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A RESULT ON A CONJECTURE OF W. LÜ, Q. LI AND C. YANG

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Abstract. In this paper, we investigate the problem of transcendental entire functions that share two values with one of their derivative. Let f be a transcendental entire function, n and k be two positive integers. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM, and $n \geq k+1$, then $(f^n)^{(k)} \equiv \frac{Q_2}{Q_1} f^n$. Furthermore, if $Q_1 = Q_2$, then $f = ce^{\frac{\lambda}{n}z}$, where Q_1, Q_2 are polynomials with $Q_1 Q_2 \not\equiv 0$, and c, λ are non-zero constants such that $\lambda^k = 1$. This result shows that the Conjecture given by W. Lü, Q. Li and C. Yang [On the transcendental entire solutions of a class of differential equations, Bull. Korean Math. Soc. 51 (2014), no. 5, 1281–1289.] is true. Also we exhibit some examples to show that the conditions of our result are the best possible.

1. Introduction definitions and results

In this paper, by a meromorphic (resp. entire) function we will always mean meromorphic (resp. entire) function in the complex plane $\mathbb{C}$. We denote by $n(r, \infty; f)$ the number of poles of f lying in $|z| < r$, the poles are counted according to their multiplicities. The quantity

$$N(r, \infty; f) = \int_0^r \frac{n(t, \infty; f) - n(0, \infty; f)}{t} dt + n(0, \infty; f) \log r$$

is called the integrated counting function or simply the counting function of poles of f.

Also $m(r, \infty; f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$ is called the proximity function of poles of f, where $\log^+ x = \log x$, if $x \geq 1$ and $\log^+ x = 0$, if $0 \leq x < 1$.

The sum $T(r, f) = m(r, \infty; f) + N(r, \infty; f)$ is called the Nevanlinna's characteristic function of f. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ except possibly a set of finite linear measure.

For $a \in \mathbb{C}$, we put $N(r, a; f) = N(r, \infty; \frac{1}{f-a})$ and $m(r, a; f) = m(r, \infty; \frac{1}{f-a})$.

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Let us denote by $\overline{n}(r,a;f)$ the number of distinct a points of f lying in $|z|<r$, where $a\in\mathbb{C}\cup\{\infty\}$. The quantity

$$\overline{N}(r,a;f)=\int_0^r\frac{\overline{n}(t,a;f)-\overline{n}(0,a;f)}{t}dt+\overline{n}(0,a;f)\log r$$

denotes the reduced counting function of a points of f (see, e.g., [5, 13]).

Let k be a positive integer and $a\in\mathbb{C}\cup\{\infty\}$. We use $N_{k)}(r,a;f)$ to denote the counting function of a-points of f with multiplicity not greater than k, $N_{(k+1}(r,a;f)$ to denote the counting function of a-points of f with multiplicity greater than k. Similarly $\overline{N}_{k)}(r,a;f)$ and $\overline{N}_{(k+1}(r,a;f)$ are their reduced functions respectively.

Let f and g be two non-constant meromorphic functions and let a be a complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM.

A meromorphic function a is said to be a small function of f provided that $T(r,a)=S(r,f)$, that is $T(r,a)=o(T(r,f))$ as $r\to\infty$ except possibly a set of finite linear measure.

During the last four decades the uniqueness theory of entire and meromorphic functions has become a prominent branch of the value distribution theory (see [13]).

Rubel-Yang [9] proposed to investigate the uniqueness of an entire function f under the assumption that f and its derivative $f^{'}$ share two complex values. Subsequently, related to one or two value sharing similar considerations have been made with respect to higher derivatives and more general (linear) differential expressions by Brück [1], Gundersen [3], Mues-Steinmetz [8], Yang [11] et al.

In this direction an interesting problem still open is the following conjecture proposed by Brück [1].

Conjecture A. *Let f be a non-constant entire function. Suppose*

$$\rho_1(f):=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and $f^{'}$ share one finite value a CM, then

$$\frac{f^{'}-a}{f-a}=c$$

for some non-zero constant c.

