semi-slant submersion such that

$$\mathcal{D}_1^R = ((\ker F_*) \oplus R(\ker F_*))^\perp \text{ and } \mathcal{D}_2 = R(\ker F_*)$$

with the almost h-v-semi-slant angle $\theta_R = \frac{\pi}{2}$ for $R \in \{I, J, K\}$, where $\{I, J, K\}$ is a quaternionic Hermitian basis.

Example 5.5. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^8$ by

$$F(x_1, \ldots, x_{12}) = (\frac{x_5 - x_7}{\sqrt{2}}, x_8, \frac{x_9 + x_{11}}{\sqrt{2}}, x_{10}, x_1, x_3, x_2, x_4).$$

Then the map F is a h-v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle \text{ and } \mathcal{D}_2 = \langle \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{11}} \rangle$$

with the h-v-semi-slant angles $\{\theta_I = \frac{\pi}{4}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{4}\}$.

Example 5.6. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^6$ by

$$F(x_1, \dots, x_{12}) = (x_5 \cos\alpha + x_7 \sin\alpha, x_6 \sin\beta - x_8 \cos\beta, x_{11}, x_{12}, x_9, x_{10}),$$

where α and β are constant. Then the map F is a h-v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \rangle$$

and

$$\mathcal{D}_2 = \langle \cos\alpha \frac{\partial}{\partial x_5} + \sin\alpha \frac{\partial}{\partial x_7}, \sin\beta \frac{\partial}{\partial x_6} - \cos\beta \frac{\partial}{\partial x_8} \rangle$$

with the h-v-semi-slant angles $\{\theta_I, \theta_J = \frac{\pi}{2}, \theta_K\}$ such that $\cos\theta_I = |\sin(\alpha - \beta)|$ and $\cos\theta_K = |\cos(\alpha - \beta)|$.

Example 5.7. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^6$ by

$$F(x_1, \dots, x_{12}) = (x_8, x_7, \dots, x_3).$$

Then the map F is an almost h-v-semi-slant submersion such that

$$\begin{aligned}
&\mathcal{D}_1^I = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \dots, \frac{\partial}{\partial x_8} \rangle, \\
&\mathcal{D}_1^J = \mathcal{D}_1^K = \langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \rangle, \\
&\mathcal{D}_2^I = 0, \quad \mathcal{D}_2^J = \mathcal{D}_2^K = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle.
\end{aligned}$$

with the almost h-v-semi-slant angles $\{\theta_I = 0, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$.

Example 5.8. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^4$ by

$$F(x_1, \dots, x_{12}) = (x_2, x_5, x_1, x_7).$$

Then the map F is an almost h-v-semi-slant submersion such that

$$\begin{aligned}
&\mathcal{D}_1^I = \mathcal{D}_2^J = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle, \\
&\mathcal{D}_1^J = \mathcal{D}_2^I = \langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} \rangle, \\
&\mathcal{D}_1^K = 0, \quad \mathcal{D}_2^K = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} \rangle, \\
&\mathcal{D}_2^I = \langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} \rangle, \quad \mathcal{D}_2^J = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle,
\end{aligned}$$

with the almost h-v-semi-slant angles $\{\theta_I = \frac{\pi}{2}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$.

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