The case that $a=0$ and that $N(r,0;f^{'})=S(r,f)$ had been proved by Brück [1] while the case that f is of finite order had been proved by Gundersen-Yang [4]. However, the corresponding conjecture for meromorphic functions fails in general (see [4]).

To the knowledge of the author perhaps Yang-Zhang [12] (see also [14]) were the first to consider the uniqueness of a power of a meromorphic (resp. entire) function $F = f^n$ and its derivative $F^{'}$ when they share certain value as this type of considerations gives most specific form of the function.

As a result during the last decade, growing interest has been devoted to this setting of entire functions. Improving all the results obtained in [12], Zhang [14] proved the following theorem.

Theorem A ([14])**.** *Let f be a non-constant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and*

$$n > k + 4,$$

then $f^n \equiv (f^n)^{(k)}$, and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

In 2009 Zhang and Yang [15] further improved the above result in the following manner.

Theorem B ([15])**.** *Let f be a non-constant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and*

$$n > k + 1.$$

Then conclusion of Theorem A holds.

In 2010 Zhang and Yang [16] further improved the above result in the following manner.

Theorem C ([16])**.** *Let f be a non-constant entire function, n and k be positive integers. Suppose f^n and $(f^n)^{(k)}$ share 1 CM and*

$$n \geq k + 1.$$

Then conclusion of Theorem A holds.

In 2011 Lü and Yi [6] proved the following theorem by using the theory of normal families.

Theorem D ([6])**.** *Let f be a transcendental entire function, n, k be two integers with $n \geq k + 1$, $F = f^n$ and $Q \not\equiv 0$ be a polynomial. If $F - Q$ and $F^{(k)} - Q$ share the value 0 CM, then $F \equiv F^{(k)}$ and $f = ce^{wz/n}$, where c and w are non-zero constants such that $w^k = 1$.*

Now observing the above theorem W. Lü, Q. Li and C. Yang [7] asked the following question.

Question. What can be said if $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share the value 0 CM ? Where Q_1, Q_2 are polynomials, $Q_1 Q_2 \not\equiv 0$.

In [7] W. Lü, Q. Li and C. Yang solved the above question for $k = 1$ by giving the transcendental entire solutions of the equation

$$F' - Q_1 = Re^{\alpha}(F - Q_2), \tag{1.1}$$

where $F = f^n$, R is a rational function and α is an entire function and they obtained the following results.

Theorem E ([7]). *Let f be a transcendental entire function and let $F = f^n$ be a solution of equation (1.1), $n \geq 2$ be an integer. Then $\frac{Q_1}{Q_2}$ is a polynomial, and*

$$f' = \frac{Q_1}{nQ_2} f.$$

Theorem F ([7]). *Let f be a transcendental entire function, $n \geq 2$ be an integer. If $f^n - Q$ and $(f^n)' - Q$ share 0 CM, where $Q \not\equiv 0$ is a polynomial, then*

$$f = ce^{z/n},$$

where c is a non-zero constant.

As well, at the end of the paper W. Lü, Q. Li and C. Yang [7] posed the following conjecture.

Conjecture B. *Let f be a transcendental entire function, n be a positive integer. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM, and $n \geq k + 1$, then $(f^n)^{(k)} = \frac{Q_2}{Q_1} f^n$. Furthermore, if $Q_1 = Q_2$, then $f = ce^{wz/n}$, where Q_1, Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$, and, c, w are non-zero constants such that $w^k = 1$.*

Our objective to write this paper is to solve the above Conjecture B. The following theorem is the main result in this paper.

Theorem 1.1. *Let f be a transcendental entire function, n and k be two positive integers. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM, and $n \geq k+1$, then $(f^n)^{(k)} = \frac{Q_2}{Q_1} f^n$. Furthermore, if $Q_1 = Q_2$, then $f = ce^{\frac{\lambda}{n}z}$, where Q_1, Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$, and, c, λ are non-zero constants such that $\lambda^k = 1$.*

Remark 1.1. It is easy to see that the condition $n \geq k + 1$ in Theorem 1.1 is sharp by the following examples.

Example 1.1. Let

$$f(z) = e^{2z} + z.$$

Then $f - Q_1$ and $f' - Q_2$ share 0 CM, but $f' \not\equiv \frac{Q_2}{Q_1} f$, where $Q_1(z) = z + 1$ and $Q_2(z) = 3$.

Example 1.2. Let

$$f(z) = e^{2z} + z^2 + z.$$

Then $f - Q_1$ and $f' - Q_2$ share 0 CM, but $f' \not\equiv \frac{Q_2}{Q_1} f$, where $Q_1(z) = z^2 + z + 1$ and $Q_2(z) = 2z + 3$.

Remark 1.2. By the following example, it is easy to see that the hypothesis of the transcendental of f in Theorem 1.1 is necessary.

Example 1.3. Let

$$f(z) = z.$$

Then $f^2 - Q_1$ and $(f^2)' - Q_2$ share 0 CM, but $(f^2)' \not\equiv \frac{Q_2}{Q_1} f^2$, where $Q_1(z) = 2z^2 + z$ and $Q_2(z) = 2z^2 + 4z$.

2. Lemmas

In this section we present the lemmas which will be needed in the sequel.

Lemma 2.1 ([2])**.** *Suppose that f is a transcendental meromorphic function and that*

$$f^n P(f) = Q(f),$$

where $P(f)$ and $Q(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients and the degree of $Q(f)$ is at most n. Then

$$m(r, \infty; P(f)) = S(r, f).$$

Lemma 2.2 ([10])**.** *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0), a_{n-1}(z), \ldots, a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.3 ([5])**.** *Let f be a non-constant meromorphic function and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then*

$$T(r, f) \le \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).$$

3. Proofs of the theorems

Proof of Theorem 1.1. Let $F_1 = \frac{f^n}{Q_1}$ and $G_1 = \frac{[f^n]^{(k)}}{Q_2}$. Clearly F_1 and G_1 share 1 CM except for the zeros of $Q_i(z)$, where $i = 1, 2$ and so

$$\overline{N}(r, 1; F_1) = \overline{N}(r, 1; G_1) + S(r, f).$$

Let

$$\Phi = \frac{F_1'(F_1 - G_1)}{F_1(F_1 - 1)}. \tag{3.1}$$

We now consider the following two cases.

Case 1. $\Phi \not\equiv 0$.

From the fundamental estimate of logarithmic derivative it follows that

$$m(r,\infty;\Phi) = S(r,f).$$

Let z_0 be a zero of f of multiplicity p such that $Q_i(z_0) \neq 0$, where $i = 1,2$. Then z_0 will be a zero of F_1 and G_1 of multiplicities np and $np-k$ respectively and so from (3.1) we get

$$\Phi(z) = O((z-z_0)^{np-k-1}). \tag{3.2}$$

Since $n \geq k+1$, it follows that Φ is holomorphic at z_0. From this and the hypotheses of Theorem 1.1 we see that

$$N(r,\infty;\Phi) = S(r,f).$$

Consequently $T(r,\Phi) = S(r,f)$. Also from (3.1) we get

$$\frac{1}{F_1} = \frac{1}{\Phi}\frac{F_1^{'}}{F_1(F_1-1)}[1 - \frac{Q_1}{Q_2}\frac{(f^n)^{(k)}}{f^n}]$$

and so

$$m(r,\infty;\frac{1}{F_1}) = S(r,f).$$

Hence

$$m(r,\infty;\frac{1}{f}) = S(r,f). \tag{3.3}$$

We consider the following two subcases.

Subcase 1.1. Let $n > k+1$.

From (3.2) we see that

$$N(r,0;f) \leq N(r,0;\Phi) \leq T(r,\frac{1}{\Phi}) \leq T(r,\Phi) + O(1) = S(r,f). \tag{3.4}$$

Now from (3.3) and (3.4) we get

$$T(r,f) = S(r,f),$$

which is a contradiction.

Subcase 1.2. Let $n = k+1$.

From (3.2) we see that

$$N_{(2}(r,0;f) \leq N(r,0;\Phi) \leq T(r,\Phi) + O(1) = S(r,f).$$

Then (3.3) gives

$$T(r,f) = N_{1)}(r,0;f) + S(r,f). \tag{3.5}$$

Let

$$F = f^n. \tag{3.6}$$

Since $F-Q_1$ and $F^{(k)}-Q_2$ share 0 CM, then there exists an entire function α, such that

$$F^{(k)}-Q_2=e^{\alpha}(F-Q_1). \tag{3.7}$$

First we suppose that α is a non-constant entire function.
By differentiation from (3.7) we get

$$F^{(k+1)}-Q_2^{'}=\alpha^{'}e^{\alpha}(F-Q_1)+e^{\alpha}(F^{'}-Q_1^{'}). \tag{3.8}$$

Now combining (3.7) and (3.8) we get

$$\begin{aligned} &F^{(k+1)}F-\alpha^{'}F^{(k)}F-F^{(k)}F^{'} \\ &=Q_1F^{(k+1)}-(\alpha^{'}Q_1+Q_1^{'})F^{(k)}-Q_2F^{'}+(Q_2^{'}-\alpha^{'}Q_2)F \\ &\quad+\alpha^{'}Q_1Q_2+Q_2Q_1^{'}-Q_1Q_2^{'}. \end{aligned} \tag{3.9}$$

Note that from (3.7) we get

$$T(r,e^{\alpha})\leq(k+2)T(r,f^n)+O(\log r)+S(r,f)=n(k+2)T(r,f)+S(r,f).$$

Since $T(r,\alpha^{'})=S(r,e^{\alpha})$, it follows that $T(r,\alpha^{'})=S(r,f)$.

By induction, we deduce from (3.6) that

$$F^{'}=nf^{n-1}f^{'},$$

$$F^{''}=n(n-1)f^{n-2}(f^{'})^2+nf^{n-1}f^{''},$$

$$F^{'''}=n(n-1)(n-2)f^{n-3}(f^{'})^3+3n(n-1)f^{n-2}f^{'}f^{''}+nf^{n-1}f^{'''}$$

and so on.

Thus in general we have

$$F^{(k)}=\sum_{\lambda}a_{\lambda}f^{l_0^{\lambda}}(f^{'})^{l_1^{\lambda}}\cdots(f^{(k)})^{l_k^{\lambda}}, \tag{3.10}$$

where l_0^{λ}, $l_1^{\lambda},\ldots,l_k^{\lambda}$ are non-negative integers satisfying $\sum_{j=0}^{k}l_j^{\lambda}=n$, $n-k\leq l_0^{\lambda}\leq n-1$ and a_{λ} are constants.

Also we set

$$F^{(k+1)}=\sum_{\lambda}b_{\lambda}f^{p_0^{\lambda}}(f^{'})^{p_1^{\lambda}}\cdots(f^{(k+1)})^{p_{k+1}^{\lambda}}, \tag{3.11}$$

where p_0^{λ}, $p_1^{\lambda},\ldots,p_{k+1}^{\lambda}$ are non-negative integers satisfying $\sum_{j=0}^{k+1}p_j^{\lambda}=n$, $n-k-1\leq p_0^{\lambda}\leq n-1$, i.e., $0\leq p_0^{\lambda}\leq n-1$ and b_{λ} are constants.

Substituting (3.6), (3.10) and (3.11) into (3.9), we have

$$f^nP(f)=Q(f), \tag{3.12}$$

where $Q(f)$ is a differential polynomial in f of degree n and

$$P(f)=\sum_{\lambda}b_{\lambda}f^{p_0^{\lambda}}(f^{'})^{p_1^{\lambda}}\cdots(f^{(k)})^{p_{k+1}^{\lambda}}-\alpha^{'}\sum_{\lambda}a_{\lambda}f^{l_0^{\lambda}}(f^{'})^{l_1^{\lambda}}\cdots(f^{(k)})^{l_k^{\lambda}} \tag{3.13}$$

$$-nf^{'}\sum_{\lambda}a_{\lambda}f^{l_0^{\lambda}-1}(f^{'})^{l_1^{\lambda}}\cdots(f^{(k)})^{l_k^{\lambda}}$$
$$=A(f^{'})^{k+1}+R_1(f),$$

is a differential polynomial in f of the degree $k+1$, where A is a suitable constant and $R_1(f)$ is a differential polynomial in f. In particular every monomial of R_1 has the form

$$R(\alpha^{'})f^{q_0^{\lambda}}(f^{'})^{q_1^{\lambda}}\cdots(f^{(k+1)})^{q_{k+1}^{\lambda}},$$

where $q_0^{\lambda},\ldots,q_{k+1}^{\lambda}$ are non-negative integers satisfying $\sum_{j=0}^{k+1}q_j^{\lambda}=n$ and $1\leq q_0^{\lambda}\leq n-1$, $R(\alpha^{'})$ is a polynomial in $\alpha^{'}$ with constant coefficients.

First we suppose $P(f)\not\equiv 0$. Then by Lemma 2.1 we get $m(r,\infty;P)=S(r,f)$ and so

$$T(r,P)=S(r,f). \tag{3.14}$$

Clearly

$$T(r,P^{'})=S(r,f). \tag{3.15}$$

Note that from (3.13) we get

$$P^{'}(f)=A(k+1)(f^{'})^{k}f^{''}+B\alpha^{'}(f^{'})^{k+1}+S_1(f), \tag{3.16}$$

is a differential polynomial in f, where B is a suitable constant and $S_1(f)$ is a differential polynomial in f. In particular every monomial of S_1 has the form

$$S(\alpha^{'})f^{r_0^{\lambda}}(f^{'})^{r_1^{\lambda}}\cdots(f^{(k+1)})^{r_{k+1}^{\lambda}},$$

where $r_0^{\lambda},\ldots,r_{k+1}^{\lambda}$ are non-negative integers satisfying $\sum_{j=0}^{k+1}r_j^{\lambda}=n$ and $1\leq r_0^{\lambda}\leq n-1$, $S(\alpha^{'})$ is a polynomial in $\alpha^{'}$ with constant coefficients.

Let z_1 be a simple zero of f. Then from (3.13) and (3.16) we have

$$P(z_1)=A\{f^{'}(z_1)\}^{k+1}$$

and

$$P^{'}(z_1)=A(k+1)\{f^{'}(z_1)\}^{k}f^{''}(z_1)+B\alpha^{'}\{f^{'}(z_1)\}^{k+1}.$$

This shows that z_1 is a zero of $Pf^{''}-[K_1P^{'}-K_2\alpha^{'}P]f^{'}$, where K_1 and K_2 are suitably constants. Let

$$\Phi_1=\frac{Pf^{''}-[K_1P^{'}-K_2\alpha^{'}P]f^{'}}{f}. \tag{3.17}$$

Clearly $\Phi_1\not\equiv 0$ and

$$T(r,\Phi_1)=S(r,f).$$

From (3.17) we obtain

$$f^{''}=\alpha_1f+\beta_1f^{'}, \tag{3.18}$$

where

$$\alpha_1 = \frac{\Phi_1}{P}, \qquad \beta_1 = K_1 \frac{P'}{P} - K_2 \alpha' \tag{3.19}$$

and

$$T(r, \alpha_1) = S(r, f), \qquad T(r, \beta_1) = S(r, f).$$

Now from (3.13) and (3.19) we get

$$\begin{aligned} P' &= (\frac{\beta_1}{K_1} + \frac{K_2}{K_1}\alpha')P \\ &= A(\frac{\beta_1}{K_1} + \frac{K_2}{K_1}\alpha')(f')^{k+1} + (\frac{\beta_1}{K_1} + \frac{K_2}{K_1}\alpha')R_1(f). \end{aligned} \tag{3.20}$$

Also from (3.16) and (3.18) we obtain

$$P' = A(k+1)\alpha_1 f(f')^k + \{A(k+1)\beta_1 + B\alpha'\}(f')^{k+1} + S_1(f). \tag{3.21}$$

By (3.20) and (3.21) we get

$$\begin{aligned} &(\frac{A}{K_1}\beta_1 - A(k+1)\beta_1 + \frac{AK_2}{K_1}\alpha' + B\alpha')(f')^{k+1} + A(k+1)\alpha_1 f(f')^k \\ &+ (\frac{\beta_1}{K_1} + \frac{K_2}{K_1}\alpha')R_1(f) + S_1(f) \equiv 0. \end{aligned} \tag{3.22}$$

Since $\alpha_1 \not\equiv 0$, from (3.22) we get

$$N_{1)}(r, 0; f) = S(r, f). \tag{3.23}$$

Therefore from (3.5) and (3.23) we have

$$T(r, f) = S(r, f),$$

which is a contradiction.

Next we suppose $P(f) \equiv 0$. Then $Q(f) \equiv 0$ from (3.12), where

$$\begin{aligned} Q(f) = Q_1 F^{(k+1)} &- (\alpha' Q_1 + Q_1')F^{(k)} - Q_2 F' + (Q_2' - \alpha' Q_2)F \\ &+ \alpha' Q_1 Q_2 + Q_2 Q_1' - Q_1 Q_2'. \end{aligned}$$

We get from (3.9) that

$$F^{(k+1)}F - \alpha' F^{(k)}F - F^{(k)}F' \equiv 0,$$

i.e.,

$$\frac{F^{(k+1)}}{F^{(k)}} \equiv \alpha' + \frac{F'}{F}. \tag{3.24}$$

By integration we have $F^{(k)} = dFe^{\alpha}$, where d is a non-zero constant. Substituting this and (3.6) into (3.7) we have

$$(d-1)f^n = \frac{Q_2 - Q_1 e^{\alpha}}{e^{\alpha}}.$$

Clearly $d \neq 1$ and all zeros of $Q_2 - Q_1 e^{\alpha}$ have the multiplicities at least n. Since $n = k+1$, by Lemma 2.3 we get

$$\begin{aligned} T(r, e^{\alpha}) &\leq \overline{N}(r, 0; e^{\alpha}) + \overline{N}(r, \infty; e^{\alpha}) + \overline{N}(r, \frac{Q_2}{Q_1}; e^{\alpha}) + S(r, e^{\alpha}) \\ &\leq \frac{1}{n} N(r, \frac{Q_2}{Q_1}; e^{\alpha}) + S(r, e^{\alpha}) \\ &\leq \frac{1}{n} T(r, e^{\alpha}) + S(r, e^{\alpha}), \end{aligned}$$

which is a contradiction since we first assume that e^{α} is a non-constant entire function.

Next we suppose that e^{α} is a non-zero constant, say D. Then from (3.7) we have

$$F^{(k)} - DF \equiv Q_2 - DQ_1. \tag{3.25}$$

Since $n = k+1$, it follows from (3.25) that

$$N(r, 0; f) = S(r, f).$$

Now by (3.3) we get

$$T(r, f) = S(r, f),$$

which is a contradiction.

Case 2. $\Phi \equiv 0$.

Now from (3.1) we get $F_1 \equiv G_1$, i.e., $(f^n)^{(k)} \equiv \frac{Q_2}{Q_1} f^n$. Furthermore if $Q_1 = Q_2$, then $f^n \equiv (f^n)^{(k)}$, and f assumes the form

$$f(z) = c e^{\frac{\lambda}{n} z},$$

where c is a nonzero constant and $\lambda^k = 1$. □

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